Glueing Algebraic Structures on a 2-Category

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Abstract

We study the glueing constructions (comma objects) on general algebraic structures on a 2-category, described in terms of 2-monads and adjunctions. Specifically, lifting theorems for the comma objects and change-of-base results on both algebras of 2-monads and adjunctions in a 2-category are presented.

As a leading example, we take the 2-monad on **Cat** whose algebras are symmetric monoidal categories, and show that many of the constructions in our previous work on models of linear type theories can be derived within this axiomatics.

1 Introduction

In the previous work [2, 3] we have considered a glueing construction for symmetric monoidal (closed) categories, for studying the logical predicates for models of linear type theories. In that construction the glueing functor is supposed to be *lax* symmetric monoidal, thus preserves the structure only up to a few coherent morphisms, not up to isomorphisms or identity.

From a view of the study of categories with algebraic structures [8] (which generalizes the study of sets with algebraic structures), symmetric monoidal categories are algebras of a 2-monad on **Cat**, while lax symmetric monoidal functors are lax morphisms between the algebras. The case study on symmetric monoidal structures suggests that, general algebraic structures determined by a 2-monad on a 2-category enjoy a glueing construction along a lax morphism. In fact this is the case, and this is the first statement of this paper (Theorem 2.1).

However, in the same work we actually dealt with the glueing of symmetric monoidal *closed* structures. As well known, symmetric monoidal closed categories are not algebras of a 2-monad on **Cat**; but the closed structure is determined as right adjoints of the symmetric monoidal products, thus their glueing can be obtained by combining the observation on glueing algebras of 2-monads together with a form of adjoint-lifting theorem. So, in Section 2 we also state a result on glueing adjunctions (Theorem 2.4). Together with Theorem 2.1, we can derive many of the results in [2, 3] in this 2-categorical axiomatics.

Section 3 is devoted to two change-of-base results of the algebras as well as adjunctions, which involve the idiom of (2-categorical generalizations of) fibrations and cofibrations. The assumptions might be seen somewhat artificial. However, since a comma object can be characterized as a pullback of the codomain (co)fibration, the setting in this section can be naturally related to that in Section 2. In any case, we use the change-of-base result for adjunctions (Theorem 3.2) for deriving Theorem 2.4.

These results show the applicability of the glueing constructions for interesting algebraic structures. In principle, we can handle any structures which arise as algebras of 2-monads on 2-categories, as well as those with some operators determined in terms of adjunctions. Examples related to computer science include cartesian closed categories, (symmetric) monoidal closed categories, symmetric monoidal adjunctions between a cartesian closed category and a symmetric monoidal closed category, and closed Freyd categories. Most cases take place in the 2-category **Cat**, except the last one for which we work in the 2-category of $\mathbf{Set}^{\rightarrow}$-categories.

As direct applications we have full completeness of the translations between various type theories whose models are described in terms of these algebraic structures. We sketch an instance in Section 4.

Preliminaries

Comma Objects. Throughout this paper we work on a 2-category $\mathcal{C}$ with comma objects ((op)lax limit [6]) $\mathbb{B}\downarrow\Gamma$ for each 1-cell $\Gamma : \mathbb{A} \rightarrow \mathbb{B}$, which is a lax cone

$$\begin{tikzcd} & \mathbb{B}\downarrow\Gamma \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[d, phantom, "\overset{\lambda}{\Leftarrow}"] & \\ \mathbb{A} \arrow[rr, "\Gamma"'] & {} & \mathbb{B} \end{tikzcd}$$

satisfying two universal properties [9]:

1. For any lax cone

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "f_0"'] \arrow[dr, "f_1"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & \\ \mathbb{A} \arrow[rr, "\Gamma"'] & {} & \mathbb{B} \end{tikzcd}$$

   there is a unique $g : \mathbb{C} \rightarrow \mathbb{B}\downarrow\Gamma$ so that its composition with λ is equal to κ (hence $d_0 \circ g = f_0$ and $d_1 \circ g = f_1$ hold).

2. If

$$
\begin{tikzcd}
& \mathbb{C} \arrow[dl, "g"'] \arrow[dr, "f"] & & \\
\mathbb{B}\downarrow\Gamma \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] \arrow[dr, "d_0"'] & & \mathbb{B}\downarrow\Gamma \arrow[dl, "d_0"] \arrow[dr, "d_1"] \arrow[d, phantom, "\overset{\lambda}{\Leftarrow}"] & \\
& \mathbb{A} \arrow[rr, "\Gamma"'] & & \mathbb{B}
\end{tikzcd}
=
\begin{tikzcd}
& & \mathbb{C} \arrow[dl, "g"'] \arrow[dr, "f"] & \\
& \mathbb{B}\downarrow\Gamma \arrow[rr, phantom, "\overset{\beta}{\Leftarrow}"] \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[d, phantom, "\overset{\lambda}{\Leftarrow}"] & & \mathbb{B}\downarrow\Gamma \arrow[dl, "d_1"] \\
\mathbb{A} \arrow[rr, "\Gamma"'] & & \mathbb{B} &
\end{tikzcd}
$$

then there is a unique $\gamma : f \Rightarrow g$ so that $\alpha = d_0 \circ \gamma$ and $\beta = d_1 \circ \gamma$.

Algebras of a 2-monad and Morphisms between Them. We suppose that T is a (strict) 2-monad on $\mathcal{C}$ (thus a $\mathcal{V}$-monad where $\mathcal{V} = \mathbf{Cat}$). We will talk about (strict) T-algebras $(\mathbb{A}, a : T\mathbb{A} \to \mathbb{A})$ (where a is asked to satisfy on the nose the usual unit and associativity laws), and about strict, strong and (op)lax T-algebra morphisms [1]. A lax T-algebra morphism between T-algebras $(\mathbb{A}, a)$ and $(\mathbb{B}, b)$ is a pair $(\Gamma : \mathbb{A} \to \mathbb{B}, \overline{\Gamma} : b \circ T\Gamma \Rightarrow \Gamma \circ a)$ subject to a few coherence laws; it is strong if $\overline{\Gamma}$ is invertible, and strict if it is the identity.

