\{\rightarrow, \rightarrow\} \textit{is Full in} \{!, \rightarrow\}^*

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Abstract

We show that the \{\rightarrow, \rightarrow\}-fragment of Intuitionistic Linear Logic is \textit{full} in the \{!, \rightarrow\}-fragment, both formulated as linear lambda calculi. The
proof is a mild extension of our previous technique used for showing the
fullness of Girard's translation from Intuitionistic Logic into Intuitionistic
Linear Logic, and makes use of double-parameterized logical predicates.

1 Linear Lambda Calculi

Our presentation is based on a dual-context type system for intuitionistic linear
logic (called DILL) due to Barber and Plotkin [1]. In this formulation of the
linear lambda calculi, a typing judgement takes the form $\Gamma ; \Delta \vdash M : \tau$ in
which $\Gamma$ represents an intuitionistic (or additive) context whereas $\Delta$ is a linear
(multiplicative) context.

A set of base types ($b$ ranges over them) is fixed throughout this paper.

1.1 $\lambda \rightarrow, \rightarrow$

The first system we consider is a fragment with connectives $\rightarrow$ (intuitionistic or
non-linear arrow type) and $\rightarrow$ (linear arrow type). We use $\lambda x^\tau . M$ and $M \otimes N$
for the non-linear lambda abstraction and application respectively, while $\lambda x^\tau . M$
and $M \otimes N$ for the linear ones.

Types and Terms

\[
\begin{align*}
\sigma &::= b \mid \sigma \rightarrow \sigma \mid \sigma \rightarrow \sigma \\
M &::= x \mid \lambda x^\tau . M \mid M \otimes M \mid \lambda x^\tau . M \mid M \otimes M
\end{align*}
\]

*The content of this paper is a very mild extension of our previous work presented in [2] –
the structure and most of the details of the proof are essentially the same (or even identical).
The reason of this reworking is rather technical; we need the main result of this paper in our
another work [3].

1
Typing

\[
\begin{align*}
\frac{\Gamma; x : \sigma \vdash x : \sigma}{\Gamma, x : \sigma, \Gamma_2 ; \emptyset \vdash x : \sigma} & \quad \frac{\Gamma, x : \sigma_1 ; \Delta \vdash M : \sigma_2}{\Gamma ; \Delta \vdash \lambda x^{n+1}. M : \sigma_1 \rightarrow \sigma_2} \\
\frac{\Gamma, x : \sigma_1 ; \Delta \vdash M : \sigma_2}{\Gamma ; \Delta \vdash M \circ N : \sigma_2} & \quad \frac{\Gamma, x : \sigma_1 \vdash M : \sigma_2}{\Gamma ; \Delta \vdash \lambda x^{n+1}. M : \sigma_1 \rightarrow \sigma_2} \\
\frac{\Gamma; \Delta, x : \sigma_1 \vdash M : \sigma_2}{\Gamma ; \Delta \vdash M : \sigma_1 \rightarrow \sigma_2} & \quad \frac{\Gamma ; \Delta_1 \vdash M : \sigma_1 \rightarrow \sigma_2 \quad \Gamma ; \Delta_2 \vdash N : \sigma_1}{\Gamma ; \Delta_1 \cap \Delta_2 \vdash M N : \sigma_2}
\end{align*}
\]

where \( \Delta_1 \cap \Delta_2 \) is a merge of \( \Delta_1 \) and \( \Delta_2 \) \[1\]. Thus, \( \Delta_1 \cap \Delta_2 \) represents one of possible merges of \( \Delta_1 \) and \( \Delta_2 \) as finite lists. We assume that, when we introduce \( \Delta_1 \cap \Delta_2 \), there is no variable occurring both in \( \Delta_1 \) and in \( \Delta_2 \). We write \( \emptyset \) for the empty context. We note that any typing judgement has a unique derivation (hence a typing judgement can be identified with its derivation).

