On Traced Monoidal Closed Categories

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The structure theorem of Joyal, Street and Verity says that every traced monoidal category $\mathcal{C}$ arises as a monoidal full subcategory of the tortile monoidal category $\mathbf{Int}\,\mathcal{C}$. In this paper we focus on a simple observation that a traced monoidal category $\mathcal{C}$ is *closed* if and only if the canonical inclusion from $\mathcal{C}$ into $\mathbf{Int}\,\mathcal{C}$ has a right adjoint. Thus, every traced monoidal closed category arises as a monoidal co-reflexive full subcategory of a tortile monoidal category. From this, we derive a series of facts on traced models of linear logic, and some on models of fixed-point computation.

1. Introduction

In (Joyal *et al.* 1996), Joyal, Street and Verity introduced the notion of traced monoidal categories. They showed that every traced monoidal category $\mathcal{C}$ fully faithfully embeds in a tortile monoidal category $\mathbf{Int}\,\mathcal{C}$, and that this Int-construction gives a left biadjoint of the forgetful 2-functor from the 2-category of tortile monoidal categories to that of traced monoidal categories. This remarkable result attracted much attention from theoretical computer scientists, particularly in connection with linear logic (Girard 1987) and Geometry of Interaction (Girard 1989; Abramsky and Jagadeesan 1994; Abramsky 1996; Haghverdi 2000; Abramsky *et al.*; Haghverdi and Scott 2006), where the special case of traced symmetric monoidal categories and compact closed categories (Kelly and Laplaza 1980) is of interest.

In this paper we shall see that the monoidal closed structure can be tied with the Int-construction in an unexpected manner. Namely, we show the following result (Theorem 4.1):

Theorem. A traced monoidal category $\mathcal{C}$ is closed if and only if the embedding from $\mathcal{C}$ into $\mathbf{Int}\,\mathcal{C}$ has a right adjoint.

Despite its simplicity, to the best of our knowledge, this fact has not been pointed out in the literature. Perhaps this is partly because the Int-construction works too nicely: tortile monoidal categories are closed, therefore every traced monoidal category embeds into a closed one just via the Int-construction. So it seems that for this reason we did not feel that traced monoidal closed categories were themselves interesting (with (Coccia *et al.* 2002) as an exception where traced monoidal closed categories with extra structure

are used for modelling higher-order cyclic shared structures). However, tortile structure (or compact closed structure in the symmetric case) describes just a very special kind of closedness, and not every traced monoidal closed category is tortile. Our result says that the relation between tortile structure and general closed structure is still in harmony: every traced monoidal closed category arises as a monoidal co-reflexive full subcategory of a tortile monoidal category. The embedding from $\mathcal{C}$ to **Int** $\mathcal{C}$ may not preserve the closed structure, but the closed structure on $\mathcal{C}$ is determined by the closed structure of **Int** $\mathcal{C}$ and the co-reflection.

The author noticed this result in July 2005, after learning from Paul-André Melliès about the traced monoidal closed category of negative Conway games (Melliès 2004). It took the author a few months to realize that it did have several applications on models of linear logic and also models of fixed-point computation. After that, Shin-ya Katsumata and Susumu Nishimura discovered a concrete example of this in the study of program transformations in 2006 (Katsumata and Nishimura 2006), and then Katsumata gave a striking application in the theory of attribute grammars in 2007 (Katsumata 2007). These discoveries prompted the author to write down the result so that it is hopefully accessible to theoretical computer scientists with mild background in category theory.

To this end, instead of just stating and proving the main result (which would need only a few pages, see Section 4), we shall include all needed definitions and results (but without proofs) on traced monoidal categories, Int-construction, monoidal closed categories and monoidal adjunctions in Section 2 and 3, making the paper largely self-contained. Section 5 is devoted for applications in models of linear logic and fixed-point computation; for the latter the author's old work on recursion created from cyclic sharing is recast as linear fixed-point operators in traced models of intuitionistic linear logic. In addition, we shall make use of the graphical presentation for monoidal categories and monoidal functors (Joyal and Street 1991; Cockett and Seely 1999; Melliès 2006) (which goes back to Penrose's diagrams for calculating with tensors (Penrose and Rindler 1984)), which is intuitively helpful and technically convenient.

While most work in computer science focus on the symmetric case (traced symmetric monoidal categories and compact closed categories), in this paper, whenever possible, we consider general traced balanced (braided) monoidal categories and tortile monoidal categories, following the original development by Joyal, Street and Verity; most of our results are valid for this generality — and also we expect (rather optimistically) that non-symmetric situations will become useful in future developments in computer science.