Fibrations in a 2-category. Later we also use a notion of fibrations and cofibrations in 2-categories, via a 'representable definition'; that is, a 1-cell $p : \mathbb{E} \to \mathbb{B}$ is a (co)fibration iff $\mathcal{C}(-, p)$ is a functor into the category of (co)fibrations and morphisms between them in the usual sense. In other words, $\mathcal{C}(\mathbb{X}, p) : \mathcal{C}(\mathbb{X}, \mathbb{E}) \to \mathcal{C}(\mathbb{X}, \mathbb{B})$ is a (co)fibration for each $\mathbb{X}$, and $\mathcal{C}(F, \mathbb{E}) : \mathcal{C}(\mathbb{Y}, \mathbb{E}) \to \mathcal{C}(\mathbb{X}, \mathbb{E})$ preserves (co)cartesian morphisms for any $F : \mathbb{Y} \to \mathbb{X}$. (See [5] for comparisons with other definitions of fibrations in 2-categories where this definition corresponds to the notion of strict fibration.)

For example, $d_0 : \mathbb{B}\downarrow\Gamma \to \mathbb{A}$ is a cofibration, and is also a fibration when $\mathbb{B}$ has pullbacks (again in the representable sense, i.e., $\mathcal{C}(-, \mathbb{B})$ is a functor into the category of categories with pullbacks and pullback-preserving functors). This observation is used in the proof of Theorem 2.4.

Examples

As a leading example, we will take the 2-category **Cat** of small categories, functors and natural transformations as $\mathcal{C}$. A comma object $\mathbb{B}\downarrow\Gamma$ for a functor $\Gamma : \mathbb{A} \to \mathbb{B}$ is just the comma category, thus its objects are tuples $(A \in \mathbb{A}, B \in \mathbb{B}, f : B \to \Gamma A)$, and an arrow from (A, B, f) to $(A', B.f')$ is a pair $(a : A \to A', b : B \to B')$ so that $\Gamma a \circ f = f' \circ b$ holds. The first projection d_0 sends (A, B, f) to A and (a, b) to a, while the second projection d_1 sends (A, B, f) to B and (a, b) to b. The natural transformation $\lambda : d_1 \Rightarrow \Gamma \circ d_0$ sends (A, B, f) to f. Also we let T be the 2-monad on **Cat** whose algebras are small symmetric monoidal categories (this is the setting used in [2, 3]). A lax T-morphism is then a lax symmetric monoidal functor (see Example 2.3).

2 Comma Objects

2.1 Lifting T-algebras Structures

The first result is easy to prove, but of fundamental use for our purpose:

Theorem 2.1 *Let $(\mathbb{A}, a)$, $(\mathbb{B}, b)$ be T-algebras and $(\Gamma, \overline{\Gamma})$ be a lax morphism from $(\mathbb{A}, a)$ to $(\mathbb{B}, b)$. Then the comma object $\mathbb{B}\downarrow\Gamma$ can be given a T-algebra structure $c : T(\mathbb{B}\downarrow\Gamma) \to \mathbb{B}\downarrow\Gamma$, so that the projections $d_0 : \mathbb{B}\downarrow\Gamma \to \mathbb{A}$ and $d_1 : \mathbb{B}\downarrow\Gamma \to \mathbb{B}$ are strict morphisms.*

Proof Sketch: We have

$$
\begin{tikzcd}
 & T(\mathbb{B}\downarrow\Gamma) \arrow[dl, "Td_0"'] \arrow[dr, "Td_1"] \arrow[d, phantom, "\overset{T\lambda}{\Leftarrow}"] & \\
T\mathbb{A} \arrow[rr, "T\Gamma"'] \arrow[d, "a"'] \arrow[drr, phantom, "\overset{\overline{\Gamma}}{\Leftarrow}"] & & T\mathbb{B} \arrow[d, "b"] \\
\mathbb{A} \arrow[rr, "\Gamma"'] & & \mathbb{B}
\end{tikzcd}
$$

By the universal property of $\mathbb{B}\downarrow\Gamma$ there exists a unique $c : T(\mathbb{B}\downarrow\Gamma) \to \mathbb{B}\downarrow\Gamma$ so that $\lambda \circ c$ agrees with this lax cone, therefore c makes the following diagram commute.

$$
\begin{tikzcd}
T\mathbb{A} \arrow[d, "a"'] & T(\mathbb{B}\downarrow\Gamma) \arrow[l, "Td_0"'] \arrow[r, "Td_1"] \arrow[d, dotted, "c"] & T\mathbb{B} \arrow[d, "b"] \\
\mathbb{A} & \mathbb{B}\downarrow\Gamma \arrow[l, "d_0"] \arrow[r, "d_1"'] & \mathbb{B}
\end{tikzcd}
$$

It is not hard to check that $(\mathbb{B}\downarrow\Gamma, c)$ is a T-algebra, i.e. the unit and associativity laws hold. □

Remark 2.2 In fact, this gives a comma object in the 2-category T-Alg_l [1] of T-algebras, lax morphisms and the appropriate 2-cells. Thus $\lambda : d_1 \Rightarrow \Gamma \circ d_0$ is a 2-cell in T-Alg_l, and satisfies the universal properties of the comma object in T-Alg_l. As a special case, for a 2-cell $\kappa : f_1 \Rightarrow \Gamma \circ f_0$ with strict morphisms

$f_0 : \mathbb{C} \to \mathbb{A}$ and $f_1 : \mathbb{C} \to \mathbb{B}$, the uniquely determined $g : \mathbb{C} \to \mathbb{B}{\downarrow}\Gamma$ is a strict morphism (this observation will be used in Section 4).