**Axioms**

\[
\begin{align*}
(\lambda x. M) N &= M[N/x] \\
\lambda x. M &= M \\
(\lambda x. M) N &= M[N/x] \\
\lambda x. M \circ x &= M \quad (x \notin FV(M))
\end{align*}
\]

The equality judgement \( \Gamma; \Delta \vdash M = N : \sigma \) is defined as usual.

### 1.2 \( \lambda! \) and \( \rightarrow \)

The second fragment is that with the exponential ! and the linear arrow type \( \rightarrow \). The same system has been used in [2] as the target calculus of Girard’s translation.

**Types and Terms**

\[
\begin{align*}
\tau &::= \ b | !\tau | \tau \rightarrow \tau \\
M &::= x | !M | \text{let } x^\tau \text{ be } M \text{ in } M | \lambda x^{n}. M | M \ M
\end{align*}
\]

Typing

\[
\begin{align*}
\frac{\Gamma; x : \tau \vdash x : \tau}{\Gamma, x : \tau, \Gamma_2 ; \emptyset \vdash x : \tau} & \quad \frac{\Gamma; \emptyset \vdash M : \tau}{\Gamma; \emptyset ! M : !\tau} \\
\frac{\Gamma; \emptyset \vdash M : \tau}{\Gamma ; \emptyset ! M : !\tau} & \quad \frac{\Gamma; \Delta, x : \tau_1 \vdash M : \tau_2}{\Gamma ; \Delta \vdash \lambda x^{n}. M : \tau_1 \rightarrow \tau_2} \\
\frac{\Gamma; \Delta_1 \vdash M : \tau_1 \rightarrow \tau_2 \quad \Gamma; \Delta_2 \vdash N : \tau_1}{\Gamma ; \Delta_1 \cap \Delta_2 \vdash M N : \tau_2} & \quad \frac{\Gamma; \Delta_1 \vdash M : \tau_1 \rightarrow \tau_2 \quad \Gamma; \Delta_2 \vdash N : \tau_1}{\Gamma ; \Delta_1 \cap \Delta_2 \vdash M N : \tau_2}
\end{align*}
\]
Axioms

\[(\lambda x.M)N \quad = \quad M[N/x] \]
\[\lambda x.Mx \quad = \quad M \]

\[\begin{align*}
\text{let } !x & \text{ be } !M \text{ in } N \quad = \quad N[M/x] \\
\text{let } !x & \text{ be } M \text{ in } !x \quad = \quad M
\end{align*}\]

\[C[\text{let } !x \text{ be } M \text{ in } N] \quad = \quad \text{let } !x \text{ be } M \text{ in } C[N]\]

where \(C[\cdot] \) is a linear context (no \(!\) binds \([-\cdot]\)); formally it is generated from the following grammar.

\[
C \quad ::= \quad [-] \mid \lambda x.C \mid CM \mid MC \mid \text{let } !x \text{ be } C \text{ in } M \mid \text{let } !x \text{ be } M \text{ in } C
\]

1.3 Translation from \(\lambda^{\to,-\to}\) to \(\lambda_{l,-\to}\)

We now give the translation \((-\cdot)^\circ\) from \(\lambda^{\to,-\to}\) to \(\lambda_{l,-\to}\), which is a straightforward extension of Girard’s translation from the simply typed lambda calculus into \(\lambda_{l,-\to}\).

\[
\begin{align*}
\Gamma^\circ & \equiv \Gamma \\
(\sigma_1 \to \sigma_2)^\circ & \equiv \sigma_1^\circ \to \sigma_2^\circ \\
x^\circ & \equiv x \\
(\lambda x^\circ . M)^\circ & \equiv \lambda x^{\sigma^\circ} . M^\circ \\
(M^\circ[N^\circ]_{\sigma_1^{\sigma_1^\circ}})^\circ & \equiv M^\circ N^\circ \\
(M^\circ[\sigma_2^\circ \sigma_1^{\sigma_1^\circ} N^\circ]_{\sigma_1^{\sigma_1^\circ\sigma_2^{\sigma_2^\circ}}})^\circ & \equiv M^\circ(!N^\circ)
\end{align*}
\]

**Proposition 1.1 (type soundness)** If \(\Gamma; \Delta \vdash M : \sigma \) is derivable in \(\lambda^{\to,-\to}\), so is \(\Gamma^\circ; \Delta^\circ \vdash M^\circ : \sigma^\circ \) in \(\lambda_{l,-\to}\).

