Logical Predicates for Intuitionistic Linear Type Theories*

Masahito Hasegawa

RIMS, Kyoto University, Kyoto 606-8502, Japan

Abstract. We develop a notion of Kripke-like parameterized logical predicates for two fragments of intuitionistic linear logic (MILL and DILL) in terms of their category-theoretic models. Such logical predicates are derived from the categorical glueing construction combined with the free symmetric monoidal cocompletion. As applications, we obtain full completeness results of translations between linear type theories.

1 Introduction

Suppose that a model of *Multiplicative Intuitionistic Linear Logic (MILL)* – the propositional fragment of linear logic [12] with I, $\otimes$ and $\multimap$ – is given. Also suppose that there is a property on elements of the model which is closed under tensor product and composition (cut) and other structural rules, and covers the interpretations of base types and constants. We show that such a property can be extended to the interpretation of all types so that it covers all MILL-definable elements. We also give a parallel result for *Dual Intuitionistic Linear Logic (DILL)* of Barber and Plotkin [5], which is an extension of MILL with the modality !. To achieve such results, we first give a suitable notion of such "predicates" on models of MILL and DILL, upon which we develop logical predicates and state the Basic Lemma. We then show that the construction above is an instance of our logical predicates.

To see why we need to introduce a property closed under tensor and so on, it would be instructive to observe that the standard logical predicates for models of simply typed lambda calculus do not work well with the linear calculi and their models. We may have a predicate $P_b \subseteq \mathcal{A}_b$ for each base type b, where $\mathcal{A}_\sigma$ is a set in which the closed terms of type σ are interpreted. As the standard logical predicates, we hope to define a predicate $P_\sigma \subseteq \mathcal{A}_\sigma$ for every type σ in an inductive way. However, we soon face a difficulty in constructing $P_{\sigma\otimes\tau}$ from P_σ and P_τ. The naive construction $P_{\sigma\otimes\tau} = \{a \otimes b \mid a \in P_\sigma, b \in P_\tau\}$ makes sense but can miss some interesting "undecomposable" elements of $\mathcal{A}_{\sigma\otimes\tau}$; in particular assume a constant of type $\sigma \otimes \tau$, then its interpretation may not belong to $P_{\sigma\otimes\tau}$ for any P_σ and P_τ. The same trouble appears when we construct $P_{!\sigma}$ from P_σ.

We solve this problem by parameterizing the predicates on the tensor-closed property (in the similar way to the Kripke logical relations [2]), so that the

* In *Proc. TLCA'99*, Lecture Notes in Computer Science 1581.

parameter indicates the linearly used resource (or the linear context). Such parameterized predicates form a model of MILL and serve as a basis for constructing logical predicates for MILL. The problem of tensor types disappears if each interesting element satisfies the tensor-closed property.

The construction is based on a few category-theoretic tools, specifically the presheaf construction (*free symmetric monoidal cocompletion* [15]) for symmetric monoidal categories and also a *glueing* (*sconing*, *Freyd covering*) construction [16, 21] on symmetric monoidal closed categories. It is known that a setting for standard logical predicates can be obtained by glueing a cartesian closed category to **Set** [21, 14]; ours is derived by glueing a symmetric monoidal closed category to the presheaf category of a small symmetric monoidal category (which specifies the tensor-closed property mentioned above). For DILL we further use a glueing construction of symmetric monoidal adjunction to accommodate the modality. However in this paper we leave these abstract idea rather implicit (except in Sect. 4) and describe all constructions concretely.

By applying our logical predicates method, we obtain the full completeness of syntactic translations between linear type theories. For instance, it is an immediate corollary of the Basic Lemma that MILL is a *full* fragment of DILL (Example 3), in the sense that, for any DILL-term $\emptyset\ ;\ \Delta \vdash M : \sigma$ with no ! in Δ nor σ, there always exists an MILL-term $\Delta \vdash N : \sigma$ such that $\emptyset\ ;\ \Delta \vdash M = N : \sigma$ holds. See Example 2 and 4 for other examples.

Though the existing syntax for linear type theories are rather diverging, their semantic models are now well-established and related each other, in terms of symmetric monoidal (closed) categories and adjunctions [6, 8, 5], and our approach based on such categorical models is likely to apply to many other linear type theories as well. In fact it is routine to modify our technique for non-commutative linear logic and monoidal (bi)closed categories (see [17]). Furthermore, by combining our approach with Hyland and Tan's double glueing construction [23] (see Example 5) we can deal with a classical linear type theory (MLL). These results, proofs and further category-theoretic analysis are reported in the full paper [13].

Also it might be fruitful to adapt our method to programming languages, see for example the complexity-parameterized logical relation used in [11]. Another interesting direction is to combine our approach to other techniques of specifying properties of semantic categories, for instance that of specification structures [1].

Acknowledgements I thank Gordon Plotkin for discussions at the initial stage of this work.

2 Multiplicative Intuitionistic Linear Logic

We recall a simple fragment of intuitionistic linear logic (Multiplicative Intuitionistic Linear Logic, MILL) together with the associated term calculus. The category-theoretic models are given as symmetric monoidal closed categories, for which soundness and completeness are known (e.g. [7]). See [10, 8] for the category-theoretic concepts used in this paper.

2.1 Syntax of MILL

We briefly recall the syntax of MILL. The detail is discussed e.g. in [7]; our presentation is chosen so that it will be compatible with DILL (Sect. 5). A set of base types (write b for one) and also a set of constants are fixed throughout this paper.

Types and Terms

$$\begin{aligned}\sigma &::= b \mid I \mid \sigma \otimes \sigma \mid \sigma \multimap \sigma \\ M &::= c(M) \mid x \mid * \mid \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ M \mid M \otimes M \mid \mathsf{let}\ x \otimes x\ \mathsf{be}\ M\ \mathsf{in}\ M \mid \\ &\quad\ \lambda x.M \mid MM\end{aligned}$$

We assume that each constant c has a fixed arity $\sigma \to \tau$, where σ and τ are types which do not involve $\multimap$. (This restriction on arity is for ease of presentation and not essential.)

Typing

$$\frac{c : \sigma \to \tau \quad \Delta \vdash M : \sigma}{\Delta \vdash c(M) : \tau}\ (\mathbf{Constant}) \qquad \frac{}{x : \sigma \vdash x : \sigma}\ (\mathbf{Variable})$$

$$\frac{}{\vdash * : I}\ (I\mathbf{I}) \qquad \frac{\Delta_1 \vdash M : I \quad \Delta_2 \vdash N : \sigma}{\Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ N : \sigma}\ (I\mathbf{E})$$

$$\frac{\Delta_1 \vdash M : \sigma \quad \Delta_2 \vdash N : \tau}{\Delta_1 \sharp \Delta_2 \vdash M \otimes N : \sigma \otimes \tau}\ (\otimes\mathbf{I}) \qquad \frac{\Delta_1 \vdash M : \sigma \otimes \tau \quad \Delta_2, x : \sigma, y : \tau \vdash N : \theta}{\Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N : \theta}\ (\otimes\mathbf{E})$$

$$\frac{\Delta, x : \sigma \vdash M : \tau}{\Delta \vdash \lambda x.M : \sigma \multimap \tau}\ (\multimap \mathbf{I}) \qquad \frac{\Delta_1 \vdash M : \sigma \multimap \tau \quad \Delta_2 \vdash N : \sigma}{\Delta_1 \sharp \Delta_2 \vdash MN : \tau}\ (\multimap \mathbf{E})$$

where $\Delta_1 \sharp \Delta_2$ is a merge of Δ_1 and Δ_2 (this notation is taken from [5]). We note that any typing judgement has a unique derivation.

Axioms

$$\begin{aligned}\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ M &= M & \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ * &= M \\ \mathsf{let}\ x \otimes y\ \mathsf{be}\ M \otimes N\ \mathsf{in}\ L &= L[M/x, N/y] & \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ x \otimes y &= M \\ (\lambda x.M)N &= M[N/x] & \lambda x.Mx &= M\end{aligned}$$

$$\begin{aligned}C[\mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ N] &= \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ C[N] \\ C[\mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N] &= \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ C[N]\end{aligned}$$

In the above $C[-]$ indicates a (well-typed) context – we assume suitable conditions on variables for avoiding undesirable captures. The equational theory of MILL is defined as the congruence relation on the terms with typing judgement generated from these axioms.