Example 2.3 Let $\mathbb{A}$, $\mathbb{B}$ be symmetric monoidal categories and $\Gamma : \mathbb{A} \to \mathbb{B}$ be a lax symmetric monoidal functor; spelling out the detail, Γ is a functor with a coherent natural transformation $m_{A,A'} : \Gamma A \otimes \Gamma A' \to \Gamma(A \otimes A')$ and a coherent arrow $m_I : \Gamma I \to I$ satisfying a few coherence diagrams. Then the comma category $\mathbb{B}{\downarrow}\Gamma$ can be given a symmetric monoidal structure, so that the projections to $\mathbb{A}$ and $\mathbb{B}$ are strict symmetric monoidal functor. Explicitly, we have the unit and tensor on $\mathbb{B}{\downarrow}\Gamma$ as

$$\begin{array}{rcl} I & = & (I, I, m_I) \\ (A, B, f) \otimes (A', B', f') & = & (A \otimes A', B \otimes B', m_{A,A'} \circ (f \otimes f')) \end{array}$$

2.2 Lifting Adjunctions

Next we turn to the problem of lifting adjunctions on the comma objects. Specifically, we consider the following problem. Let $\mathbb{B}_1 {\downarrow} \Gamma_1$, $\mathbb{B}_2 {\downarrow} \Gamma_2$ be the comma objects of $\Gamma_1 : \mathbb{A}_1 \to \mathbb{B}_1$ and $\Gamma_2 : \mathbb{A}_2 \to \mathbb{B}_2$ respectively. Also suppose that there are adjunctions $\mathbb{A}_1 \underset{U^{\mathbb{A}}}{\overset{F^{\mathbb{A}}}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathbb{A}_2$ and $\mathbb{B}_1 \underset{U^{\mathbb{B}}}{\overset{F^{\mathbb{B}}}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathbb{B}_2$. Consider $F : \mathbb{B}_1{\downarrow}\Gamma_1 \to \mathbb{B}_2{\downarrow}\Gamma_2$ satisfying $d_0 \circ F = F^{\mathbb{A}} \circ d_0$ and $d_1 \circ F = F^{\mathbb{B}} \circ d_1$ (such an F, if exists, is uniquely determined). We ask if F has a right adjoint U, so that the first projections give a map of adjunction from $F \dashv U$ to $F^{\mathbb{A}} \dashv U^{\mathbb{A}}$. Note that we do not want to assume $F^{\mathbb{B}} \circ \Gamma_1 = \Gamma_2 \circ F^{\mathbb{A}}$ (or $F^{\mathbb{B}} \circ \Gamma_1 \simeq \Gamma_2 \circ F^{\mathbb{A}}$) which excludes many interesting examples; an acceptable assumption we can make is the existence of a 2-cell from $F^{\mathbb{B}} \circ \Gamma_1$ to $\Gamma_2 \circ F^{\mathbb{A}}$.

Theorem 2.4 *Suppose that*

- *there are adjunctions* $\mathbb{A}_1 \underset{U^{\mathbb{A}}}{\overset{F^{\mathbb{A}}}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathbb{A}_2$ *and* $\mathbb{B}_1 \underset{U^{\mathbb{B}}}{\overset{F^{\mathbb{B}}}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathbb{B}_2$,
- *1-cells* $\Gamma_1 : \mathbb{A}_1 \to \mathbb{B}_1$, $\Gamma_2 : \mathbb{A}_2 \to \mathbb{B}_2$, *and a 2-cell* $\sigma : F^{\mathbb{B}} \circ \Gamma_1 \Rightarrow \Gamma_2 \circ F^{\mathbb{A}}$,
- *and moreover* $\mathbb{B}_1$ *has pullbacks.*

Then there exists an adjunction $\mathbb{B}_1{\downarrow}\Gamma_1 \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathbb{B}_2{\downarrow}\Gamma_2$ *so that the first projections give a map of adjunction from* $F \dashv U$ *to* $F^{\mathbb{A}} \dashv U^{\mathbb{A}}$.

Proof Sketch: We first observe that there exists an adjunction $\mathbb{B}_1{\downarrow}\mathbb{B}_1 \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathbb{B}_2{\downarrow}\mathbb{B}_2$ so that the first projections (d_0's) form a map of adjunction from $F \dashv U$ to $F^{\mathbb{B}} \dashv U^{\mathbb{B}}$. Note that these projections are automatically cofibrations (the codomain cofibrations in the case of **Cat**). Since $\mathbb{B}_1$ has pullbacks, it follows that the first projection from $\mathbb{B}_1 \downarrow \mathbb{B}_1$ to $\mathbb{B}_1$ is a fibration too. Finally, we note that $\mathbb{B}_i \downarrow \Gamma_i$ is obtained by pulling back the first projection from $\mathbb{B}_i \downarrow \mathbb{B}_i$ to $\mathbb{B}_i$ along Γ_i, so we can apply Theorem 3.2 to obtain an adjunction between $\mathbb{B}_1 \downarrow \Gamma_1$ and $\mathbb{B}_2 \downarrow \Gamma_2$. □