**Proof:** For instance, the derivation

\[
\begin{array}{c}
\Gamma, x : \sigma_1 \quad \vdash \Delta \vdash M : \sigma_2 \\
\end{array}
\]

is sent to

\[
\begin{array}{c}
\Gamma^\circ, y : \sigma_1^\circ \quad \vdash y : \sigma_1^\circ \\
\Gamma^\circ, x : \sigma_1^\circ \quad \Delta^\circ \vdash M^\circ : \sigma_2^\circ \\
\end{array}
\]

while

\[
\begin{array}{c}
\Gamma \quad \vdash \Delta \vdash M : \sigma_1 \quad \Gamma \quad \emptyset \vdash N : \sigma_1 \\
\end{array}
\]

is sent to

\[
\begin{array}{c}
\Gamma^\circ \quad \vdash \Delta^\circ \vdash M \circ N : \sigma_2 \\
\end{array}
\]
is sent to

\[
\begin{align*}
\vdots & \vdots \\
\Gamma^0 \vdash \Delta^0 \vdash M^0 : !\sigma_1 \rightarrow !\sigma_2 & \Rightarrow \Gamma^0 \vdash \emptyset ! N^0 : \sigma_1^2 \\
\Gamma^0 \vdash \Delta^0 \vdash (M \preceq N)^0 & \Rightarrow M^0 \vdash ! N^0 : \sigma_2^2
\end{align*}
\]

Lemma 1.2 (compatibility with substitution) \(M^0[N^0/x] = (M[N/x])^0\).

Proof: Easy.

Proposition 1.3 (equational soundness) If \(\Gamma \vdash \Delta \vdash M = N : \sigma\) holds in \(\lambda^{\rightarrow, \rightarrow}\), so does \(\Gamma^0 \vdash \Delta^0 \vdash M^0 = N^0 : \sigma^0\) in \(\lambda^{1, 1}\).

Proof: The crucial cases are:

\[
((\lambda x. M) \preceq N)^0 = (\lambda y. \text{let } ! x \text{ be } y \text{ in } M^0) (! N^0)
\]

let ! x be ! N^0 in M^0

= M^0[N^0/x]

= (M[N/x])^0

\[
(\lambda x. M \preceq a x)^0 = \lambda y. \text{let } ! x \text{ be } y \text{ in } M^0 (! x)
\]

= \lambda y. M^0 (let ! x be y in ! x)

= \lambda y. M^0 y

= \lambda

\[
M
\]

Proposition 1.4 (equational completeness) If \(\Gamma^0 \vdash \Delta^0 \vdash M^0 = N^0 : \sigma^0\) holds in \(\lambda^{1, 1}\), then \(\Gamma \vdash \Delta \vdash M = N : \sigma\) holds in \(\lambda^{\rightarrow, \rightarrow}\).

Proof: There is an obvious translation \((-)^*\) from \(\lambda^{1, 1}\) to the simply typed lambda calculus \(\lambda^{\rightarrow}\) which forgets all the information regarding the linearity (see [2]). Since the composition \(\lambda^{\rightarrow, \rightarrow} \overset{-1}{\xrightarrow{(-)^*}} \lambda^{1, 1} \overset{-1}{\xrightarrow{(-)^*}} \lambda^{\rightarrow}\) obviously reflects the equality, so does \((-)^0\).

