

Weak adjoint functor theorems

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joint work with John Bourke and Lukáš Vokřínek

Adjoint Functor Theorems

ur-AFT

category \mathcal{B} with **all** limits

$U: \mathcal{B} \rightarrow \mathcal{A}$ preserves them

U has left adjoint

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General AFT

category \mathcal{B} with small limits

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Solution Set Condition

U has left adjoint

more Adjoint Functor Theorems

General AFT (Freyd)

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SSC

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Enriched AFT (Kelly)

\mathcal{V} -category \mathcal{B} with small limits

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Weak AFT (Kainen)

category \mathcal{B} with small products

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U has **weak** left adjoint

$$\begin{array}{ccc} A & \xrightarrow{\eta} & UFA \\ & \searrow f & \downarrow Uf' \\ & & UB \end{array} \quad \begin{array}{c} FA \\ \downarrow \exists f' \\ B \end{array}$$

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Very (!) General AFT

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Enriched weakness

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 \qquad
 \begin{array}{ccc}
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 B & &
 \end{array}
 \qquad
 \mathcal{B}(FA, B) \xrightarrow{\text{surj.}} \mathcal{A}(A, UB)$$

- ▶ enriched categories have homs $\mathcal{C}(C, D)$ lying in \mathcal{V}
- ▶ (Lack-Rosicky) “Enriched Weakness” uses class \mathcal{E} of morphisms in \mathcal{V} to play the role of surjections
- ▶ $\mathcal{V} = \mathbf{Set}$, $\mathcal{E} = \{\text{surjections}\}$ gives unenriched weakness
- ▶ $\mathcal{E} = \{\text{isomorphisms}\}$ gives “non-weak weakness”

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$U: \mathcal{B} \rightarrow \mathcal{A}$ preserves them

SSC

U has \mathcal{E} -weak left adjoint

Examples

$$\mathcal{B}(FA, B) \xrightarrow{\mathcal{E}} \mathcal{A}(A, UB)$$

\mathcal{V}	\mathcal{E}	\mathcal{E} -weak left adjoint
Set	isos	left adjoint
Set	surjections	weak left adjoint
\mathcal{V}	isos	(enriched) left adjoint
Cat	equivalences	left biadjoint

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Mr Retract Equivalence



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sSet	shrinkable morphisms	

Definition

A morphism $p: X \rightarrow Y$ of simplicial sets is *shrinkable* (dual strong deformation retract) if it is contractible in **sSet**/ Y :

- ▶ it has a section s
- ▶ with a homotopy $s \circ p \sim 1_X$
- ▶ such that induced homotopy $p \circ s \circ p \sim p$ is trivial.

The setting

Let \mathcal{V} be a monoidal model category with cofibrant unit I

...

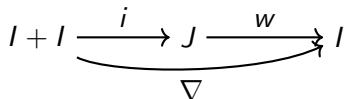
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An *interval* in \mathcal{V} is a factorization

$$I + I \xrightarrow{i} J \xrightarrow{w} I$$


of the codiagonal with $i \in \mathcal{I}$ and $w \in \mathcal{W}$.

- ▶ A morphism $A \rightarrow B$ in a \mathcal{V} -category \mathcal{C} is a morphism $I \rightarrow \mathcal{C}(A, B)$.
- ▶ A homotopy between morphisms $A \rightarrow B$ is a morphism $J \rightarrow \mathcal{C}(A, B)$ (for some interval)
- ▶ A homotopy is trivial if it factorizes through w
- ▶ A morphism p is shrinkable if there exist s with $p \circ s = 1$, and homotopy $s \circ p \sim 1$ with $p \circ s \circ p \sim p$ trivial.

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Examples (of \mathcal{V})

\mathcal{V}	\mathcal{I}	\mathcal{W}	shrinkable morphisms
Set	all	isos	isos
\mathcal{V}	all	isos	isos
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Cat	inj obj	equiv	retract equivalences
sSet	mono	wk hty equiv	(Kan) shrinkable
sSet	mono	wk cat equiv	(Joyal) shrinkable
2-Cat		biequivalences	surj, full biequivalences

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In general, for $f: X \rightarrow Y$ in \mathcal{V} :

- ▶ trivial fibration \Rightarrow shrinkable
(if X, Y cofibrant)
- ▶ shrinkable \Rightarrow weak equiv
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\mathcal{V} -category \mathcal{B} with \square **limits**
 $U: \mathcal{B} \rightarrow \mathcal{A}$ preserves them
 SSC

U has \mathcal{E} -weak left adjoint
 for $\mathcal{E} = \{\text{shrinkables}\}$

The limits in question

Limit of a functor $S: \mathcal{D} \rightarrow \mathcal{B}$ is defined by natural isos

$$\mathcal{B}(B, \lim S) \cong [\mathcal{D}, \mathbf{Set}](\Delta 1, \mathcal{B}(B, S)).$$

Limit of \mathcal{V} -functor $S: \mathcal{D} \rightarrow \mathcal{B}$ weighted by $G: \mathcal{D} \rightarrow \mathcal{V}$ defined by

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The power of $A \in \mathcal{B}$ by $X \in \mathcal{V}$ defined by

$$\mathcal{B}(B, A^X) \cong \mathcal{V}(X, \mathcal{B}(B, A))$$

A weight $G: \mathcal{D} \rightarrow \mathcal{V}$ is cofibrant
if it is projective with respect to
the pointwise trivial fibrations:

A \mathcal{V} -category \mathcal{B} has enough cofibrant limits if for any weight G
there is a cofibrant G' with a pointwise trivial fibration $G' \rightarrow G$ for
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$$\begin{array}{ccc}
 & H & HD \\
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 G & K & KD
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 \quad D \in \mathcal{P}$$

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A \mathcal{V} -category \mathcal{B} has **enough cofibrant limits** if for any weight G there is a cofibrant G' with a pointwise trivial fibration $G' \rightarrow G$ for which \mathcal{B} has G' -weighted limits.

The VGAF

Let \mathcal{V} be a monoidal model category with cofibrant unit, and \mathcal{E} the shrinkable morphisms.

V(ery) G(eneral) AFT

\mathcal{V} -category \mathcal{B} with all powers and enough cofibrant limits

$U: \mathcal{B} \rightarrow \mathcal{A}$ preserves them

SSC

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The VGAF

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\mathcal{B}_0 and \mathcal{A}_0 are accessible, U_0 accessible functor (unenriched)

U has an \mathcal{E} -weak left adjoint

Applications (to 2-categories)

This involves the case $\mathcal{V} = \mathbf{Cat}$.

A 2-category will have enough cofibrant limits if it has PIE limits:
that is, if it has

- ▶ products
- ▶ inserters
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Mr PIE



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Theorem

Let \mathcal{B} be a 2-category with PIE limits, and let $U: \mathcal{B} \rightarrow \mathcal{A}$ preserve them. If U satisfies SSC, then it has a left biadjoint.

Theorem

Any accessible 2-category with PIE limits has bicolimits.

Application (to Riehl-Verity ∞ -cosmoi)

An ∞ -cosmos is a **sSet**-category with all powers, enough cofibrant limits, and certain further structure.

These are intended to be a model-independent framework in which to study the totality of $(\infty, 1)$ -categories and related structures.

Corollary

Any accessible ∞ -cosmos has weak colimits.

A cosmological functor is an enriched functor between ∞ -cosmoi which preserves this structure.

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Any cosmological functor satisfying the SSC has a weak left adjoint.

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