In the case of $\mathcal{C} = \mathbf{Cat}$, the construction can be described explicitly as follows. First, we have $F : \mathbb{B}_1 \downarrow \Gamma_1 \to \mathbb{B}_2 \downarrow \Gamma_2$ by $F(A, B, f) = (F^{\mathbb{A}}A, F^{\mathbb{B}}B, \sigma_A \circ F^{\mathbb{B}}f)$ and $F(a, b) = (F^{\mathbb{A}}a, F^{\mathbb{B}}b)$. Its right adjoint $U : \mathbb{B}_2 \downarrow \Gamma_2 \to \mathbb{B}_1 \downarrow \Gamma_1$ sends (A, B, f) to $(U^{\mathbb{A}}A, P(A), p(f))$ given by the top line of the following pullback

$$\begin{tikzcd}
P(A) \arrow[r, "p(f)"] \arrow[d] & \Gamma_1 U^{\mathbb{A}} A \arrow[d, "\tau_A"] \\
U^{\mathbb{B}} B \arrow[r, "U^{\mathbb{B}} f"'] & U^{\mathbb{B}} \Gamma_2 A
\end{tikzcd}$$

where $\tau : \Gamma_1 \circ U^{\mathbb{A}} \Rightarrow U^{\mathbb{B}} \circ \Gamma_2$ is derived from σ in the obvious way (c.f. Proof of Theorem 3.2).

Example 2.5 (Lemma 3.1 of [3], see also [2]) Let $\mathbb{A}$ and $\mathbb{B}$ be symmetric monoidal closed categories and $\Gamma : \mathbb{A} \to \mathbb{B}$ be a lax symmetric monoidal functor. Moreover suppose that $\mathbb{B}$ has pullbacks. Then the comma category $\mathbb{B} \downarrow \Gamma$ can be given a symmetric monoidal closed structure, so that the projection to $\mathbb{A}\,\mathbb{B}$ is a strict symmetric monoidal closed functor. To give a right adjoint of $(-) \otimes (A, B, f) : \mathbb{B} \downarrow \Gamma \to \mathbb{B} \downarrow \Gamma$, we have to consider adjunctions $(-) \otimes A \dashv A \multimap (-)$ and $(-) \otimes B \dashv B \multimap (-)$, with a natural transformation $m_{-,A} \circ (- \otimes f) : \Gamma(-) \otimes B \Rightarrow \Gamma(- \otimes A)$. Then the exponent from (A, B, f) to (A', B', f') in $\mathbb{B} \downarrow \Gamma$ is given by the top line of the following pullback [2, 3].

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d] & \Gamma(A \multimap A') \arrow[d, "(f \multimap \Gamma A') \circ \Lambda(\Gamma(\mathsf{ev}_{A,A'} \circ m_{A \multimap A', A}))"] \\
B \multimap B' \arrow[r, "B \multimap f'"'] & B \multimap \Gamma A'
\end{tikzcd}$$

3 Change-of-base

In [3], it is observed that pulling back a strict symmetric monoidal closed bifibration along a lax symmetric monoidal functor results a strict symmetric monoidal closed bifibration (see Example 3.3 below). In this section we give the 2-categorical generalizations of this situation, for T-algebras and for adjunctions.

3.1 Change-of-base for T-algebras

Theorem 3.1 *Let $(\mathbb{A}, a)$, $(\mathbb{B}, b)$, $(\mathbb{E}, e)$ be T-algebras, $(\Gamma, \overline{\Gamma})$ be a lax morphism from $(\mathbb{A}, a)$ to $(\mathbb{B}, b)$, and $p : \mathbb{E} \to \mathbb{B}$ be an oplax morphism from $(\mathbb{E}, e)$ to $(\mathbb{B}, b)$ which is also a cofibration. Suppose that there is a pullback:*

$$\begin{tikzcd}
\mathbb{D} \arrow[r] \arrow[d, "q"'] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{A} \arrow[r, "\Gamma"'] & \mathbb{B}
\end{tikzcd}$$

Then $\mathbb{D}$ can be given a T-algebra structure $d : T\mathbb{D} \to \mathbb{D}$ so that q is a strict morphism from $(\mathbb{D}, d)$ to $(\mathbb{A}, a)$.

Proof Sketch: We first observe

$$\begin{tikzcd}
 & T\mathbb{D} \arrow[dl, "Tq"'] \arrow[dr, "Tr"] & \\
T\mathbb{A} \arrow[r, "T\Gamma"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{\Gamma} \Leftarrow"] & T\mathbb{B} \arrow[d, "b"] \arrow[dr, phantom, "\overline{p} \Leftarrow"] & T\mathbb{E} \arrow[l, "Tp"'] \arrow[d, "e"] \\
\mathbb{A} \arrow[r, "\Gamma"'] & \mathbb{B} & \mathbb{E} \arrow[l, "p"]
\end{tikzcd}$$

By the universal property of $p{\downarrow}\mathbb{B}$ we have

$$\begin{tikzcd}
 & T\mathbb{D} \arrow[dl, "e \circ Tr"'] \arrow[d, dotted, "h"] \arrow[dr, "\Gamma \circ a \circ Tq"] & \\
\mathbb{E} & p{\downarrow}\mathbb{B} \arrow[l, "d_0"] \arrow[r, "d_1"'] & \mathbb{B}
\end{tikzcd}$$

Therefore we obtain a commutative diagram

$$
\begin{tikzcd}
T\mathbb{D} \arrow[r, "h"] \arrow[d, "Tq"'] & p{\downarrow}\mathbb{B} \arrow[r, "\alpha"] \arrow[dr, "d_1"'] & \mathbb{E} \arrow[d, "p"] \\
T\mathbb{A} \arrow[r, "a"'] & \mathbb{A} \arrow[r, "\Gamma"'] & \mathbb{B}
\end{tikzcd}
$$

where $\alpha : p{\downarrow}\mathbb{B} \rightarrow \mathbb{E}$ is given by the cocartesian lifting. Hence we have $d : T\mathbb{D} \rightarrow \mathbb{D}$ as the unique arrow determined by

$$
\begin{tikzcd}
T\mathbb{D} \arrow[dr, dotted, "d"] \arrow[drr, "\alpha \circ h"] \arrow[ddr, "a \circ Tq"'] & & \\
& \mathbb{D} \arrow[r, "r"] \arrow[d, "q"] & \mathbb{E} \arrow[d, "p"] \\
& \mathbb{A} \arrow[r, "\Gamma"'] & \mathbb{B}
\end{tikzcd}
$$

Some calculation shows that $(\mathbb{D}, d)$ is a T-algebra. $\square$

3.2 Change-of-base for Adjunctions

Now we give the most involved result in this note.