2 Dual Logical Predicates

In addition to the case of Girard’s translation from the simply typed lambda calculus to \(\lambda^{1, 1}\), we have to deal with the linearity in the source calculus as well. To this end we add one more parameter for linear contexts to the logical predicates used in [2] for showing the fullness of Girard’s translation.

Notations We write \(\Lambda^{\rightarrow, \rightarrow}\) and \(\Lambda^{1, 1}\) for the sets of well-typed terms of \(\lambda^{\rightarrow, \rightarrow}\) and \(\lambda^{1, 1}\) respectively, and use the following notations.

\[
\begin{align*}
\Lambda^{\rightarrow, \rightarrow} (\Gamma ; \Delta) & = \{ M \in \Lambda^{\rightarrow, \rightarrow} \mid \Gamma ; \Delta \vdash M : \sigma \} \\
\Lambda^{1, 1} (\Gamma ; \Delta) & = \{ M \in \Lambda^{1, 1} \mid \Gamma ; \Delta \vdash M : \tau \} \\
\Gamma_1 \leq \Gamma_2 & \text{ iff } \Gamma_2 = \Gamma, \Gamma_1 \text{ for some } \Gamma
\end{align*}
\]
**Definition 2.1 (dual predicate)** Let $\tau$ be a type of $\lambda^{1,-\omega}$. A family $P$ of sets indexed by the contexts of $\lambda^{1,-\omega}$ is called a dual predicate on $\tau$ when $P(\Gamma; \Delta) \subseteq \Lambda^{1,-\omega}(\Gamma^0; \Delta^0)$ and closed under

- renaming: $M \in P(\Gamma; \Delta)$ implies $M\rho \in P(\Gamma\rho; \Delta\rho)$ for any renaming $\rho$
- weakening: $\Gamma \leq \Gamma'$ implies $P(\Gamma; \Delta) \subseteq P(\Gamma'; \Delta)$
- equality: $M \in P(\Gamma; \Delta)$ and $M = M'$ imply $M' \in P(\Gamma; \Delta)$

**Lemma 2.2 (linear implication on dual predicates)** Let $P_1$, $P_2$ be dual predicates on $\tau_1$ and $\tau_2$ respectively. Then there is a dual predicate $P_1 \rightarrow P_2$ on $\tau_1 \rightarrow \tau_2$ defined by

$$P_1 \rightarrow P_2(\Gamma; \Delta) = \left\{ \begin{array}{c}
\left\{ M \in \Lambda^{1,-\omega}(\Gamma^0; \Delta^0) \mid N \in P_1(\Gamma'; \Delta') \text{ and } \Gamma \leq \Gamma' \text{ imply } M N \in P_2(\Gamma'; \Delta\Delta') \right\} \\
N \subseteq P(\Gamma; \emptyset) \end{array} \right\}$$

**Proof:** Easy. $\square$

**Lemma 2.3 (exponential on dual predicates)** Let $P$ be a dual predicate on $\tau$. Then there is a dual predicate $!P$ on $\tau$ defined by

$$!P(\Gamma; \Delta) = \left\{ \begin{array}{c}
\left\{ M \in \Lambda^{1,-\omega}(\Gamma^0; \emptyset) \mid M \models N \text{ for some } N \in P(\Gamma; \emptyset) \right\} \text{ if } \Delta = \emptyset \\
\emptyset \text{ otherwise}
\end{array} \right\}$$

**Proof:** Easy. $\square$

Now we are ready to repeat the same story as given in [2]:

**Definition 2.4** A family $\{P_\tau\}$ of dual predicates is called a logical dual predicate when each $P_\tau$ is a dual predicate on $\tau$, and $P_{\tau_1 \rightarrow \tau_2} = P_\tau_1 \rightarrow P_\tau_2$ as well as $P_{\tau} =!P_\tau$ hold.