2. Preliminaries

2.1. *Monoidal Categories*

A *monoidal category* (Mac Lane 1971) (or *tensor category* (Joyal and Street 1993)) $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $l_A : I \otimes A \xrightarrow{\sim} A$ and

$r_A : A \otimes I \xrightarrow{\sim} A$ such that the following two diagrams commute for all A, B, C and D:

$$
\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\;\;a\;\;} & (A \otimes B) \otimes (C \otimes D) \\
\downarrow{\scriptstyle a \otimes D} & & \downarrow{\scriptstyle a} \\
(A \otimes (B \otimes C)) \otimes D \xrightarrow{\;a\;} A \otimes ((B \otimes C) \otimes D) \xrightarrow{\;A \otimes a\;} & & A \otimes (B \otimes (C \otimes D))
\end{array}
$$

$$
\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\;\;a\;\;} & A \otimes (I \otimes B) \\
& {\scriptstyle r \otimes B} \searrow \quad \swarrow {\scriptstyle A \otimes l} & \\
& A \otimes B &
\end{array}
$$

Mac Lane's coherence theorem (Mac Lane 1971; Joyal and Street 1993) states that all diagrams built from a, l and r commute. It follows that, in principle, most results stated for the *strict* monoidal categories (where a, l, r are the identity arrows and $(A \otimes B) \otimes C$ is identified with $A \otimes (B \otimes C)$, and similarly for tensor unit) can be reformulated and proved without strictness. In many places of this paper, for ease of presentation, we avoid putting brackets on tensor products when rigour is guaranteed by the coherence theorem.

A *braiding* is a natural isomorphism $c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ such that both c and c^{-1} satisfy the following "bilinearity" diagrams (the case for c^{-1} is omitted):

$$
\begin{array}{ccc}
(A \otimes B) \otimes C \xrightarrow{\;a\;} A \otimes (B \otimes C) \xrightarrow{\;c\;} (B \otimes C) \otimes A \\
\downarrow{\scriptstyle c \otimes C} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle a} \\
(B \otimes A) \otimes C \xrightarrow{\;a\;} B \otimes (A \otimes C) \xrightarrow[\;B \otimes c\;]{} B \otimes (C \otimes A)
\end{array}
$$

A *symmetry* is a braiding such that $c_{A,B} = c_{B,A}^{-1}$. A *braided/symmetric monoidal category* is a monoidal category equipped with a braiding/symmetry.

A *twist* or a *balance* for a braided monoidal category is a natural isomorphism $\theta_A : A \xrightarrow{\sim} A$ such that $\theta_I = id_I$ and $\theta_{A \otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}$ hold. A *balanced monoidal category* is a braided monoidal category with a twist. Note that a symmetric monoidal category is a balanced monoidal category with $\theta_A = id_A$ for every A.

2.2. *Monoidal Functors and Natural Transformations*

For monoidal categories $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ and $\mathcal{C}' = (\mathcal{C}', \otimes', I', a', l', r')$, a *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a tuple (F, m, m_I) where F is a functor from $\mathcal{C}$ to $\mathcal{C}'$, m is a natural transformation from $F(-) \otimes' F(=)$ to $F(- \otimes =)$ and $m_I : I' \to FI$ is an arrow

in $\mathcal{C}'$, satisfying the coherence conditions below.

$$
\begin{array}{ccccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{m \otimes' FC} & F(A \otimes B) \otimes' FC & \xrightarrow{m} & F((A \otimes B) \otimes C) \\
\downarrow{\scriptstyle a'} & & & & \downarrow{\scriptstyle Fa} \\
FA \otimes' (FB \otimes' FC) & \xrightarrow[FA \otimes' m]{} & FA \otimes' F(B \otimes C) & \xrightarrow[m]{} & F(A \otimes (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
I' \otimes' FA & \xrightarrow{l'} & FA \\
\downarrow{\scriptstyle m_I \otimes FA} & & \uparrow{\scriptstyle Fl} \\
FI \otimes' FA & \xrightarrow[m]{} & F(I \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
FA \otimes' I' & \xrightarrow{r'} & FA \\
\downarrow{\scriptstyle FA \otimes m_I} & & \uparrow{\scriptstyle Fr} \\
FA \otimes' FI & \xrightarrow[m]{} & F(A \otimes I)
\end{array}
$$

It is called *strong* if $m_{A,B}$ and m_I are all isomorphisms.

A *balanced monoidal functor* from a balanced $\mathcal{C}$ to another $\mathcal{C}'$ is a monoidal functor (F, m, m_I) which additionally satisfies the following condition

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{c'} & FB \otimes' FA \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
F(A \otimes B) & \xrightarrow[Fc]{} & F(B \otimes A)
\end{array}
$$

and $F(\theta_A) = \theta_{FA}$.

For monoidal functors (F, m, m_I), (G, n, n_I) with the same source and target monoidal categories, a *monoidal natural transformation* from (F, m, m_I) to (G, n, n_I) is a natural transformation $\varphi : F \to G$ such that the following diagrams commute:

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{m} & F(A \otimes B) \\
\downarrow{\scriptstyle \varphi \otimes' \varphi} & & \downarrow{\scriptstyle \varphi} \\
GA \otimes' GB & \xrightarrow[n]{} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
 & I' & \\
{\scriptstyle m_I}\swarrow & & \searrow{\scriptstyle n_I} \\
FI & \xrightarrow[\varphi]{} & GI
\end{array}
$$

A *(balanced/symmetric) monoidal adjunction* between (balanced/symmetric) monoidal categories is an adjunction in which both of the functors are (balanced/symmetric) monoidal and the unit and counit are monoidal natural transformations.