Theorem 3.2 *Suppose that*

- *there are adjunctions* $\mathbb{A}_1 \underset{U^{\mathbb{A}}}{\overset{F^{\mathbb{A}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{A}_2$, $\mathbb{B}_1 \underset{U^{\mathbb{B}}}{\overset{F^{\mathbb{B}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{B}_2$ *and* $\mathbb{E}_1 \underset{U^{\mathbb{E}}}{\overset{F^{\mathbb{E}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{E}_2$,
- *1-cells* $\Gamma_1 : \mathbb{A}_1 \to \mathbb{B}_1$ *and* $\Gamma_2 : \mathbb{A}_2 \to \mathbb{B}_2$, *and a 2-cell* $\sigma : F^{\mathbb{B}} \circ \Gamma_1 \Rightarrow \Gamma_2 \circ F^{\mathbb{A}}$,
- *fibration* $p_1 : \mathbb{E}_1 \to \mathbb{B}_1$ *and a cofibration* $p_2 : \mathbb{E}_2 \to \mathbb{B}_2$,
- *so that* p_1 *and* p_2 *form a map of adjunction from* $F^{\mathbb{E}} \dashv U^{\mathbb{E}}$ *to* $F^{\mathbb{B}} \dashv U^{\mathbb{B}}$.

Now suppose that the following pullbacks exist:

$$\begin{tikzcd} \mathbb{D}_1 \arrow[r, "r_1"] \arrow[d, "q_1"'] & \mathbb{E}_1 \arrow[d, "p_1"] \\ \mathbb{A}_1 \arrow[r, "\Gamma_1"'] & \mathbb{B}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}_2 \arrow[r, "r_2"] \arrow[d, "q_2"'] & \mathbb{E}_2 \arrow[d, "p_2"] \\ \mathbb{A}_2 \arrow[r, "\Gamma_2"'] & \mathbb{B}_2 \end{tikzcd}$$

Then there exists an adjunction $\mathbb{D}_1 \underset{U^{\mathbb{D}}}{\overset{F^{\mathbb{D}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{D}_2$ *so that* q_1 *and* q_2 *form a map of adjunction from* $F^{\mathbb{D}} \dashv U^{\mathbb{D}}$ *to* $F^{\mathbb{A}} \dashv U^{\mathbb{A}}$.

<u>Proof:</u> We use $\eta^{\mathbb{A}}$, $\varepsilon^{\mathbb{A}}$, $\eta^{\mathbb{B}}$, $\varepsilon^{\mathbb{B}}$, $\eta^{\mathbb{E}}$ and $\varepsilon^{\mathbb{E}}$ for units and counits of adjunctions $F^{\mathbb{A}} \dashv U^{\mathbb{A}}$, $F^{\mathbb{B}} \dashv U^{\mathbb{B}}$ and $F^{\mathbb{E}} \dashv U^{\mathbb{E}}$ respectively. Since p_1, p_2 form a map of adjunction, we have $p_1 \circ \eta^{\mathbb{E}} = \eta^{\mathbb{B}} \circ p_1$ and also $p_2 \circ \varepsilon^{\mathbb{E}} = \varepsilon^{\mathbb{B}} \circ p_2$.

Define a 2-cell $\tau : \Gamma_1 \circ U^{\mathbb{A}} \Rightarrow U^{\mathbb{B}} \circ \Gamma_2$ by

$$\begin{tikzcd} & & \mathbb{A}_2 \arrow[dl, "U^{\mathbb{A}}"'] \arrow[dd, "id"] \\ & \mathbb{A}_1 \arrow[dl, "\Gamma_1"'] \arrow[dr, "F^{\mathbb{A}}"] \arrow[r, phantom, "\overset{\varepsilon^{\mathbb{A}}}{\Rightarrow}"] & {} \\ \mathbb{B}_1 \arrow[dr, "F^{\mathbb{B}}"] \arrow[dd, "id"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & \mathbb{A}_2 \arrow[dl, "\Gamma_2"] \\ {} & \mathbb{B}_2 \arrow[dl, "U^{\mathbb{B}}"] \arrow[l, phantom, "\overset{\eta^{\mathbb{B}}}{\Rightarrow}"] & \\ \mathbb{B}_1 & & \end{tikzcd}$$

Right Adjoint. Consider the 2-cell $\tau \circ q_2 : \Gamma_1 \circ U^{\mathbb{A}} \circ q_2 \Rightarrow U^{\mathbb{B}} \circ \Gamma_2 \circ q_2 \in \mathcal{C}(\mathbb{D}_2, \mathbb{B}_1)$. By noting that $U^{\mathbb{B}} \circ \Gamma_2 \circ q_2 = p_1 \circ U^{\mathbb{E}} \circ r_2$, we have a cartesian lifting $\alpha : f \Rightarrow U^{\mathbb{E}} \circ r_2 \in \mathcal{C}(\mathbb{D}_2, \mathbb{E}_1)$ so that