**Proposition 2.5 (Basic Lemma)** Let $\{P_\tau\}$ be a logical dual predicate. For $M_i \in P_{\tau_i}(\Gamma; \Delta_i)$ (1 $\leq i \leq m$), $M_j' \in P_{\tau_j'}(\Gamma; \Delta_j')$ (1 $\leq j \leq n$) and $x_1 : \tau_1, \ldots, x_m : \tau_m ; y_1 : \tau_1', \ldots, y_n : \tau_n ' \vdash N : \tau$, it follows that

let $x_1$ be $M_1$ in ... let $!x_m$ be $M_m$ in $N[M_1/x_1, \ldots, M_m/x_m] \in P_\tau(\Gamma; \Delta)$

where $\Delta = \Delta_1 \ldots \Delta_m \Delta'_1 \ldots \Delta'_n$.

Note that this can be equally stated as follows:

**Proposition 2.6** Let $\{P_\tau\}$ be a logical dual predicate. For $M_i \in P_{\tau_i}(\Gamma; \emptyset)$ (1 $\leq i \leq m$), $M_j' \in P_{\tau_j'}(\Gamma; \Delta_j')$ (1 $\leq j \leq n$) and $x_1 : \tau_1, \ldots, x_m : \tau_m ; y_1 : \tau_1', \ldots, y_n : \tau_n ' \vdash N : \tau$, it follows that

$$N[M_1/x_1, \ldots, M_m/x_m, M_j'/y_1, \ldots, M_n'/y_n] \in P_\tau(\Gamma; \Delta)$$

where $\Delta = \Delta_1 \ldots \Delta_m \Delta'_1 \ldots \Delta'_n$.  

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Proof: Induction on the derivation of terms (equivalently typing judgements) - actually identical as the proof of Basic Lemma in [2], except that we have to add the parameters for linear contexts. □

Corollary 2.7 (Basic Lemma for closed terms) Let \( \{ P_\tau \} \) be a logical dual predicate. Then, for any \( \emptyset \); \( \emptyset \vdash N : \tau \), it follows that \( N \in P_\tau (\emptyset ; \emptyset ) \).

3 Fullness

Definition 3.1 For a type \( \sigma \) and context \( \Gamma \); \( \Delta \) of \( \lambda^{\ast,-\ast} \), define
\[
P_\sigma (\Gamma ; \Delta) = \{ N \in \Lambda_{\lambda^{\ast,-\ast}}^{\sigma,\tau}(\Gamma^\circ ; \Delta^\circ) \mid N = M^\circ \text{ for some } M \in \Lambda_{\lambda^{\ast,-\ast}}^{\sigma,\tau}(\Gamma ; \Delta) \}
\]

Lemma 3.2 \( P \) is a dual predicate on \( \sigma^\circ \).

Proof: Obvious. □

Lemma 3.3 (crucial lemma) \( P_{\sigma_1 \rightarrow \sigma_2} = \lnot P_{\sigma_1} \rightarrow P_{\sigma_2} \).

Proof: Suppose that \( M \in P_{\sigma_1 \rightarrow \sigma_2}(\Gamma ; \Delta) \), i.e. \( M = M^\circ \) for some \( \Gamma ; \Delta \vdash M' : \sigma_1 \rightarrow \sigma_2 \). Let \( N \in \lnot P_{\sigma_1}(\Gamma^\prime ; \Delta^\prime) \) - by definition we may safely assume that \( \Delta^\prime = \emptyset \). This means that \( N = \lnot N' \) for some \( N' \in P_{\sigma_1}(\Gamma^\prime ; \emptyset) \). Therefore \( N = \lnot N'' \) for some \( \Gamma^\prime ; \emptyset \vdash N'' : \sigma_1 \). It then follows that
\[
MN = M^\circ (\lnot N''^\circ) = (M^\circ \cdot N''^\circ)^\circ
\]
Hence \( MN \in P_{\sigma_2}(\Gamma^\prime ; \Delta^\prime) \). Therefore \( M \in \lnot P_{\sigma_1} \rightarrow P_{\sigma_2}(\Gamma ; \Delta) \).