2.3. *Geometry of Monoidal Categories*

In this paper we make use of the graphical language for monoidal categories, known as string diagrams or Penrose diagrams, developped by Joyal and Street (Joyal and Street 1991). A morphism $f : A_1 \otimes A_2 \otimes \ldots \otimes A_m \to B_1 \otimes B_2 \otimes \ldots \otimes B_n$ in a monoidal category can be drawn as (from left to right)
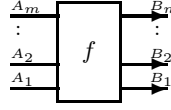
Morphisms can be composed, either sequentially —

$$f : X \to Y, \ g : Y \to Z \qquad\qquad g \circ f : X \to Z$$

or in parallel:

$$f : A \to B, \ g : C \to D \qquad f \otimes g : A \otimes C \to B \otimes D$$

Braids are expressed by crossing: $c = $, $c^{-1} = $.   Twists are drawn as

$\theta = $ and $\theta^{-1} = $.

Remark 2.1. Links in these pictures should be regarded as "ribbons" or "framed tangles", as stressed in (Joyal and Street 1991; Joyal and Street 1993; Shum 1994). Our notation for twists is intended to be a reasonable alternative for the ribbon twisting notation used in the literature.

Monoidal functors and monoidal natural transformations also allow concise graphical presentations, as demonstrated by Cockett and Seely (Cockett and Seely 1999) and Melliès (Melliès 2006). Consider a monoidal functor $F = (F, m, m_I) : \mathcal{C} \to \mathcal{D}$. Given $f : A \otimes B \to C$ in $\mathcal{C}$, we may draw a picture with "box" (here with the dark shadow, should not be confused with the square around f)
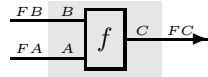
which represents $FA \otimes FB \overset{m_{A,B}}{\to} F(A \otimes B) \overset{Ff}{\to} FC$. The dark area is in $\mathcal{C}$, where as the white area in $\mathcal{D}$. Similarly, given $a : I \to A$, the picture
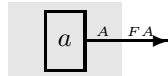
represents $I \overset{m_I}{\to} FI \overset{Fa}{\to} FA$. The three coherence conditions of monoidal functors ensure that this notation works well for general $f : A_1 \otimes \ldots \otimes A_n \to B$, where the grouping of A_i's does not matter. In addition, if F is strong monoidal, we can do the same for tensors in the codomain. It is a pleasant exercise to write down the coherence conditions for braids and monoidal natural transformations using this box notation. For instance, one of the diagrams for monoidal natural transformations can be shown as follows.

$$=$$

See (Cockett and Seely 1999) and (Melliès 2006) for further details and examples.

3. Traced Monoidal Categories

3.1. *Traced Monoidal Categories*

We present a slightly simplified definition of traced monoidal categories, where trace is defined in an object-wise manner. Such a theory of object-wise trace has been developped by Blute, Cockett and Seely for linearly distributive categories (Blute *et al.* 2000); Milner also gave a similar axiomatization for his reflexive action calculi (Milner 1994).
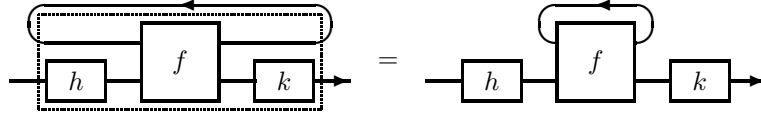
A *traced monoidal category* (Joyal *et al.* 1996) is a balanced monoidal category $\mathcal{C}$ equipped with a family of functions, called *trace* operator

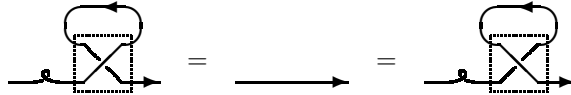$$Tr_{A,B}^X : \mathcal{C}(A \otimes X, B \otimes X) \longrightarrow \mathcal{C}(A,B)$$
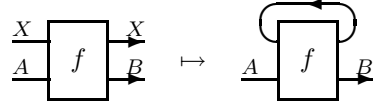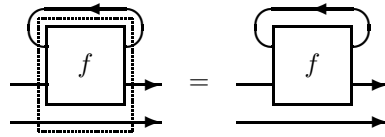

subject to the following four coherence axioms.

Tightening (Naturality) $Tr_{A',B'}^X((k \otimes id_X) \circ f \circ (h \otimes id_X)) = k \circ Tr_{A,B}^X(f) \circ h$

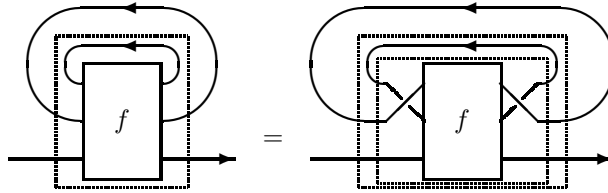*Yanking* $Tr_{X,X}^X(c_{X,X}) \circ \theta_X^{-1} = id_X = Tr_{X,X}^X(c_{X,X}^{-1}) \circ \theta_X$

Superposing $Tr_{C \otimes A, C \otimes B}^X(id_C \otimes f) = id_C \otimes Tr_{A,B}^X(f)$

Exchange
$Tr_{A,B}^X(Tr_{A \otimes X, B \otimes X}^Y(f)) = Tr_{A,B}^Y(Tr_{A \otimes Y, B \otimes Y}^X((id_B \otimes c_{Y,X}) \circ f \circ (id_A \otimes c_{Y,X}^{-1})))$