$$
\begin{tikzcd}
 & \mathbb{D}_2 \arrow[d, "q_2"] & \\
 & \mathbb{A}_2 \arrow[dl, "U^{\mathbb{A}}"'] \arrow[dr, "\Gamma_2"] & \\
\mathbb{A}_1 \arrow[dr, "\Gamma_1"'] \arrow[rr, phantom, "\overset{\tau}{\Rightarrow}"] & & \mathbb{B}_2 \arrow[dl, "U^{\mathbb{B}}"] \\
 & \mathbb{B}_1 &
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{D}_2 \arrow[dr, "r_2"] \arrow[dd, "f"'] & \\
 & \mathbb{E}_2 \arrow[dl, "U^{\mathbb{E}}"] \arrow[l, phantom, "\overset{\alpha}{\Rightarrow}"] \\
\mathbb{E}_1 \arrow[d, "p_1"] & \\
\mathbb{B}_1 &
\end{tikzcd}
$$

holds. We then define $U : \mathbb{D}_2 \to \mathbb{D}_1$ by the universal property of the following pullback

$$
\begin{tikzcd}
\mathbb{D}_2 \arrow[d, "q_2"'] \arrow[dr, dotted, "U"] \arrow[drr, "f"] & & \\
\mathbb{A}_2 \arrow[dr, "U^{\mathbb{A}}"'] & \mathbb{D}_1 \arrow[r, "r_1"] \arrow[d, "q_1"] & \mathbb{E}_1 \arrow[d, "p_1"] \\
 & \mathbb{A}_1 \arrow[r, "\Gamma_1"'] & \mathbb{B}_1
\end{tikzcd}
$$

Therefore we have

$$
\begin{tikzcd}
 & \mathbb{D}_2 \arrow[dl, "U"'] \arrow[d, "q_2"] \arrow[dr, "r_2"] & \\
\mathbb{D}_1 \arrow[d, "q_1"'] & \mathbb{A}_2 \arrow[dl, "U^{\mathbb{A}}"'] \arrow[dr, "\Gamma_2"] & \mathbb{E}_2 \arrow[d, "p_2"] \\
\mathbb{A}_1 \arrow[dr, "\Gamma_1"'] \arrow[rr, phantom, "\overset{\tau}{\Rightarrow}"] & & \mathbb{B}_2 \arrow[dl, "U^{\mathbb{B}}"] \\
 & \mathbb{B}_1 &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathbb{D}_2 \arrow[dl, "U"'] \arrow[dr, "r_2"] & \\
\mathbb{D}_1 \arrow[d, "q_1"'] \arrow[dr, "r_1"'] \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] & & \mathbb{E}_2 \arrow[dl, "U^{\mathbb{E}}"] \arrow[d, "p_2"] \\
\mathbb{A}_1 \arrow[dr, "\Gamma_1"'] & \mathbb{E}_1 \arrow[d, "p_1"] & \mathbb{B}_2 \arrow[dl, "U^{\mathbb{B}}"] \\
 & \mathbb{B}_1 &
\end{tikzcd}
$$

Left Adjoint. Consider the 2-cell $\sigma \circ q_1 : F^{\mathbb{B}} \circ \Gamma_1 \circ q_1 \Rightarrow \Gamma_2 \circ F^{\mathbb{A}} \circ q_1 \in \mathcal{C}(\mathbb{D}_1, \mathbb{B}_2)$. By noting that $F^{\mathbb{B}} \circ \Gamma_1 \circ q_1 = p_2 \circ F^{\mathbb{E}} \circ r_1$, we have a cocartesian lifting $\beta : F^{\mathbb{E}} \circ r_1 \Rightarrow g \in \mathcal{C}(\mathbb{D}_1, \mathbb{E}_2)$ so that

$$
\begin{tikzcd}
& \mathbb{D}_1 \arrow[d, "q_1"] & \\
& \mathbb{A}_1 \arrow[dl, "\Gamma_1"'] \arrow[dr, "F^{\mathbb{A}}"] & \\
\mathbb{B}_1 \arrow[dr, "F^{\mathbb{B}}"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & \mathbb{A}_2 \arrow[dl, "\Gamma_2"] \\
& \mathbb{B}_2 &
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{D}_1 \arrow[dl, "r_1"'] \arrow[dd, "g"] \\
\mathbb{E}_1 \arrow[dr, "F^{\mathbb{E}}"'] \arrow[r, phantom, "\overset{\beta}{\Rightarrow}"] & \\
& \mathbb{E}_2 \arrow[d, "p_2"] \\
& \mathbb{B}_2
\end{tikzcd}
$$

holds. We then have $F : \mathbb{D}_1 \to \mathbb{D}_2$ by

$$
\begin{tikzcd}
\mathbb{D}_1 \arrow[d, "q_1"'] \arrow[dr, dotted, "F"'] \arrow[drr, "g"] & & \\
\mathbb{A}_1 \arrow[dr, "F^{\mathbb{A}}"'] & \mathbb{D}_2 \arrow[r, "r_2"] \arrow[d, "q_2"'] & \mathbb{E}_2 \arrow[d, "p_2"] \\
& \mathbb{A}_2 \arrow[r, "\Gamma_2"'] & \mathbb{B}_2
\end{tikzcd}
$$

Therefore we obtain

$$
\begin{tikzcd}
& \mathbb{D}_1 \arrow[dl, "r_1"'] \arrow[d, "q_1"] \arrow[dr, "F"] & \\
\mathbb{E}_1 \arrow[d, "p_1"'] & \mathbb{A}_1 \arrow[dl, "\Gamma_1"'] \arrow[dr, "F^{\mathbb{A}}"] & \mathbb{D}_2 \arrow[d, "q_2"] \\
\mathbb{B}_1 \arrow[dr, "F^{\mathbb{B}}"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & \mathbb{A}_2 \arrow[dl, "\Gamma_2"] \\
& \mathbb{B}_2 &
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{D}_1 \arrow[dl, "r_1"'] \arrow[dr, "F"] & \\
\mathbb{E}_1 \arrow[d, "p_1"'] \arrow[dr, "F^{\mathbb{E}}"'] \arrow[rr, phantom, "\overset{\beta}{\Rightarrow}"] & & \mathbb{D}_2 \arrow[d, "q_2"] \arrow[dl, "r_2"] \\
\mathbb{B}_1 \arrow[dr, "F^{\mathbb{B}}"'] & \mathbb{E}_2 \arrow[d, "p_2"] & \mathbb{A}_2 \arrow[dl, "\Gamma_2"] \\
& \mathbb{B}_2 &
\end{tikzcd}
$$