2.2 Semantics of MILL

Let $\mathbb{C}$ be a symmetric monoidal closed category with tensor product $\otimes$, unit object I and exponent $\multimap$. Assume that there is an object $[\![b]\!]$ for each base type b and an arrow $[\![c]\!] : [\![\sigma]\!] \to [\![\tau]\!]$ for each constant $c : \sigma \to \tau$, where $[\![\sigma]\!]$ is defined by $[\![I]\!] = I$, $[\![\sigma\otimes\tau]\!] = [\![\sigma]\!]\otimes[\![\tau]\!]$ and $[\![\sigma \multimap \tau]\!] = [\![\sigma]\!] \multimap [\![\tau]\!]$. For each typing judgement $\Delta \vdash M : \tau$, we define its interpretation $[\![\Delta \vdash M : \tau]\!] : [\![|\Delta|]\!] \to [\![\tau]\!]$ in $\mathbb{C}$ as follows, where $|\Delta| = (\ldots([\![\sigma_1]\!] \otimes [\![\sigma_2]\!])\ldots) \otimes [\![\sigma_n]\!]$ for $\Delta \equiv x_1 : \sigma_1, x_2 : \sigma_2, \ldots, x_n : \sigma_n$.

$$[\![\Delta \vdash c(M) : \tau]\!] = [\![|\Delta|]\!] \xrightarrow{[\![\Delta\vdash M:\sigma]\!]} [\![\sigma]\!] \xrightarrow{[\![c]\!]} [\![\tau]\!]$$

$$[\![x : \sigma \vdash x : \sigma]\!] = [\![\sigma]\!] \xrightarrow{id_{[\![\sigma]\!]}} [\![\sigma]\!]$$

$$[\![\vdash * : I]\!] = I \xrightarrow{id_I} I$$

$$[\![\Delta_1\sharp\Delta_2 \vdash \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ N : \sigma]\!] = [\![|\Delta_1\sharp\Delta_2|]\!] \xrightarrow{\simeq} [\![|\Delta_1|]\!] \otimes [\![|\Delta_2|]\!] \xrightarrow{[\![\Delta_1\vdash M:I]\!]\otimes[\![\Delta_2\vdash N:\sigma]\!]} I \otimes [\![\sigma]\!] \xrightarrow{\simeq} [\![\sigma]\!]$$

$$[\![\Delta_1\sharp\Delta_2 \vdash M\otimes N : \sigma \otimes \tau]\!] = [\![|\Delta_1\sharp\Delta_2|]\!] \xrightarrow{\simeq} [\![|\Delta_1|]\!] \otimes [\![|\Delta_2|]\!] \xrightarrow{[\![\Delta_1\vdash M:\sigma]\!]\otimes[\![\Delta_2\vdash N:\tau]\!]} [\![\sigma]\!] \otimes [\![\tau]\!]$$

$$[\![\Delta_1\sharp\Delta_2 \vdash \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N : \theta]\!] = [\![|\Delta_1\sharp\Delta_2|]\!] \xrightarrow{\simeq} [\![|\Delta_1|]\!] \otimes [\![|\Delta_2|]\!] \xrightarrow{[\![\Delta_1\vdash M:\sigma\otimes\tau]\!]\otimes id_{[\![|\Delta_2|]\!]}} ([\![\sigma]\!] \otimes [\![\tau]\!]) \otimes [\![|\Delta_2|]\!] \xrightarrow{\simeq} ([\![|\Delta_2|]\!] \otimes [\![\sigma]\!]) \otimes [\![\tau]\!] \xrightarrow{[\![\Delta_2,x:\sigma,y:\tau\vdash N:\theta]\!]} [\![\theta]\!]$$

$$[\![\Delta \vdash \lambda x.M : \sigma \multimap \tau]\!] = [\![|\Delta|]\!] \xrightarrow{\Lambda([\![\Delta,x:\sigma\vdash M:\tau]\!])} [\![\sigma]\!] \multimap [\![\tau]\!]$$

$$[\![\Delta_1\sharp\Delta_2 \vdash MN : \tau]\!] = [\![|\Delta_1\sharp\Delta_2|]\!] \xrightarrow{\simeq} [\![|\Delta_1|]\!] \otimes [\![|\Delta_2|]\!] \xrightarrow{[\![\Delta_1\vdash M:\sigma\multimap\tau]\!]\otimes[\![\Delta_2\vdash N:\sigma]\!]} ([\![\sigma]\!] \multimap [\![\tau]\!]) \otimes [\![\sigma]\!] \xrightarrow{\mathsf{ev}} [\![\tau]\!]$$

where "$\simeq$" denotes a (uniquely determined) canonical isomorphism. We write ev for the counit of the adjunction $- \otimes C \dashv C \multimap -$, and $\Lambda(f) : A \to C \multimap B$ for the adjoint mate of $f : A \otimes C \to B$.

Proposition 1. *This semantics is sound and complete.* □

3 Logical Predicates for MILL

We introduce parameterized predicates on objects of a symmetric monoidal closed category, and show that such predicates give rise to another symmetric monoidal closed category. We then define the logical predicates as type-indexed families of the predicates (inductively determined on the type structure), and state the Basic Lemma. We also give the canonically determined logical predicate which is used in showing full completeness of translations between linear type theories. We conclude this section by sketching the generalization to logical relations.

3.1 $\mathbb{C}_0$-Predicates

Let $\mathbb{C}_0$ be a small symmetric monoidal category, $\mathbb{C}_1$ a locally small symmetric monoidal closed category and $\mathbb{I}$ be a strict symmetric monoidal functor from $\mathbb{C}_0$ to $\mathbb{C}_1$.

Definition 1. *An* Obj($\mathbb{C}_0$)*-indexed set* $P = \{P(X)\}_{X \in \mathbb{C}_0}$ *is a* $\mathbb{C}_0$-predicate on $A \in \mathbb{C}_1$ *when*

- $P(X) \subseteq \mathbb{C}_1(\mathbb{I}X, A)$ *for* $X \in \mathbb{C}_0$*, and*
- *for* $f \in \mathbb{C}_0(X, Y)$*,* $g \in P(Y)$ *implies* $g \circ \mathbb{I}f \in P(X)$. □

We may intuitively think that $\mathbb{C}_1(\mathbb{I}X, A)$ represents the set of proofs of a sequent $X \vdash A$, and $\mathbb{C}_0$ (imported into $\mathbb{C}_1$ via $\mathbb{I}$) determines a property on proofs which is closed under tensor, composition and structural constructions. Unlike the traditional non-linear calculi and logical predicates over them, we explicitly state the "resource" X, which plays some significant role in our work. Then, for a $\mathbb{C}_0$-predicate P on A, $P(X)$ is a predicate on the proofs of $X \vdash A$. The second condition tells us that P is stable under the change of resource along a proof of $X \vdash Y$, provided that it satisfies the property $\mathbb{C}_0$.