Readers familiar with the paper by Joyal, Street and Verity (Joyal *et al.* 1996) should find no difficulty in seeing that these are all derivable from the original axiomatization. Conversely, original axioms are derivable from our axioms; we shall give a slightly non-trivial derivation of the *Sliding* and *Vanishing for tensor* in an appendix. The remaining *Vanishing for unit* is in fact redundant in the original axiomatisation (as demonstrated in an appendix), so our axioms are equivalent to the original axioms in *ibid.*
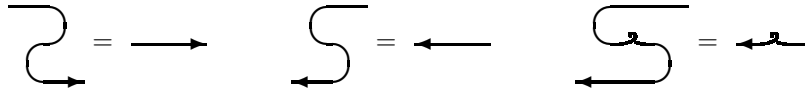
3.2. *Tortile Monoidal Categories*

A *tortile monoidal category* (Shum 1994) (also *ribbon category* (Yetter 2001)) is a balanced monoidal category with an object A^* for each object A, unit $\eta_A : I \to A \otimes A^*$ and counit $\varepsilon_A : A^* \otimes A \to I$ such that each of

$$A \overset{l_A^{-1}}{\to} I \otimes A \overset{\eta_A \otimes id_A}{\to} (A \otimes A^*) \otimes A \overset{a_{A,A^*,A}}{\to} A \otimes (A^* \otimes A) \overset{id_A \otimes \varepsilon_A}{\to} A \otimes I \overset{r_A}{\to} A$$

$$A^* \overset{r_{A^*}^{-1}}{\to} A^* \otimes I \overset{id_{A^*} \otimes \eta_A}{\to} A^* \otimes (A \otimes A^*) \overset{a_{A^*,A,A^*}^{-1}}{\to} (A^* \otimes A) \otimes A^* \overset{\varepsilon_A \otimes id_{A^*}}{\to} I \otimes A^* \overset{l_{A^*}}{\to} A^*$$

is the identity and moreover $\theta_A^* = \theta_{A^*}$ holds, where, for $f : A \to B$, $f^* : B^* \to A^*$ is given by (omitting l, r and a)

$$B^* \overset{id_{B^*} \otimes \eta_A}{\to} B^* \otimes A \otimes A^* \overset{id_{B^*} \otimes f \otimes id_{A^*}}{\to} B^* \otimes B \otimes A^* \overset{\varepsilon_B \otimes id_{A^*}}{\to} A^*.$$

It follows that $(-)^*$ extends to a contravariant equivalence, $A^{**} \simeq A$, and the functor $(-) \otimes A$ is left adjoint to $(-) \otimes A^*$ with unit $id_X \otimes \eta_A : X \to X \otimes A \otimes A^*$ and counit $id_X \otimes \varepsilon_A : X \otimes A^* \otimes A \to X$. Note that tortile symmetric monoidal categories (in which braiding is a symmetry and twist is the identity) are familiar compact closed categories (Kelly and Laplaza 1980). The unit and counit in tortile monoidal categories can be drawn as and respectively. With them, three axioms are expressed as follows.


The importance of tortile monoidal categories in knot theory comes from the following result:

Theorem 3.1. (Shum 1994) The tortile monoidal category freely generated by a single object is equivalent to the category of framed tangles.

Therefore, tortile monoidal categories give rise to invariants (or models) for tangles, just in the same sense that cartesian closed categories give rise to models of the simply typed lambda calculi (Lambek and Scott 1986).

Any tortile monoidal category has a unique trace (called *canonical trace* (Joyal *et al.* 1996)), hence is also a traced monoidal category. The canonical trace is given by combining η, ε, c and θ:

$$Tr_{A,B}^X f = (id_B \otimes (\varepsilon_X \circ (id_{X^*} \otimes \theta_X) \circ c_{X,X^*})) \circ (f \otimes id_{X^*}) \circ (id_A \otimes \eta_X)$$
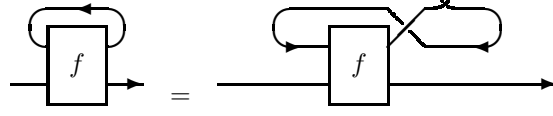
It follows that a monoidal full subcategory of a tortile monoidal category is traced.

3.3. *The Int-Construction*

In fact, every traced monoidal category arises in this way: given a traced monoidal category $\mathcal{C}$, we can construct a tortile monoidal category $\mathbf{Int}\,\mathcal{C}$ to which $\mathcal{C}$ fully faithfully embeds, via the Int-construction of Joyal, Street and Verity — an abstract version of "Geometry of Interaction" of Girard/Abramsky. Below we briefly recall its ingredients.