Unit and Counit. Consider a 2-cell $\beta^* : r_1 \Rightarrow U^{\mathbb{E}} \circ r_2 \circ F$ given by

$$\begin{tikzcd}
& \mathbb{D}_1 \arrow[dl, "r_1"'] \arrow[dr, "F"] & \\
\mathbb{E}_1 \arrow[dd, "id"'] \arrow[dr, "F^{\mathbb{E}}"] & \overset{\beta}{\Rightarrow} & \mathbb{D}_2 \arrow[dl, "r_2"] \\
& \mathbb{E}_1 \arrow[dl, "U^{\mathbb{E}}"] \arrow[l, phantom, "\overset{\eta^{\mathbb{E}}}{\Rightarrow}"] & \\
\mathbb{E}_2 & &
\end{tikzcd}$$

Since $\alpha : r_1 \circ U \Rightarrow U^{\mathbb{E}} \circ r_2$ is cartesian over $\tau \circ q_2$, so is $\alpha \circ F : r_1 \circ U \circ F \Rightarrow U^{\mathbb{E}} \circ r_2 \circ F$ over $\tau \circ q_2 \circ F$. As $p_1 \circ \beta^* = (\tau \circ q_2 \circ F)(\Gamma_1 \circ \eta^{\mathbb{A}} \circ q_1)$, there is a unique $\kappa : r_1 \Rightarrow r_1 \circ U \circ F$ so that $(\alpha \circ F)\kappa = \beta^*$ and $p_1 \circ \kappa = \Gamma_1 \circ \eta^{\mathbb{A}} \circ q_1$ hold.

Therefore we have 2-cells $\eta^{\mathbb{A}} \circ q_1 : q_1 \Rightarrow q_1 \circ U \circ F$ and $\kappa : r_1 \Rightarrow r_1 \circ U \circ F$ so that $p_1 \circ \kappa = \Gamma_1 \circ \eta^{\mathbb{A}} \circ q_1$. By the 2-dimensional universal property of the pullback there is a unique $\eta : id \Rightarrow U \circ F$ such that $q_1 \circ \eta = \eta^{\mathbb{A}} \circ q_1$ and $r_1 \circ \eta = \kappa$ hold.

Dually, consider a 2-cell $\alpha_* : F^{\mathbb{E}} \circ r_1 \circ U \Rightarrow r_2$ given by

$$\begin{tikzcd}
& \mathbb{D}_2 \arrow[dl, "U"'] \arrow[dr, "r_2"] & \\
\mathbb{D}_1 \arrow[dr, "r_1"'] & \overset{\alpha}{\Rightarrow} & \mathbb{E}_2 \arrow[dl, "U^{\mathbb{E}}"'] \arrow[dd, "id"] \\
& \mathbb{E}_1 \arrow[dr, "F^{\mathbb{E}}"'] \arrow[r, phantom, "\overset{\varepsilon^{\mathbb{E}}}{\Rightarrow}"] & \\
& & \mathbb{E}_2
\end{tikzcd}$$

Since $\beta : F^{\mathbb{E}} \circ r_1 \Rightarrow r_2 \circ F$ is cocartesian over $\sigma \circ q_1$, so is $\beta \circ U : F^{\mathbb{E}} \circ r_1 \circ U \Rightarrow r_2 \circ F \circ U$ over $\sigma \circ q_1 \circ U$. As $p_2 \circ \alpha_* = (\Gamma_2 \circ \varepsilon^{\mathbb{A}} \circ q_2)(\sigma \circ q_1 \circ U)$, there is a unique $\theta : r_2 \circ F \circ U \Rightarrow r_2$ so that $\theta(\beta \circ U) = \alpha_*$ and $p_2 \circ \theta = \Gamma_2 \circ \varepsilon^{\mathbb{A}} \circ q_2$ hold.

Therefore we have 2-cells $\varepsilon^{\mathbb{A}} \circ q_2 : q_2 \circ F \circ U \Rightarrow q_2$ and $\theta : r_2 \circ F \circ U \Rightarrow r_2$ so that $p_2 \circ \theta = \Gamma_2 \circ \varepsilon^{\mathbb{A}} \circ q_2$. By the 2-dimensional universal property of the pullback there is a unique $\varepsilon : F \circ U \Rightarrow id$ such that $q_2 \circ \varepsilon = \varepsilon^{\mathbb{A}} \circ q_1$ and $r_2 \circ \varepsilon = \theta$ hold.