Definition 2. *Define the category of* $\mathbb{C}_0$*-predicates* $\mathbb{C}_0\mathsf{PRED}$ *as follows:*

- *an object of* $\mathbb{C}_0\mathsf{PRED}$ *is a pair* (P, A) *where* P *is a* $\mathbb{C}_0$*-predicate on* $A \in \mathbb{C}_1$*;*
- *an arrow from* (P, A) *to* (Q, B) *is an arrow* $h \in \mathbb{C}_1(A, B)$ *such that* $g \in P(X)$ *implies* $h \circ g \in Q(X)$. □

Definition 3. *For* $\mathbb{C}_0$*-predicates* P *on* A *and* Q *on* B*, define* $\mathbb{C}_0$*-predicates* $P \otimes Q$ *on* $A \otimes B$ *and* $P \multimap Q$ *on* $A \multimap B$ *as follows.*

$$(P \otimes Q)(X) = \left\{ ((g \otimes h) \circ \mathbb{I}f) \;\middle|\; \begin{array}{l} \exists Y, Z \in \mathbb{C}_0 \quad f \in \mathbb{C}_0(X, Y \otimes Z), \\ g \in P(Y),\ h \in Q(Z) \end{array} \right\}$$

$$(P \multimap Q)(X) = \left\{ f \in \mathbb{C}_1(\mathbb{I}X, A \multimap B) \;\middle|\; \begin{array}{l} \forall Y \in \mathbb{C}_0 \quad a \in P(Y) \text{ implies} \\ \mathsf{ev} \circ (f \otimes a) \in Q(X \otimes Y) \end{array} \right\}$$

□

The definition of $P \otimes Q$ above is derived from a few category-theoretic tools, which will be explained in Sect. 4; for now, we shall give a proof-theoretic explanation. A sequent $X \vdash A \otimes B$ can be derived as

$$\dfrac{\begin{array}{c} \Pi_f \\ \vdots \\ X \vdash Y \otimes Z \end{array} \qquad \dfrac{\begin{array}{c} \Pi_g \\ \vdots \\ Y \vdash A \end{array} \quad \begin{array}{c} \Pi_h \\ \vdots \\ Z \vdash B \end{array}}{Y, Z \vdash A \otimes B}\,(\otimes\mathbf{I})}{X \vdash A \otimes B}\,(\otimes\mathbf{E})$$

where $X \vdash Y \otimes Z$ splits a resource X to Y and Z which are used to prove A and B respectively. In general, such a splitting of resource is not unique, so we consider all possible cases such that (i) the proof Π_f of the splitting satisfies the "tensor-closed property" $\mathbb{C}_0$ and (ii) the proofs Π_g of $Y \vdash A$ and Π_h of $Z \vdash B$ satisfy the predicates $P(Y)$ and $Q(Z)$ respectively – in such cases we say that the derivation satisfies the property $(P \otimes Q)(X)$.

The definition of $P \multimap Q$ is in spirit the same as the usual definition of logical predicates; $M : A \Rightarrow B$ satisfies $P \Rightarrow Q$ if and only if $MN : B$ belongs to Q for any $N : A$ satisfying P. However, since our type theory is linear, we have to deal with the resources of terms linearly, and we explicitly state them in the definition: intuitively, $\Delta \vdash M : A \multimap B$ satisfies $P \multimap Q$ if and only if $\Delta, \Delta' \vdash MN : B$ satisfies Q for any $\Delta' \vdash N : A$ satisfying P.

Lemma 1. *For each $X, A \in \mathbb{C}_0$ define $\mathbb{P}_A(X) = \{\mathbb{I}f \mid f \in \mathbb{C}_0(X, A)\}$. Then*

- *$\mathbb{P}_A$ is a $\mathbb{C}_0$-predicate on $\mathbb{I}A$.*
- *$f : (\mathbb{P}_A, \mathbb{I}A) \to (\mathbb{P}_B, \mathbb{I}B)$ in $\mathbb{C}_0\mathsf{PRED}$ iff $f = \mathbb{I}g$ for some $g \in \mathbb{C}_0(A, B)$.*
- *$\mathbb{P}_A \otimes \mathbb{P}_B = \mathbb{P}_{A \otimes B}$.* □

Proposition 2. *$\mathbb{C}_0\mathsf{PRED}$ forms a symmetric monoidal closed category by the following data: the unit object is $(\mathbb{P}_I, I)$, tensor is given by $(P, A) \otimes (Q, B) = (P \otimes Q, A \otimes B)$, and exponent $(P, A) \multimap (Q, B) = (P \multimap Q, A \multimap B)$. Moreover $\mathbb{P}$ extends to a strict symmetric monoidal functor from $\mathbb{C}_0$ to $\mathbb{C}_0\mathsf{PRED}$ which is full.* □

Remark 1. If $\mathbb{C}_0$ is closed and $\mathbb{I}$ preserves exponents strictly, then so is $\mathbb{P}$ – in particular we have $\mathbb{P}_{A \multimap B} = \mathbb{P}_A \multimap \mathbb{P}_B$. □

Example 1 (Subsconing). If $\mathbb{C}_0$ is equivalent to the one object one arrow category, a $\mathbb{C}_0$-predicate on A is just a subset of $\mathbb{C}_1(I, A)$, thus is a predicate on the global elements of A. For predicates P on A and Q on B, we have

$$\begin{aligned} P \otimes Q &= \{(g \otimes h) \circ \simeq \ \mid g \in P, h \in Q\} \\ P \multimap Q &= \{f \in \mathbb{C}_1(I, A \multimap B) \mid \mathsf{ev} \circ (f \otimes g) \circ \simeq \in Q \text{ for any } g \in P\} \end{aligned}$$

where $\simeq$ indicates the canonical isomorphism $I \xrightarrow{\simeq} I \otimes I$. Following [21] we call this category of predicates the *subsconing* of $\mathbb{C}_1$ and write $\widetilde{\mathbb{C}_1}$ for it. □

3.2 Logical $\mathbb{C}_0$-Predicates

Suppose that we have $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{I} : \mathbb{C}_0 \to \mathbb{C}_1$ as before. Also we fix an interpretation $[\![-]\!]_1$ of MILL in $\mathbb{C}_1$.

Definition 4. *A type-indexed family $\{P_\sigma\}$ is a* logical $\mathbb{C}_0$-predicate *if*

- *P_σ is a $\mathbb{C}_0$-predicate on $[\![\sigma]\!]_1$,*
- *$P_I = \mathbb{P}_I$, $P_{\sigma \otimes \tau} = P_\sigma \otimes P_\tau$, $P_{\sigma \multimap \tau} = P_\sigma \multimap P_\tau$, and*
- *$[\![c]\!]_1 : (P_\sigma, [\![\sigma]\!]_1) \to (P_\tau, [\![\tau]\!]_1)$ for each constant $c : \sigma \to \tau$.* □

Note that a logical $\mathbb{C}_0$-predicate is determined by its instances at base types. Given a logical $\mathbb{C}_0$-predicate $\{P_\sigma\}$, we can interpret MILL in $\mathbb{C}_0\,\mathsf{PRED}$ by $[\![b]\!] = (P_b, [\![b]\!]_1)$ for each base type b and $[\![c]\!] = [\![c]\!]_1 : (P_\sigma, [\![\sigma]\!]_1) \to (P_\tau, [\![\tau]\!]_1)$ for each constant $c : \sigma \to \tau$. Thus we have

Lemma 2 (Basic Lemma for MILL). *Let $\{P_\sigma\}$ be a logical $\mathbb{C}_0$-predicate. Then, for any term $\Delta \vdash M : \tau$, $[\![\Delta \vdash M : \tau]\!]_1 : (P_{|\Delta|}, [\![|\Delta|]\!]_1) \to (P_\tau, [\![\tau]\!]_1)$ holds.* □

$\mathbb{C}_0$ itself determines a logical $\mathbb{C}_0$-predicate in a canonical way, provided that

- for each base type b there is an object $[\![b]\!]_0 \in \mathbb{C}_0$, and
- for each constant $c : \sigma \to \tau$ there is an arrow $[\![c]\!]_0 \in \mathbb{C}_0([\![\sigma]\!]_0, [\![\tau]\!]_0)$

where $[\![\sigma]\!]_0$ is defined inductively by $[\![I]\!]_0 = I$ and $[\![\sigma \otimes \tau]\!]_0 = [\![\sigma]\!]_0 \otimes [\![\tau]\!]_0$. Then we automatically have an interpretation $[\![-]\!]_1$ in $\mathbb{C}_1$ determined by $[\![b]\!]_1 = \mathbb{I}([\![b]\!]_0)$ and $[\![c]\!]_1 = \mathbb{I}([\![c]\!]_0)$. Now define the *canonical logical* $\mathbb{C}_0$*-predicate* $\{\mathbb{P}^*_\sigma\}$ by $\mathbb{P}^*_b = \mathbb{P}_{[\![b]\!]_0}$. Basic Lemma for the canonical logical $\mathbb{C}_0$-predicate implies that, at $\multimap$-free types (at any types if $\mathbb{C}_0$ and $\mathbb{I}$ are closed) a definable element is in the image of $\mathbb{I}$.