Objects of $\mathbf{Int}\,\mathcal{C}$ are pairs of objects of $\mathcal{C}$. An arrow from (A^+, A^-) to (B^+, B^-) in $\mathbf{Int}\,\mathcal{C}$ is an arrow from $A^+ \otimes B^-$ to $B^+ \otimes A^-$ in $\mathcal{C}$. The identity on (A^+, A^-) is $id_{A^+} \otimes \theta_{A^-}^{-1} \in \mathcal{C}(A^+ \otimes A^-, A^+ \otimes A^-)$. The composition of $f \in \mathbf{Int}\,\mathcal{C}((A^+, A^-), (B^+, B^-)) = \mathcal{C}(A^+ \otimes B^-, B^+ \otimes A^-)$ and $g \in \mathbf{Int}\,\mathcal{C}((B^+, B^-), (C^+, C^-)) = \mathcal{C}(B^+ \otimes C^-, C^+ \otimes B^-)$ is given by
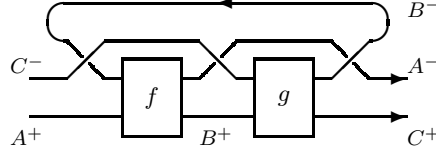
Next we look at the monoidal structure. On objects, we define tensor and unit by $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, B^- \otimes A^-)$ and $I = (I, I)$. On arrows, for $f_1 : (A_1^+, A_1^-) \to (B_1^+, B_1^-)$ and $f_2 : (A_2^+, A_2^-) \to (B_2^+, B_2^-)$, define $f_1 \otimes f_2$ by
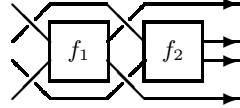
Braids and twists in $\mathbf{Int}\,\mathcal{C}$ are not quite obvious:

$$c = \qquad\qquad \theta =$$
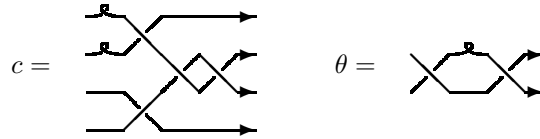
(It is an interesting exercise to write down c^{-1} and θ^{-1} explicitly.)

Finally we describe the duality, which is not so hard: $(A^+, A^-)^* = (A^-, A^+)$. The unit $\eta_{(A^+, A^-)} : I \to (A^+, A^-) \otimes (A^+, A^-)^*$ is given by $id_{A^+} \otimes \theta_{A^-}^{-1}$. The counit $\varepsilon_{(A^+, A^-)} : (A^+, A^-)^* \otimes (A^+, A^-) \to I$ is $id_{A^-} \otimes \theta_{A^+}$. We can extend $(-)^*$ to be a contravariant equivalence on $\mathbf{Int}\,\mathcal{C}$ — on arrows $f : (A^+, A^-) \to (B^+, B^-)$ define $f^* : (B^+, B^-)^* \to (A^+, A^-)^*$ as

Theorem 3.2. (Joyal *et al.* 1996) These data determine a tortile monoidal structure on

Int $\mathcal{C}$. Moreover, the functor $\mathcal{N} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ sending A to (A, I) strongly preserves the traced monoidal structure, and is full faithful.

Explicitly, the canonical trace on **Int** $\mathcal{C}$ can be given as follows. (It is not entirely obvious for the non-symmetric case.) For $f : (A^+, A^-) \otimes (X^+, X^-) \to (B^+, B^-) \otimes (X^+, X^-)$, its trace $Tr^{(X^+, X^-)} f : (A^+, A^-) \to (B^+, B^-)$ is
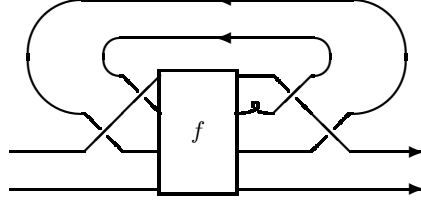
It easily follows that $\mathcal{N}$ preserves trace up to canonical isomorphisms.

In fact, Int-construction is *universal*, as shown in *ibid.*: it gives a left biadjoint to the forgetful 2-functor from the 2-category of tortile monoidal categories to that of traced monoidal categories.

3.4. *Trace-Fixpoint Correspondence*

The following correspondence between traces and fixed-point operators on categories with finite products, noticed by Martin Hyland and the author independently[†], shows that traces and fixed-point operators commonly used in computer science are very closely related.

Let $\mathcal{C}$ be a category with finite products. A parameterized fixed-point operator on $\mathcal{C}$ is a family of functions

$$(-)^\dagger : \mathcal{C}(A \times X, X) \to \mathcal{C}(A, X)$$

which is natural in A and satisfies the *fixed-point equation* $f^\dagger = f \circ \langle id_A, f^\dagger \rangle : A \to X$ for any $f : A \times X \to X$. It is called a *Conway fixed-point operator* if it satisfies the dinaturality

$$(f \circ \langle \pi_{A,X}, g \rangle)^\dagger = f \circ \langle id_A, (g \circ \langle \pi_{A,Y}, f \rangle)^\dagger \rangle : A \to X$$

for $f : A \times Y \to X$ and $g : A \times X \to Y$, and the diagonal property

$$(f \circ (id_A \times \Delta_X))^\dagger = (f^\dagger)^\dagger : A \to X$$

for $f : A \times X \times X \to X$, where $\Delta_X : X \to X \times X$ is the diagonal map.

Theorem 3.3. (Hasegawa 1997; Hasegawa 1999) For any category with finite products, to give a Conway operator is to give a trace (where finite products are taken as the monoidal structure).


[†] We should note that mathematically equivalent observations were made by several authors before the notion of traced monoidal categories was introduced, in particular by Bloom and Ésik (Bloom and Ésik 1993) and Ştefănescu (Ştefănescu 2000).