Finally, we need to show $(U \circ \varepsilon)(\eta \circ U) = U$ and $(\varepsilon \circ F)(F \circ \eta) = F$. By the 2-dimensional universal property of the pullbacks, it suffices to show

$$\begin{array}{rcl}
r_1 \circ ((U \circ \varepsilon)(\eta \circ U)) & = & r_1 \circ U \\
q_1 \circ ((U \circ \varepsilon)(\eta \circ U)) & = & q_1 \circ U \\
r_2 \circ ((\varepsilon \circ F)(F \circ \eta)) & = & r_2 \circ F \\
q_2 \circ ((\varepsilon \circ F)(F \circ \eta)) & = & q_2 \circ F
\end{array}$$

The second and fourth equations are easily verified. Since α is cartesian, the first equation follows from $\alpha(r_1 \circ ((U \circ \varepsilon)(\eta \circ U))) = \alpha$ which is routinely checked. Similarly, since β is cocartesian, the third equation follows from $(r_2 \circ ((\varepsilon \circ F)(F \circ \eta)))\beta = \beta$, which again is easily shown. □

Example 3.3 (Proposition 3.2 of [3]) Let $\mathbb{A}, \mathbb{B}, \mathbb{E}$ be symmetric monoidal closed categories, $\Gamma : \mathbb{A} \to \mathbb{B}$ a lax symmetric monoidal functor, and $p : \mathbb{E} \to \mathbb{B}$ be a strict symmetric monoidal closed functor which is also a bifibration. Now consider a pullback:

$$\begin{tikzcd} \mathbb{D} \arrow[r] \arrow[d, "q"'] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{A} \arrow[r, "\Gamma"'] & \mathbb{B} \end{tikzcd}$$

By Theorem 3.1 and 3.2, $\mathbb{D}$ can be given a symmetric monoidal closed structure, for which q is strict.

4 Full Completeness via Glueing

We conclude this note by sketching how a glueing construction can be used for relating algebraic structures on **Cat** (and the corresponding type theories); further examples are found in [2, 3] and papers cited there, in particular [7].

Let **SMCat** be the category of small symmetric monoidal categories and strict symmetric monoidal functors, and **SMCCat** be that of small symmetric monoidal closed categories and strict symmetric monoidal closed functors. The forgetful functor from **SMCCat** to **SMCat** has a left adjoint $\mathcal{F} : \mathbf{SMCat} \to \mathbf{SMCCat}$, which sends a symmetric monoidal category $\mathbb{C}$ to the symmetric monoidal closed category $\mathcal{F}\mathbb{C}$ obtained by freely adding exponents to $\mathbb{C}$. The $\mathbb{C}$-component of the unit of this adjunction is the obvious strict symmetric monoidal functor $j : \mathbb{C} \to \mathcal{F}\mathbb{C}$. We show that j is fully faithful – this amounts to the full completeness of the $\otimes$, I-fragment of intuitionistic linear logic in the $\otimes$, I, $\multimap$-fragment [2].

Before proving the result, let us note an observation on the glueing construction into the presheaf categories. Let $j : \mathbb{C} \to \mathbb{D}$ be any functor. Let $\Gamma : \mathbb{D} \to \mathbf{Set}^{\mathbb{C}^{op}}$ be the functor sending $D \in \mathbb{D}$ to $\mathbb{D}(j-, D) \in \mathbf{Set}^{\mathbb{C}^{op}}$, and $\mathcal{Y} : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{op}}$ be the Yoneda embedding. Then there is a natural transformation $\overline{j} : \mathcal{Y} \Rightarrow \Gamma \circ j$ where $(\overline{j}_X)_Y$ sends $f \in \mathbb{C}(Y, X)$ to $j(f) \in \mathbb{D}(jY, jX)$. Let $\mathbb{G}$ be the comma category $\mathbf{Set}^{\mathbb{C}^{op}} \downarrow \Gamma$; by the universal property of the comma object, there is a functor $h : \mathbb{C} \to \mathbb{G}$ so that $d_0 \circ h = j$ and $d_1 \circ h = \mathcal{Y}$, uniquely determined by the lax cone $\overline{j}$. It follows that, by the Yoneda lemma, this h is fully faithful.

Now the proof of the full faithfulness of $j : \mathbb{C} \to \mathcal{F}\mathbb{C}$. Consider the functor $\Gamma : \mathcal{F}\mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{op}}$ as above. Recall that $\mathbf{Set}^{\mathbb{C}^{op}}$ is a symmetric monoidal cocompletion of $\mathbb{C}$ [4], thus can be given a symmetric monoidal closed structure, for which the Yoneda embedding $\mathcal{Y} : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{op}}$ is strict symmetric monoidal and Γ is lax symmetric monoidal. From Example 2.5, we know that the comma category $\mathbb{G} = \mathbf{Set}^{\mathbb{C}^{op}} \downarrow \Gamma$ can be given a symmetric monoidal closed structure, for which the projection $d_0 : \mathbb{G} \to \mathcal{F}\mathbb{C}$ is strict symmetric monoidal closed. By the previous observation, there is a fully faithful functor $h : \mathbb{C} \to \mathbb{G}$ so that $d_0 \circ h = j$. From an observation in Remark 2.2, we also know that h is strict symmetric monoidal. Since j is a unit, there is a unique strict symmetric monoidal closed functor $g : \mathcal{F}\mathbb{C} \to \mathbb{G}$ so that $g \circ j = h$ and $d_0 \circ g = id$. Therefore we have the following diagram in **SMCat**.

$$\begin{tikzcd}
 & & \mathcal{F}\mathbb{C} \arrow[dl, "g"'] \arrow[dd, "id"] \\
\mathbb{C} \arrow[urr, "j"] \arrow[r, "h"] \arrow[drr, "j"'] & \mathbb{G} \arrow[dr, "d_0"] & \\
 & & \mathcal{F}\mathbb{C}
\end{tikzcd}$$

As h is fully faithful and g is faithful, it follows that j is fully faithful. (Here we are slightly sloppy about the size issue – to be precise, we had to cut down $\mathbf{Set}^{\mathbb{C}^{op}}$ to be small while keeping the needed structure, to work within **SMCat**.)

Remark 4.1 Obviously this proof works only for algebras which are closed under the exponentiation with **Set**. To examine the limitation of this approach, it seems important to identify the 2-monads whose algebras enjoy this property.

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