3.3 Binary Logical $\mathbb{C}_0$-Relations

It is straightforward to generalize (or specialize) our logical predicates to multiple arguments, i.e. *logical relations*, in the same way as demonstrated in [21]. Here we spell out the case of binary ones. Suppose that $\mathbb{C}_0$ is a small symmetric monoidal category, $\mathbb{C}_1$ and $\mathbb{C}_2$ are locally small symmetric monoidal closed categories and that $\mathbb{I}_1 : \mathbb{C}_0 \to \mathbb{C}_1$ and $\mathbb{I}_2 : \mathbb{C}_0 \to \mathbb{C}_2$ are strict symmetric monoidal functors. A binary $\mathbb{C}_0$-relation is just a $\mathbb{C}_0$-predicate obtained by replacing $\mathbb{C}_1$ by $\mathbb{C}_1 \times \mathbb{C}_2$ and $\mathbb{I}$ by $\langle \mathbb{I}_1, \mathbb{I}_2 \rangle : \mathbb{C}_0 \to \mathbb{C}_1 \times \mathbb{C}_2$. Explicitly:

Definition 5. *An* $\mathrm{Obj}(\mathbb{C}_0)$*-indexed set* $R = \{R(X)\}_{X \in \mathbb{C}_0}$ *is a* $\mathbb{C}_0$-relation on $(A, B) \in \mathbb{C}_1 \times \mathbb{C}_2$ *when* $R(X) \subseteq \mathbb{C}_1(\mathbb{I}_1 X, A) \times \mathbb{C}_2(\mathbb{I}_2 X, B)$ *for* $X \in \mathbb{C}_0$*, and, for* $f \in \mathbb{C}_0(X, Y)$*,* $(g, h) \in P(Y)$ *implies* $(g \circ \mathbb{I}_1 f, h \circ \mathbb{I}_2 f) \in P(X)$*.* □

Definition 6. *Define the category of* $\mathbb{C}_0$*-relations* $\mathbb{C}_0\,\mathsf{REL}$ *as follows: an object of* $\mathbb{C}_0\,\mathsf{REL}$ *is a triple* (A, B, R) *where* R *is a* $\mathbb{C}_0$*-relation on* (A, B)*; and an arrow from* (A, B, R) *to* (A', B', R') *is a pair* $(h \in \mathbb{C}_1(A, A'), k \in \mathbb{C}_2(B, B'))$ *such that* $(f, g) \in R(X)$ *implies* $(h \circ f, k \circ g) \in R'(X)$*.* □

Proposition 2 tells us that $\mathbb{C}_0\,\mathsf{REL}$ is a symmetric monoidal closed category. More explicitly, for $\mathbb{C}_0$-relations R on (A, B) and R' on (A', B'), we have $\mathbb{C}_0$-relations $R \otimes R'$ on $(A \otimes A', B \otimes B')$ and $R \multimap R'$ on $(A \multimap A', B \multimap B')$ as follows.

$$(R \otimes R')(X) = \left\{ ((g \otimes g') \circ \mathbb{I}_1 f, (h \otimes h') \circ \mathbb{I}_2 f) \;\middle|\; \begin{array}{l} \exists Y, Z \in \mathbb{C}_0 \;\; f \in \mathbb{C}_0(X, Y \otimes Z), \\ (g, h) \in R(Y), (g', h') \in R'(Z) \end{array} \right\}$$

$$(R \multimap R')(X) = \left\{ (f, g) \;\middle|\; \begin{array}{l} \forall Y \in \mathbb{C}_0 \;\; (a, b) \in R(Y) \text{ implies} \\ (\mathsf{ev} \circ (f \otimes a), \mathsf{ev} \circ (g \otimes b)) \in R'(X \otimes Y) \end{array} \right\}$$

Now fix interpretations $[\![-]\!]_1$ and $[\![-]\!]_2$ of MILL in $\mathbb{C}_1$ and $\mathbb{C}_2$ respectively.

Definition 7. *A type-indexed family* $\{R_\sigma\}$ *is a* logical $\mathbb{C}_0$-relation *if*

- R_σ *is a* $\mathbb{C}_0$*-relation on* $([\![\sigma]\!]_1, [\![\sigma]\!]_2)$,
- $R_I(X) = \{(\mathbb{I}_1 f, \mathbb{I}_2 f) \mid f \in \mathbb{C}_0(X, I)\}$, $R_{\sigma\otimes\tau} = R_\sigma \otimes R_\tau$, $R_{\sigma\multimap\tau} = R_\sigma \multimap R_\tau$ *and*
- $([\![c]\!]_1, [\![c]\!]_2) : ([\![\sigma]\!]_1, [\![\sigma]\!]_2, R_\sigma) \to ([\![\tau]\!]_1, [\![\tau]\!]_2, R_\tau)$ *for each constant* $c : \sigma \to \tau$. □

Lemma 3 (Basic Lemma, binary version). *Let* $\{R_\sigma\}$ *be a logical* $\mathbb{C}_0$*-relation. Then, for any* $\Delta \vdash M : \tau$, $([\![\Delta \vdash M : \tau]\!]_1, [\![\Delta \vdash M : \tau]\!]_2) : ([\![|\Delta|]\!]_1, [\![|\Delta|]\!]_2, R_{|\Delta|}) \to ([\![\tau]\!]_1, [\![\tau]\!]_2, R_\tau)$ *holds.* □

4 Categorical Glueing

We sketch the categorical glueing constructions used in our development; the detailed category-theoretic analysis is found in [13].

We write $(\mathbb{D} \downarrow \Gamma)$ for the comma category [19] (or the "glued category") of a functor $\Gamma : \mathbb{C} \to \mathbb{D}$. An object of $(\mathbb{D} \downarrow \Gamma)$ is a triple $(D \in \mathbb{D}, C \in \mathbb{C}, f : D \to \Gamma C)$. An arrow from (D, C, f) to (D', C', f') is a pair $(d : D \to D', c : C \to C')$ satisfying $\Gamma c \circ f = f' \circ d$. We note that there is a projection functor $p : (\mathbb{D} \downarrow \Gamma) \to \mathbb{C}$ given by $p(D, C, f) = C$ and $p(d, c) = c$.

Lemma 4. *Suppose that* $\mathbb{C}$ *and* $\mathbb{D}$ *are symmetric monoidal closed categories and that* $\Gamma : \mathbb{C} \to \mathbb{D}$ *is a symmetric monoidal functor. Moreover suppose that* $\mathbb{D}$ *has pullbacks. Then the comma category* $\mathcal{G} \equiv (\mathbb{D} \downarrow \Gamma)$ *can be given a symmetric monoidal closed structure, so that the projection* $p : \mathcal{G} \to \mathbb{C}$ *is strict symmetric monoidal closed.*

Proof (sketch). We define the symmetric monoidal structure on $\mathcal{G}$ by

$$\begin{array}{c} I_{\mathcal{G}} \equiv (I_{\mathbb{D}}, I_{\mathbb{C}}, m_I) \\ (D, C, f) \otimes (D', C', f') \equiv (D \otimes D', C \otimes C', m_{C,C'} \circ (f \otimes f')) \\ (d, c) \otimes (d', c') \equiv (d \otimes d', c \otimes c') \end{array}$$

where $m_I : I_{\mathbb{D}} \to \Gamma I_{\mathbb{C}}$ and $m_{C,C'} : \Gamma C \otimes \Gamma C' \to \Gamma(C \otimes C')$ are the coherent morphisms of the symmetric monoidal functor Γ. Exponents are defined as

$$(D, C, f) \multimap (D', C', f') \equiv ((D \multimap D') \times_{D \multimap \Gamma C'} \Gamma(C \multimap C'),\ C \multimap C',\ \pi_2)$$

which is given by the following pullback in $\mathbb{D}$.

$$\begin{array}{ccc} (D \multimap D') \times_{D \multimap \Gamma C'} \Gamma(C \multimap C') & \xrightarrow{\pi_2} & \Gamma(C \multimap C') \\ & & \Big\downarrow {\scriptstyle \Lambda(\Gamma \mathsf{ev}_{C,C'} \circ m_{C \multimap C', C})} \\ \Big\downarrow {\scriptstyle \pi_1} & & \Gamma C \multimap \Gamma C' \\ & & \Big\downarrow {\scriptstyle f \multimap \Gamma C'} \\ D \multimap D' & \xrightarrow[D \multimap f']{} & D \multimap \Gamma C' \end{array}$$

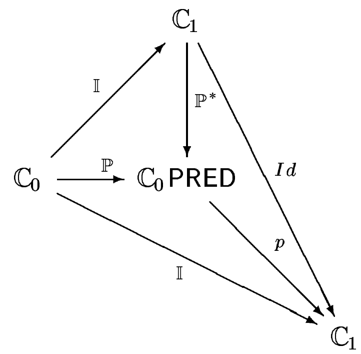
□

This result seems to be folklore. Notice that the glueing functor Γ does not have to be strong.