Conversely, suppose that \( M \in \lnot P_{\sigma_1} \rightarrow P_{\sigma_2}(\Gamma ; \Delta) \). Since \( \lnot z \in \lnot P_{\sigma_1}(z : \sigma_1, \Gamma ; \emptyset) \), we have \( M(\lnot z) \in P_{\sigma_2}(z : \sigma_1, \Gamma ; \Delta) \). Hence there exists \( z : \sigma_1, \Gamma ; \Delta \vdash N : \sigma_2 \) such that \( M(\lnot z) = N^\circ \).

\[
M = \lambda y^{\sigma_1}.M y = \lambda y^{\sigma_1}.M (\text{let } z^{\sigma_1} \text{ be } y \text{ in } \lnot z) = \lambda y^{\sigma_1}.\text{let } z^{\sigma_1} \text{ be } y \text{ in } M(\lnot z) = \lambda y^{\sigma_1}.\text{let } z^{\sigma_1} \text{ be } y \text{ in } N^\circ = (\lambda y^{\sigma_1}.N)^\circ
\]
So we conclude that \( M \in P_{\sigma_1 \rightarrow \sigma_2}(\Gamma ; \Delta) \). □

Proposition 3.4 Consider a family \( \{ P_\tau \} \) indexed by the types of \( \lambda^{\ast,-\ast} \), such that \( P_\tau \) is a dual predicate on \( \sigma^\circ \) and satisfies \( P_{\tau_1 \rightarrow \tau_2} = \lnot P_{\tau_1} \rightarrow P_{\tau_2} \). Then there is a logical dual predicate \( \{ P_\tau^\ast \} \) such that \( P_\tau^\ast = P_\tau \) holds for any type \( \tau \) of \( \lambda^{\ast,-\ast} \) - explicitly, such a \( \{ P_\tau^\ast \} \) is given by \( P^\ast_\emptyset = P^\ast_\emptyset \), \( P^\ast_{\tau_1 \rightarrow \tau_2} = P^\ast_{\tau_1} \rightarrow P^\ast_{\tau_2} \) and \( P^\ast_{\lnot \tau} = \lnot P^\ast_\tau \).
Proof: Induction on the types of $\lambda^{\rightarrow,-\sigma}$. The only nontrivial case is that of arrow types $\sigma_1 \rightarrow \sigma_2$:

$$P^*_{(\sigma_1 \rightarrow \sigma_2)^*} = P^*_{(\sigma_1 \rightarrow \sigma_2)^* - \sigma_2}$$

$$= |P^*_{\sigma_1} - o P^*_{\sigma_2}| \quad \text{definition of } P^*$$

$$= |P^*_{\sigma_1} - o P^*_{\sigma_2}| \quad \text{induction hypothesis}$$

$$= P^*_{\sigma_1 \rightarrow \sigma_2}$$

$\square$

**Corollary 3.5** Define a logical dual predicate $\{P^*_c\}$ by $P^*_{h} = P_h$, $P^*_{r_1, r_2} = P^*_{r_1} - o P^*_{r_2}$ and $P^*_{c} = |P^*_c|$. Then $P^*_{c} = P_c$ holds for any type $c$ of $\lambda^{\rightarrow,-\sigma}$.

**Theorem 3.6 (fullness)** Suppose that $\Gamma^0 ; \Delta^0 \vdash N ; \sigma^0$ is derivable in $\lambda^{1,-\sigma}$. Then there exists $\Gamma ; \Delta \vdash M : c$ derivable in $\lambda^{\rightarrow,-\sigma}$ such that $\Gamma^0 ; \Delta^0 \vdash M^0 = N ; \sigma^0$ holds.

Proof: Apply the Basic Lemma to the logical dual predicate $\{P^*_c\}$. $\square$

**References**