Here we shall just recall the constructions of this bijective correspondence:

$$\frac{f : A \times X \to X}{f^\dagger = Tr^X_{A,X}(\Delta_X \circ f) : A \to X}$$

$$\frac{g : A \times X \to B \times X}{Tr^X_{A,B}(g) = \pi_{B,X} \circ (g \circ (id_A \times \pi'_{B,X}))^\dagger : A \to B}$$

The construction of Conway operator from a trace can be drawn as follows.
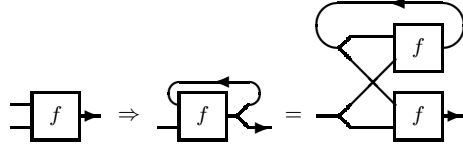
Thus many categories in denotational/algebraic semantics are traced. For example, the category **Cppo** of pointed cpo's and continuous functions is traced, where trace determined by the least fixed-points.

4. Traced Monoidal Closed Categories

So far we have not thought much about *closed structure*, or higher-types. Recall that a monoidal category $\mathcal{C}$ is *closed* if $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $A \multimap -$:

$$\mathcal{C}(X \otimes A, Y) \simeq \mathcal{C}(X, A \multimap Y)$$

In particular, tortile monoidal categories are closed, with $A \multimap B = B \otimes A^*$. We will denote (the Y-component of) the counit of this adjunction by $\mathsf{ev}_{A,Y} : (A \multimap Y) \otimes A \longrightarrow Y$, and for $f : X \otimes A \longrightarrow Y$ let $\Lambda(f) : X \longrightarrow A \multimap Y$ be the unique arrow satisfying $\mathsf{ev}_{A,Y} \circ (\Lambda(f) \otimes id_A) = f$.

In the context of linear logic (Girard 1987), being symmetric monoidal closed means that we can interpret the intuitionistic multiplicative fragment (tensor $\otimes$, unit $\top$, and linear implication $\multimap$) in $\mathcal{C}$. In the rest of this paper, we will see that, for a traced monoidal category, closedness has yet another reading in terms of the Int-construction, which in turn is also related to the modality ! and linear decomposition $A \to B = !A \multimap B$ in linear logic.

4.1. *Monoidal Closed Categories and Co-reflection*

It is known that a monoidal co-reflective full subcategory of a monoidal closed category is also closed (although the closed structure may not be preserved by the inclusion):

Lemma 4.1. (folklore[‡]). Let $\mathcal{C} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{D}$ be a monoidal adjunction. If $\mathcal{F}$ is full faithful and $\mathcal{D}$ is closed, then $\mathcal{C}$ is also closed, with $A \multimap_{\mathcal{C}} B = \mathcal{U}(\mathcal{F}A \multimap_{\mathcal{D}} \mathcal{F}B)$.

[‡] Martin Hyland (private communication) attributes this result to Brian Day.

Proof.

$$
\begin{aligned}
\mathcal{C}(C \otimes A, B) & \simeq & \mathcal{D}(\mathcal{F}(C \otimes A), \mathcal{F}B) && \text{$\mathcal{F}$ is full faithful} \\
& \simeq & \mathcal{D}(\mathcal{F}C \otimes \mathcal{F}A, \mathcal{F}B) && \text{$\mathcal{F}$ is strong monoidal} \\
& \simeq & \mathcal{D}(\mathcal{F}C, \mathcal{F}A \multimap \mathcal{F}B) && \text{$\mathcal{D}$ is closed} \\
& \simeq & \mathcal{C}(C, \mathcal{U}(\mathcal{F}A \multimap \mathcal{F}B)) && \text{$\mathcal{F} \dashv \mathcal{U}$}
\end{aligned}
$$

Here we appeal to the fact that an adjunction $\mathcal{F} \dashv \mathcal{U}$ is a monoidal adjunction if and only if $\mathcal{F}$ is strong monoidal (Kelly 1974). □

4.2. *Main Observation*

Below we present a variation for traced monoidal categories. It characterizes closedness in terms of an adjunction associated to the Int-construction.

Theorem 4.1. (main observation) Let $\mathcal{C}$ be a traced monoidal category, and $\mathcal{N} : \mathcal{C} \to \textbf{Int}\,\mathcal{C}$ be the canonical inclusion from $\mathcal{C}$ into $\textbf{Int}\,\mathcal{C}$ (i.e. $\mathcal{N}(A) = (A, I)$). Then $\mathcal{C}$ is closed if and only if $\mathcal{N}$ has a right adjoint.

Corollary 4.1. Every traced monoidal closed category is equivalent to a monoidal co-reflexive full subcategory of a tortile monoidal category.