In the situation of the last section, by letting $\Gamma : \mathbb{C}_1 \to \mathbf{Set}^{\mathbb{C}_0^{op}}$ be the functor which sends X to $\mathbb{C}_1(\mathbb{I}-, X)$, we obtain the setting for the category of $\mathbb{C}_0$-predicates. The symmetric monoidal closed structure of $\mathbf{Set}^{\mathbb{C}_0^{op}}$ is given by $I(-) = \mathbb{C}_0(-, I)$, $(F \otimes G)(-) = \int^{X,Y} FX \times GY \times \mathbb{C}_0(-, X \otimes Y)$ and $(F \multimap G)(-) = \mathbf{Set}^{\mathbb{C}_0^{op}}(F(=), G(- \otimes =))$ (see [15]), for which Γ becomes symmetric monoidal. For describing the predicates, we are interested in the full subcategory of the glued category whose objects are subobjects in $\mathbf{Set}^{\mathbb{C}_0^{op}}$. This is precisely the category $\mathbb{C}_0\mathsf{PRED}$, which is again symmetric monoidal closed; the definition of unit and tensor are patched in the obvious way (this is possible because $\mathbf{Set}^{\mathbb{C}_0^{op}}$ admits epi-mono factorization), resulting the concrete descriptions in Sect. 3.

Lafont has shown that, using the glueing for cartesian closed categories, a small cartesian category fully and faithfully embeds to the cartesian closed category freely generated from the former [16]. We can use $\mathbb{C}_0\mathsf{PRED}$ for showing a parallel result:

Example 2. Let $\mathbb{C}_0$ be a small symmetric monoidal category and $\mathbb{C}_1$ be the symmetric monoidal closed category freely generated from $\mathbb{C}_0$. Then the embedding $\mathbb{I} : \mathbb{C}_0 \to \mathbb{C}_1$ is full faithful. Faithfulness is easily shown by constructing a symmetric monoidal closed category to which $\mathbb{C}_0$ faithfully embeds. Fullness follows from the commutative diagrams



where $\mathbb{P}^*$ is the uniquely determined strict symmetric monoidal closed functor making the upper triangle commute, and the right triangle commutes because of the universal property of $\mathbb{I}$. Since both $\mathbb{P}$ and p are full, so is $\mathbb{I} = p \circ \mathbb{P}$. $\square$

Syntactically, this implies that the I, $\otimes$-fragment of MILL is full in MILL; we can show it by applying the Basic Lemma to the canonical logical predicate (where $\mathbb{C}_0$ is the term model of the I, $\otimes$-fragment), which in fact is a concrete reworking of Example 2.

For interpreting the modality ! of DILL in the following section, we will need to determine a symmetric monoidal adjunction between the glued categories:

Lemma 5. *Suppose that* $\mathbb{C}_1 \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{C}_2$ *and* $\mathbb{D}_1 \underset{U'}{\overset{F'}{\rightleftarrows}} \mathbb{D}_2$ *are (symmetric monoidal) adjunctions, with (symmetric monoidal) functors* $\Gamma_1 : \mathbb{C}_1 \to \mathbb{D}_1$ *and* $\Gamma_2 : \mathbb{C}_2 \to \mathbb{D}_2$ *together with a (monoidal) natural isomorphism* $\tau : U'\Gamma_2 \simeq \Gamma_1 U$. *For* $\mathcal{G}_1 \equiv (\mathbb{D}_1 \downarrow \Gamma_1)$ *and* $\mathcal{G}_2 \equiv (\mathbb{D}_2 \downarrow \Gamma_2)$, *there are functors* $\mathcal{F} : \mathcal{G}_1 \to \mathcal{G}_2$ *and* $\mathcal{U} : \mathcal{G}_2 \to \mathcal{G}_1$ *given by*

$$\begin{aligned} \mathcal{F}(D, C, f) &= (F'D,\ FC,\ \sigma_C \circ F'f), \quad \mathcal{F}(d, c) = (F'd, Fc), \\ \mathcal{U}(Y, X, g) &= (U'Y,\ UX,\ \tau_X \circ U'g), \quad \mathcal{U}(y, x) = (U'y, Ux) \end{aligned}$$

where $\sigma_C = \varepsilon'_{\Gamma_2 FC} \circ F'\tau^{-1}_{FC} \circ F'\Gamma_1\eta_C : F'\Gamma_1 C \to \Gamma_2 FC$ *(*η *is the unit of* $F \dashv U$ *and* ε' *is the counit of* $F' \dashv U'$*).* $\mathcal{F}$ *is (strong symmetric monoidal and) left adjoint to* $\mathcal{U}$. *Moreover the projections* $p_1 : \mathcal{G}_1 \to \mathbb{C}_1$ *and* $p_2 : \mathcal{G}_2 \to \mathbb{C}_2$ *give a map of adjunction* [19] *from* $\mathcal{G}_1 \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{G}_2$ *to* $\mathbb{C}_1 \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{C}_2$. □

5 Dual Intuitionistic Linear Logic

Now we enrich our logic and calculus with the modality !. There are many possible choices for this, see for instance [7]. Here we choose the formulation due to Barber and Plotkin, called Dual Intuitionistic Linear Logic (DILL) [5] for its simple syntax and equational theory, as well as for the well-established category-theoretic models of DILL in terms of symmetric monoidal adjunctions. Alternatively we could use Benton's Linear Non-Linear Logic (LNL Logic) [6] which has essentially the same class of category-theoretic models as DILL. In DILL a typing judgement takes the form $\Gamma\ ;\ \Delta \vdash M : \sigma$ in which Γ represents an intuitionistic (or additive) context whereas Δ is a linear (multiplicative) context.

5.1 Syntax of DILL

Types and Terms

$$\begin{aligned} \sigma &::= b \mid I \mid \sigma \otimes \sigma \mid \sigma \multimap \sigma \mid\ !\sigma \\ M &::= c(M) \mid x \mid * \mid \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ M \mid M \otimes M \mid \mathsf{let}\ x \otimes x\ \mathsf{be}\ M\ \mathsf{in}\ M \mid \\ &\quad\ \lambda x.M \mid MM \mid\ !M \mid \mathsf{let}\ !x\ \mathsf{be}\ M\ \mathsf{in}\ M \end{aligned}$$

Typing

$$\frac{c : \sigma \to \tau \quad \Gamma\ ;\ \Delta \vdash M : \sigma}{\Gamma\ ;\ \Delta \vdash c(M) : \tau}\ (\mathbf{Constant}) \qquad \frac{}{\Gamma\ ;\ x : \sigma \vdash x : \sigma}\ (\mathbf{Variable_{lin}})$$

$$\frac{}{\Gamma\ ;\ \emptyset \vdash * : I}\ (I\mathbf{I}) \qquad \frac{\Gamma\ ;\ \Delta_1 \vdash M : I \quad \Gamma\ ;\ \Delta_2 \vdash N : \sigma}{\Gamma\ ;\ \Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ N : \sigma}\ (I\mathbf{E})$$

$$\frac{\Gamma\ ;\ \Delta_1 \vdash M : \sigma \quad \Gamma\ ;\ \Delta_2 \vdash N : \tau}{\Gamma\ ;\ \Delta_1 \sharp \Delta_2 \vdash M \otimes N : \sigma \otimes \tau}\ (\otimes\mathbf{I}) \qquad \frac{\begin{array}{c}\Gamma\ ;\ \Delta_1 \vdash M : \sigma \otimes \tau \\ \Gamma\ ;\ \Delta_2, x : \sigma, y : \tau \vdash N : \theta\end{array}}{\Gamma\ ;\ \Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N : \theta}\ (\otimes\mathbf{E})$$

$$\frac{\Gamma\ ;\ \Delta, x : \sigma \vdash M : \tau}{\Gamma\ ;\ \Delta \vdash \lambda x.M : \sigma \multimap \tau}\ (\multimap\mathbf{I}) \qquad \frac{\Gamma\ ;\ \Delta_1 \vdash M : \sigma \multimap \tau \quad \Gamma\ ;\ \Delta_2 \vdash N : \sigma}{\Gamma\ ;\ \Delta_1 \sharp \Delta_2 \vdash MN : \tau}\ (\multimap\mathbf{E})$$

$$\frac{}{\Gamma_1, x:\sigma, \Gamma_2 \ ; \ \emptyset \vdash x:\sigma} \ (\mathbf{Variable_{int}})$$

$$\frac{\Gamma \ ; \ \emptyset \vdash M:\sigma}{\Gamma \ ; \ \emptyset \vdash !M :!\sigma} \ (!\mathbf{I}) \qquad \frac{\Gamma \ ; \ \Delta_1 \vdash M :!\sigma \quad \Gamma, x:\sigma \ ; \ \Delta_2 \vdash N:\tau}{\Gamma \ ; \ \Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ !x \ \mathsf{be}\ M \ \mathsf{in}\ N:\tau} \ (!\mathbf{E})$$

Axioms

$$\begin{array}{rr} \mathsf{let}\ * \ \mathsf{be}\ * \ \mathsf{in}\ M = M & \mathsf{let}\ * \ \mathsf{be}\ M \ \mathsf{in}\ * = M \\ \mathsf{let}\ x\otimes y \ \mathsf{be}\ M\otimes N \ \mathsf{in}\ L = L[M/x, N/y] & \mathsf{let}\ x\otimes y \ \mathsf{be}\ M \ \mathsf{in}\ x\otimes y = M \\ (\lambda x.M)N = M[N/x] & \lambda x.Mx = M \\ \mathsf{let}\ !x \ \mathsf{be}\ !M \ \mathsf{in}\ N = N[M/x] & \mathsf{let}\ !x \ \mathsf{be}\ M \ \mathsf{in}\ !x = M \end{array}$$

$$\begin{array}{rl} C[\mathsf{let}\ * \ \mathsf{be}\ M \ \mathsf{in}\ N] &= \mathsf{let}\ * \ \mathsf{be}\ M \ \mathsf{in}\ C[N] \\ C[\mathsf{let}\ x\otimes y \ \mathsf{be}\ M \ \mathsf{in}\ N] &= \mathsf{let}\ x\otimes y \ \mathsf{be}\ M \ \mathsf{in}\ C[N] \\ C[\mathsf{let}\ !x \ \mathsf{be}\ M \ \mathsf{in}\ N] &= \mathsf{let}\ !x \ \mathsf{be}\ M \ \mathsf{in}\ C[N] \end{array}$$

where $C[-]$ is a linear context (no ! binds $[-]$).

5.2 Semantics of DILL

Let $\mathbb{C}$ be a cartesian category (category with finite products), $\mathbb{D}$ a symmetric monoidal closed category and $\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$ (with $F \dashv U$) a symmetric monoidal adjunction; we understand that the symmetric monoidal structure on $\mathbb{C}$ is given by (a choice of) the terminal object and binary product. Assume that there is an object $[\![b]\!] \in \mathbb{D}$ for each base type b and an arrow $[\![c]\!] \in \mathbb{D}([\![\sigma]\!], [\![\tau]\!])$ for each constant $c:\sigma \to \tau$, where $[\![\sigma]\!] \in \mathbb{D}$ is inductively defined by $[\![I]\!] = I$, $[\![\sigma\otimes\tau]\!] = [\![\sigma]\!]\otimes[\![\tau]\!]$, $[\![\sigma \multimap \tau]\!] = [\![\sigma]\!] \multimap [\![\tau]\!]$ and $[\![!\sigma]\!] = FU[\![\sigma]\!]$. For each typing judgement $\Gamma \ ; \ \Delta \vdash M:\sigma$, we define $[\![\Gamma \ ; \ \Delta \vdash M:\sigma]\!] : [\![|\Gamma \ ; \ \Delta|]\!] \to [\![\tau]\!]$ in $\mathbb{D}$ as follows, where $|\Gamma \ ; \ \Delta| = |!\Gamma, \Delta|$ in which $!\Gamma = x_1 :!\sigma_1, \dots, x_n :!\sigma_n$ for $\Gamma \equiv x_1:\sigma_1,\dots,x_n:\sigma_n$. First eight cases are dealt with as in MILL, with care for discarding or duplicating the intuitionistic context, using

$$\begin{array}{ll} \mathsf{discard}_{\Gamma,\Delta} & : [\![|\Gamma \ ; \ \Delta|]\!] \to [\![|\Delta|]\!] \\ \mathsf{split}_{\Gamma,\Delta_1,\Delta_2} & : [\![|\Gamma \ ; \ \Delta_1\sharp\Delta_2|]\!] \to [\![|\Gamma \ ; \ \Delta_1|]\!] \otimes [\![|\Gamma \ ; \ \Delta_2|]\!] \end{array}$$

which are defined in terms of projections and diagonal maps in $\mathbb{C}$ and imported into $\mathbb{D}$ via F. For last three cases we have

$$[\![\Gamma_1, x:\sigma, \Gamma_2 \ ; \ \emptyset \vdash x:\sigma]\!] =$$
$$[\![|\Gamma_1, x:\sigma, \Gamma_2|]\!] \overset{\cong}{\to} F(\dots \times U[\![\sigma]\!] \times \dots) \xrightarrow{F\mathsf{proj}} FU[\![\sigma]\!] \xrightarrow{\varepsilon} [\![\sigma]\!]$$

$$[\![\Gamma \ ; \ \emptyset \vdash !M :!\sigma]\!] = [\![|\Gamma \ ; \ \emptyset|]\!] \overset{\cong}{\to} \textstyle\bigotimes_i FU[\![\sigma_i]\!] \xrightarrow{\bigotimes_i \delta} \bigotimes_i FUFU[\![\sigma_i]\!] \xrightarrow{m}$$
$$FU(\textstyle\bigotimes_i FU[\![\sigma_i]\!]) \overset{\cong}{\to} FU[\![|\Gamma \ ; \ \emptyset|]\!] \xrightarrow{FU[\![\Gamma \ ; \ \emptyset\vdash M:\sigma]\!]} FU[\![\sigma]\!]$$

$$[\![\Gamma \ ; \ \Delta_1\sharp\Delta_2 \vdash \mathsf{let}\ !x \ \mathsf{be}\ M \ \mathsf{in}\ N:\tau]\!] =$$
$$[\![|\Gamma \ ; \ \Delta_1\sharp\Delta_2|]\!] \xrightarrow{\mathsf{split}} [\![|\Gamma \ ; \ \Delta_1|]\!] \otimes [\![|\Gamma \ ; \ \Delta_2|]\!] \xrightarrow{[\![\Gamma \ ; \ \Delta_1\vdash M:!\sigma]\!]\otimes id}$$
$$[\![!\sigma]\!] \otimes [\![|\Gamma \ ; \ \Delta_2|]\!] \overset{\cong}{\to} [\![|\Gamma, x:\sigma \ ; \ \Delta_2|]\!] \xrightarrow{[\![\Gamma, x:\sigma \ ; \ \Delta_2\vdash N:\tau]\!]} [\![\tau]\!]$$

where proj is a suitable projection in $\mathbb{C}$, ε and δ are the counit and comultiplication of the comonad FU while m is an induced coherent morphism.

Proposition 3. *This semantics is sound and complete* [5]. □

5.3 Logical Predicates for DILL

Consider the following commutative diagram of functors

$$\begin{array}{ccc} \mathbb{C}_0 & \xrightarrow{F_0} & \mathbb{D}_0 \\ \Big\downarrow{\scriptstyle \mathbb{I}} & & \Big\downarrow{\scriptstyle \mathbb{J}} \\ \mathbb{C}_1 & \xrightarrow[F_1]{} & \mathbb{D}_1 \end{array}$$

in which $\mathbb{C}_0$ and $\mathbb{C}_1$ are cartesian categories, $\mathbb{D}_0$ symmetric monoidal and $\mathbb{D}_1$ symmetric monoidal closed; and F_0, F_1 are strong symmetric monoidal while $\mathbb{I}$, $\mathbb{J}$ are strict symmetric monoidal. Moreover assume that F_1 has a right adjoint $U_1 : \mathbb{D}_1 \to \mathbb{C}_1$.