Proof of Theorem 4.1. "if" follows from the previous folklore lemma, as $\mathcal{N}$ is full faithful and strong symmetric monoidal. Note that, by the lemma,

$$
A \multimap_{\mathcal{C}} B \simeq \mathcal{U}(\mathcal{N}A \multimap_{\textbf{Int}\,\mathcal{C}} \mathcal{N}B) = \mathcal{U}((A, I)^* \otimes (B, I)) \simeq \mathcal{U}(B, A)
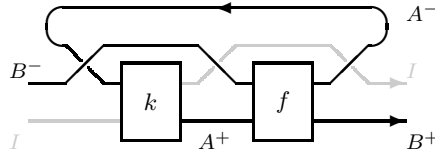$$

where $\mathcal{U}$ is right adjoint to $\mathcal{N}$. This suggests how we proceed to show the "only if" part. That is, if $\mathcal{C}$ is closed, define $\mathcal{U}(A^+, A^-) = A^- \multimap A^+$. For $f : (A^+, A^-) \to (B^+, B^-)$, let $\mathcal{U}(f) : (A^- \multimap A^+) \to (B^- \multimap B^+)$ be

$$
\Lambda(Tr^{A^-}_{(A^- \multimap A^+) \otimes B^-, B^+}(f \circ (\mathsf{ev}_{A^-, A^+} \otimes id_{B^-}) \circ (id_{A^- \multimap A^+} \otimes c_{B^-, A^-})))
$$

or, more internally, a map sending $k : A^- \to A^+$ to $Tr^{A^-}_{B^-, B^+}(f \circ (k \otimes B^-) \circ c_{B^-, A^-}) : B^- \to B^+$, see the picture below.
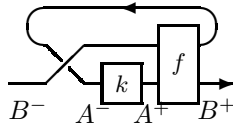
In other words, $\mathcal{U}(f)$ describes "composition with f in $\textbf{Int}\,\mathcal{C}$" in terms of $\mathcal{C}$.

From this, it is immediate to see that $\mathcal{U}$ is indeed a functor.

Finally, it is easy to see the adjointness:

$$
\begin{aligned}
\mathbf{Int}\,\mathcal{C}(\mathcal{N}(A), (B^+, B^-)) &= \mathbf{Int}\,\mathcal{C}((A, I), (B^+, B^-)) \\
&= \mathcal{C}(A \otimes B^-, B^+ \otimes I) \\
&\simeq \mathcal{C}(A,\ B^- \multimap B^+) \\
&= \mathcal{C}(A,\ \mathcal{U}(B^+, B^-)).
\end{aligned}
$$

The (A^+, A^-)-component of the counit of the adjunction is given by $\mathrm{ev}_{A^-, A^+} : (A^- \multimap A^+) \otimes A^- \to A^+$, while the unit is trivial. $\qquad\square$

Note that the adjunction in the theorem above gives rise to an idempotent balanced monoidal comonad $\mathcal{N}\mathcal{U}$ on $\mathbf{Int}\,\mathcal{C}$ which sends (A^+, A^-) to $(A^- \multimap A^+, I)$. Recall that, for an idempotent comonad K on a category $\mathcal{C}$, its co-Kleisli category $\mathcal{C}_K$ and co-Eilenberg-Moore category $\mathcal{C}^K$ are both equivalent to the co-reflexive full subcategory of $\mathcal{C}$ whose objects are in the image of K. Thus we have:

Corollary 4.2. For any traced monoidal closed category $\mathcal{C}$, there is an idempotent balanced monoidal comonad on $\mathbf{Int}\,\mathcal{C}$ such that its co-Kleisli category is equivalent to $\mathcal{C}$.

Thus all traced monoical closed categories come from tortile monoidal categories with an idempotent balanced monoidal comonad. The converse is not true, however: there is a tortile monoidal category $\mathcal{Y}$ with an idempotent balanced monoidal comonad ! such that $\mathbf{Int}\,(\mathcal{Y}_!)$ is not equivalent to $\mathcal{Y}$ — see Section 6.2.

Remark 4.1. It might be the case that Int-construction and co-Kleisli construction give rise to a biadjunction between the 2-category of traced monoidal closed categories and a suitable 2-category of tortile monoidal categories with idempotent balanced monoidal comonad, but for now we do not know the answer.

5. Applications

5.1. *Models of Linear Logic*

We have already noted that symmetric monoidal closed categories are the models of multiplicative fragment of intuitionistic linear logic (IMLL). Here we shall quickly recall additional structures needed for modelling other elements of linear logic (Seely 1989; Barr 1991; Benton 1995; Bierman 1995; Barber and Plotkin 1997; Hyland and Schalk 2003; Melliès 2003).

A symmetric monoidal closed category with an object $\bot$ so that the canonical map from A to $(A \multimap \bot) \multimap \bot$ is invertible for all objects A is called a $*$-*autonomous category* (Barr 1991). $*$-autonomous categories are the models of multiplicative fragment of classical linear logic (MLL), where $\bot$ (called a dualizing object) models the falsity and $A \multimap \bot$ the linear negation of A. Compact closed categories (= tortile symmetric monoidal categories) are special instances of $*$-autonomous categories, so they also are models of MLL.

To interpret additive conjunctions and disjunctions of linear logic, it suffices just to assume finite products and finite coproducts. However, they are not particularly impor-

tant in this paper; we do not know useful conditions to ensure the existence of finite (bi)products[§] on **Int** $\mathcal{C}$, so for now additives do not have good place in our story.