As in Sect. 3, we define the categories of $\mathbb{C}_0$- and $\mathbb{D}_0$-predicates – let us call them $\mathbb{C}_0\mathsf{PRED}$ and $\mathbb{D}_0\mathsf{PRED}$ respectively. Note that $\mathbb{C}_0\mathsf{PRED}$ is a cartesian category with products given by $(P \times Q)(X) = \{\langle f, g\rangle \mid f \in P(X), g \in Q(X)\}$ for $\mathbb{C}_0$-predicates P and Q (which coincides with $P \otimes Q$ in Definition 3).

Now we give functors between $\mathbb{C}_0\mathsf{PRED}$ and $\mathbb{D}_0\mathsf{PRED}$. For a $\mathbb{C}_0$-predicate P on $A \in \mathbb{C}_1$, define a $\mathbb{D}_0$-predicate $L(P)$ on $F_1 A \in \mathbb{D}_1$ by

$$L(P)(Y) \;=\; \{F_1 g \circ \mathbb{J} f \mid \exists X \in \mathbb{C}_0\ f \in \mathbb{D}_0(Y, F_0 X), g \in P(X)\}$$

and, for a $\mathbb{D}_0$-predicate Q on $B \in \mathbb{D}_1$, a $\mathbb{C}_0$-predicate $\widehat{F_0}(Q)$ on $U_1 B \in \mathbb{C}_1$ by

$$\widehat{F_0}(Q)(X) \;=\; \{f^* \in \mathbb{C}_1(\mathbb{I}X, U_1 B) \mid f \in Q(F_0 X) \subseteq \mathbb{D}_1(\mathbb{J}F_0 X, B) = \mathbb{D}_1(F_1 \mathbb{I} X, B)\}$$

where $f^* : \mathbb{I}X \to U_1 B$ is the adjoint mate of $f : F_1 \mathbb{I} X \to B$.

Proposition 4. *L and $\widehat{F_0}$ extend to functors between $\mathbb{C}_0\mathsf{PRED}$ and $\mathbb{D}_0\mathsf{PRED}$. Moreover L is strong symmetric monoidal, and left adjoint to $\widehat{F_0}$.* □

Therefore we have a symmetric monoidal adjunction between a cartesian category $\mathbb{C}_0\mathsf{PRED}$ and a symmetric monoidal closed category $\mathbb{D}_0\mathsf{PRED}$. Let ! be the induced comonad on $\mathbb{D}_0\mathsf{PRED}$, that is, we define a $\mathbb{D}_0$-predicate $!P$ on $F_1 U_1 A$ by

$$(!P)(Y) \;=\; \{F_1 g^* \circ \mathbb{J} f \mid \exists X \in \mathbb{C}_0\ f \in \mathbb{D}_0(Y, F_0 X), g \in P(F_0 X)\}$$

for a $\mathbb{D}_0$-predicate P on A. These are derived from a category-theoretic construction (left Kan extension [19] gives a left adjoint of $(-) \circ F_0 : \mathbf{Set}^{\mathbb{D}_0^{\mathrm{op}}} \to \mathbf{Set}^{\mathbb{C}_0^{\mathrm{op}}}$) together with Lemma 5 (for glueing $\mathbb{C}_1 \underset{U_1}{\overset{F_1}{\rightleftarrows}} \mathbb{D}_1$ to $\mathbf{Set}^{\mathbb{C}_0^{\mathrm{op}}} \underset{(-)\circ F_0}{\rightleftarrows} \mathbf{Set}^{\mathbb{D}_0^{\mathrm{op}}}$), but here let us motivate $!P$ more intuitively. A sequent $\emptyset\ ;\ Y \vdash !A$ can be proved as

$$\dfrac{\begin{array}{c}\Pi_f \\ \vdots \\ \emptyset\ ;\ Y \vdash !X\end{array} \quad \dfrac{\begin{array}{c}\Pi_g \\ \vdots \\ X\ ;\ \emptyset \vdash A\end{array}}{X\ ;\ \emptyset \vdash !A}\,(!\mathbf{I})}{\emptyset\ ;\ Y \vdash !A}\,(!\mathbf{E})$$

where $\emptyset\ ;\ Y \vdash !X$ converts a linear resource Y to $!X$ which is used non-linearly in $X\ ;\ \emptyset \vdash !A$ to produce $!A$. Taking all such possible cases into account, we say that the proof satisfies $(!P)(Y)$ when Π_f belongs to $\mathbb{D}_0$ and Π_g satisfies $P(X)$.

Now let us fix an interpretation $[\![-]\!]_1$ of DILL in $\mathbb{C}_1 \underset{U_1}{\overset{F_1}{\rightleftarrows}} \mathbb{D}_1$.

Definition 8. *A type-indexed family* $\{P_\sigma\}$ *is a* logical $(\mathbb{C}_0 \overset{F_0}{\to} \mathbb{D}_0)$-predicate *if*

- P_σ *is a* $\mathbb{D}_0$*-predicate on* $[\![\sigma]\!]_1$,
- $P_I = \mathbb{P}_I$, $P_{\sigma\otimes\tau} = P_\sigma \otimes P_\tau$, $P_{\sigma\multimap\tau} = P_\sigma \multimap P_\tau$ *and* $P_{!\sigma} = !P_\sigma$ *hold, and*
- $[\![c]\!]_1 : (P_\sigma, [\![\sigma]\!]_1) \to (P_\tau, [\![\tau]\!]_1)$ *for each constant* $c : \sigma \to \tau$. □

Lemma 6 (Basic Lemma for DILL). *Let* $\{P_\sigma\}$ *be a logical* $(\mathbb{C}_0 \overset{F_0}{\to} \mathbb{D}_0)$*-predicate. Then, for* $\Gamma\ ;\ \Delta \vdash M : \tau$, $[\![\Gamma\ ;\ \Delta \vdash M : \tau]\!]_1 : (P_{|\Gamma\ ;\ \Delta|}, [\![|\Gamma\ ;\ \Delta|]\!]_1) \to (P_\tau, [\![\tau]\!]_1)$ *holds.* □

$(\mathbb{C}_0 \overset{F_0}{\to} \mathbb{D}_0)$ itself determines the *canonical logical* $(\mathbb{C}_0 \overset{F_0}{\to} \mathbb{D}_0)$*-predicate* when

- for each base type b there is an object $[\![b]\!]_0 \in \mathbb{D}_0$, and
- for each constant $c : \sigma \to \tau$ there is an arrow $[\![c]\!]_0 \in \mathbb{D}_0([\![\sigma]\!]_0, [\![\tau]\!]_0)$

where $[\![\sigma]\!]_0$ is defined inductively by $[\![I]\!]_0 = I$ and $[\![\sigma\otimes\tau]\!]_0 = [\![\sigma]\!]_0 \otimes [\![\tau]\!]_0$. In such cases we automatically have an interpretation $[\![-]\!]_1$ in $\mathbb{D}_1$ determined by $[\![b]\!]_1 = \mathbb{J}([\![b]\!]_0)$ and $[\![c]\!]_1 = \mathbb{J}([\![c]\!]_0)$, and the canonical logical $(\mathbb{C}_0 \overset{F_0}{\to} \mathbb{D}_0)$-predicate $\{\mathbb{P}^*_\sigma\}$ is determined by $\mathbb{P}^*_b = \mathbb{P}_{[\![b]\!]_0}$.

Example 3 (From MILL to DILL). Let $\mathbb{D}_0$ be the term model of MILL and $\mathbb{C}_0$ equivalent to the one object one arrow category, and $\mathbb{C}_1 \underset{U_1}{\overset{F_1}{\rightleftarrows}} \mathbb{D}_1$ be the term model of DILL with the same base types and constants. Applying the Basic Lemma to the canonical logical $(\mathbb{C}_0 \to \mathbb{D}_0)$-predicate it follows that MILL is a full fragment of DILL; note that $\mathbb{P}_{\sigma\multimap\tau} = \mathbb{P}_\sigma \multimap \mathbb{P}_\tau$ holds for !-free types σ and τ (see Remark 1). □

Example 4 (From action calculi to DILL). Suppose that $\mathbb{C}_0 \overset{F_0}{\to} \mathbb{D}_0$ is the term model of an *action calculus* [20, 22] and $\mathbb{C}_1 \underset{U_1}{\overset{F_1}{\rightleftarrows}} \mathbb{D}_1$ is that of the corresponding DILL (alternatively the LNL Logic of Benton [6]), with $\mathbb{I}$ and $\mathbb{J}$ induced by the translation from the action calculus to DILL. If we have only non-parameterized constants, Basic Lemma applied to the canonical logical predicate implies that the translation is full. In fact we can deal with parameterized constants (control operators) as well (see [13]), so together with the conservativity [4] we have the full completeness of DILL (LNL) over (static) action calculi. □

6 Related Work, Further Work

6.1 Categorical Logical Predicates

Our treatment of logical predicates in category-theoretic framework is inspired by Hermida's work on fibrations and logical predicates [14], and also influenced by Mitchell and others' work, in particular [21]. However, all these results are for typed lambda calculi. Blute and Scott [9] do consider a linear variant, and the intuition behind their work seems close to ours, though their work is on classical linear logic and better understood in connection with Tan's recent work (see below). We also note that Ambler [3] has studied some relevant idea. The fact that our construction yields (bi)fibrations has some significance in our glueing constructions; we leave this categorical analysis to the full paper [13].

6.2 Classical Linear Type Theories

So far we have only considered "intuitionistic" linear type theories. It is natural to expect that our construction works equally well in the settings with duality, i.e., classical linear theories. Here is a relevant construction explored by Tan:

Example 5 (Double Glueing). An attractive use of categorical glueing is developed in Tan's thesis [23]. Let $\mathbb{C}$ be a $*$-autonomous category (typically a compact closed category). Because of the duality, $\mathbb{C}^{\mathrm{op}}$ is also $*$-autonomous and we have subscones (Example 1) $\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{C}^{\mathrm{op}}}$ with projections $p_1 : \widetilde{\mathbb{C}} \to \mathbb{C}$ and $p_2 : \widetilde{\mathbb{C}^{\mathrm{op}}} \to \mathbb{C}^{\mathrm{op}}$. Hyland noticed that the category $\mathbf{G}\mathbb{C}$ obtained by the following pullback is a $*$-autonomous category.

$$\begin{array}{ccc} \mathbf{G}\mathbb{C} & \longrightarrow & \widetilde{\mathbb{C}^{\mathrm{op}}}^{\mathrm{op}} \\ \downarrow & & \downarrow {\scriptstyle p_2^{\mathrm{op}}} \\ \widetilde{\mathbb{C}} & \xrightarrow[p_1]{} & \mathbb{C} \end{array}$$

Explicitly, $\mathbf{G}\mathbb{C}$'s object is a triple $A = (|A| \in \mathbb{C}, A_s \subseteq \mathbb{C}(I, |A|), A_t \subseteq \mathbb{C}(|A|, I))$ and an arrow $f : A \to B$ in $\mathbf{G}\mathbb{C}$ is an arrow $f : |A| \to |B|$ in $\mathbb{C}$ satisfying $f \circ a \in B_s$ for $a \in A_s$ and also $b \circ f \in A_t$ for $b \in B_t$ (this generalizes Loader's "linear logical predicates" [18]). The duality between $\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{C}^{\mathrm{op}}}^{\mathrm{op}}$ induces a duality on $\mathbf{G}\mathbb{C}$ which determines a $*$-autonomous structure. Tan calls this construction a *double glueing*, from which she has obtained various full completeness results for multiplicative linear logic (MLL). $\square$

In fact it makes sense to replace the subscones in double glueing by $\mathbb{C}_0\,\mathsf{PRED}$ for some suitably chosen symmetric monoidal category $\mathbb{C}_0$. Using this we can derive a notion of logical predicates for MLL and, for example, can show that MILL is a full fragment of MLL. See [13] for an exposition.

References

1. Abramsky, S., Gay, S.J., and Nagarajan, R. (1996), Specification structures and propositions-as-types for concurrency, *in* "Proceedings, 8th Banff Higher Order Workshop", Springer LNCS 1043, pp. 5–40.
2. Alimohamed, M. (1995), A characterization of lambda definability in categorical models of implicit polymorphism, *Theoret. Comp. Sci.* **146**, 5–23.
3. Ambler, S.J. (1992), "First Order Linear Logic in Symmetric Monoidal Closed Categories", Ph.D. thesis, ECS-LFCS-92-194, University of Edinburgh.
4. Barber, A., Gardner, P., Hasegawa, M., and Plotkin, G. (1998), From action calculi to linear logic, *in* "Computer Science Logic (CSL'97), Selected Papers", Springer LNCS 1414, pp. 78–97.
5. Barber, A., and Plotkin, G. (1997), Dual intuitionistic linear logic, submitted.
6. Benton, N. (1995), A mixed linear non-linear logic: proofs, terms and models, *in* "Computer Science Logic (CSL'94), Selected Papers", Springer LNCS 933, pp. 121–135.
7. Benton, N., Bierman, G.M., de Paiva, V., and Hyland, J.M.E. (1993), Linear lambda-calculus and categorical models revisited, *in* "Computer Science Logic (CSL'92), Selected Papers", Springer LNCS 702, pp. 61–84.
8. Bierman, G.M. (1995), What is a categorical model of intuitionistic linear logic? *in* "Proceedings, Typed Lambda Calculi and Applications (TLCA'95)", Springer LNCS 902, pp. 78–93.
9. Blute, R.F., Scott, P.J. (1996), Linear Läuchli semantics, *Ann. Pure Appl. Logic* **77**, 229–296.
10. Eilenberg, S., and Kelly, G.M. (1966), Closed categories, *in* "Proceedings, Categorical Algebra (La Jolla 1965)", pp. 421-562, Springer-Verlag.
11. Garrigue, J., and Minamide, Y. (1998), On the runtime complexity of type-directed unboxing, *in* "Proceedings, International Conference on Functional Programming (ICFP'98)", pp. 1–12, ACM Press.
12. Girard, J.-Y. (1987), Linear logic, *Theoret. Comp. Sci.* **50**, 1–102.
13. Hasegawa, M. (1999), Categorical glueing and logical predicates for models of linear logic, technical report, RIMS, Kyoto University.
14. Hermida, C. (1994), "Fibrations, Logical Predicates and Indeterminates", Ph.D. thesis, ECS-LFCS-93-277, University of Edinburgh.
15. Im, G.B., and Kelly, G.M. (1986), A universal property of the convolution monoidal structure, *J. Pure Appl. Algebra* **43**, 75–88.
16. Lafont, Y. (1988), "Logiques, Catégories et Machines", Thèse de Doctorat, Université Paris VII.
17. Lambek, J. (1995), Bilinear logic in algebra and linguistics, *in* "Advances in Linear Logic", pp. 43–59, Cambridge University Press.
18. Loader, R. (1994), "Models of Lambda Calculi and Linear Logic: Structural, Equational and Proof-Theoretic Characterisations", Ph.D. thesis, Oxford University.
19. Mac Lane, S. (1971), "Categories for the Working Mathematician", Graduate Texts in Mathematics 5, Springer-Verlag.
20. Milner, R. (1996), Calculi for interaction, *Acta Inform.* **33**(8), 707–737.
21. Mitchell, J.C., and Scedrov, A. (1992), Notes on sconing and relators, *in* "Computer Science Logic (CSL'92), Selected Papers", Springer LNCS 702, pp. 352–378.
22. Power, A.J. (1996), Elementary control structures, *in* "Proceedings, Concurrency Theory (CONCUR'96)", Springer LNCS 1119, pp. 115–130.
23. Tan, A.M. (1997), "Full Completeness for Models of Linear Logic", Ph.D. thesis, University of Cambridge.