A symmetric monoidal adjunction between a category with finite products and a symmetric monoidal category gives rise to a comonad on the symmetric monoidal category, which models the exponential ! of linear logic (Barber and Plotkin 1997; Bierman 1995). Such a comonad is called a *linear exponential comonad* (Hyland and Schalk 2003). Explicitly, a linear exponential comonad is a symmetric monoidal comonad ! on a symmetric monoidal category $\mathcal{C}$ such that the category of its coalgebras is a category of commutative comonoids, which means

— there are specified monoidal natural transformations $e_A : !A \to I$ and $d_A : !A \to !A \otimes !A$ which form a commutative comonoid $(!A, e_A, d_A)$ in $\mathcal{C}$ and also are coalgebra morphisms from $(!A, \delta_A)$ to (I, m_I) and $(!A \otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$ respectively, and
— any coalgebra morphism from $(!A, \delta_A)$ to $(!B, \delta_B)$ is also a comonoid morphism from $(!A, e_A, d_A)$ to $(!B, e_B, d_B)$.

In summary, a model of multiplicative exponential fragment of intuitionistic linear logic (IMELL) is a symmetric monoidal closed category with a linear exponential comonad. A model of multiplicative exponential fragment of classical linear logic (MELL) is a $*$-autonomous category with a linear exponential comonad.

Returning to our study on traced categories, if $\mathcal{C}$ is a traced cartesian closed category, our main theorem implies that $\mathcal{C} \xrightleftharpoons[\mathcal{U}]{\mathcal{N}} \mathbf{Int}\,\mathcal{C}$ is a symmetric monoidal adjunction, thus $\mathcal{N}\mathcal{U}$ is a linear exponential comonad on **Int** $\mathcal{C}$.

Corollary 5.1. For any traced cartesian closed category $\mathcal{C}$, there is an idempotent linear exponential comonad on **Int** $\mathcal{C}$ such that its co-Kleisli category is equivalent to $\mathcal{C}$.

Explicitly, this comonad sends (A^+, A^-) to $(A^- \Rightarrow A^+, 1)$.

Together with the trace-fixpoint correspondence:

Corollary 5.2. Any cartesian closed category with a Conway fixed-point operator is equivalent to one arising from a compact closed model of MELL via the co-Kleisli construction.

We can go further. Since monoidal adjunctions are closed under composition, Int-construction actually sends a traced model for IMELL to a compact closed model for MELL:

Corollary 5.3. Let $\mathcal{C}$ be a traced symmetric monoidal closed category with a linear exponential comonad ! (i.e. a model of IMELL). Then **Int** $\mathcal{C}$ is equipped with a linear exponential comonad !' given by $!'(A^+, A^-) = \mathcal{N}(!(\mathcal{U}(A^+, A^-))) = (!(A^- \multimap A^+), I)$.

$$! \;\; \mathrel{\reflectbox{$\circlearrowright$}} \;\; \mathcal{C} \xrightleftharpoons[\mathcal{U}]{\mathcal{N}} \mathbf{Int}\,\mathcal{C}$$

This gives an alternative possibility for interpreting exponentials in Geometry of Inter-action — not quite the same as Girard's (Girard 1989) or the treatment by Abramsky, Haghverdi, Scott et al (Abramsky and Jagadeesan 1994; Haghverdi 2000; Abramsky *et al.*; Haghverdi and Scott 2006).

5.2. *Linear Fixed-points: Recursion from Cyclic Sharing Revisited*

In (Hasegawa 1997; Hasegawa 1999), it is shown that[¶]

Theorem 5.1. Given a symmetric monoidal adjunction $\mathcal{C} \xrightarrow[\overset{\perp}{\longleftarrow}]{F} \mathcal{D}$ between a category $\mathcal{C}$ with finite products (taken as the monoidal structure) and a traced symmetric monoidal category $\mathcal{D}$, there exists a family of functions

$$(-)^\dagger : \mathcal{D}(FA \otimes X, X) \longrightarrow \mathcal{D}(FA, X)$$

which is natural in A and dinatural in X.

Explicitly, $f^\dagger$ is given by

$$\varepsilon_X \circ Tr_{FA,FUX}^{FUX}({m_{UX,UX}^F}^{-1} \circ F(\Delta_{UX} \circ Uf \circ m_{FA,X}^U \circ (\eta_A \times id_{UX})) \circ m_{A,UX}^F)$$

where η and ε are the unit and counit of the adjunction (should not be confused with those for tortile monoidal categories). In particular, $f^\dagger$ satisfies the fixed-point equation:

$$f^\dagger = f \circ (id_{FA} \otimes f^\dagger) \circ {m^F}^{-1} \circ F\Delta_A$$

This result has been used for providing semantics of recursion in lambda calculi with cyclic sharing (Hasegawa 1997; Hasegawa 1999).

Using the box notation (Cockett and Seely 1999; Melliès 2006), this $f^\dagger$ can be nicely expressed as follows.
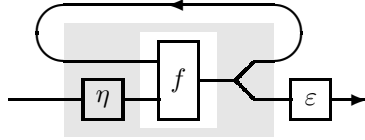
In this picture, the inner box and outer box correspond to functors U and F respectively, and the components in the gray zone belong to $\mathcal{C}$ while those in white to $\mathcal{D}$. Happily, we do not have to be bothered by the coherence morphisms m^F and m^U. In an appendix we give a graphical derivation of the fixed-point equation using the box notation, which replaces the lengthy calculation in (Hasegawa 1999).

If $\mathcal{D}$ is closed, the operator $(-)^\dagger$ can be replaced by a family of arrows of $\mathcal{D}(FU(X \multimap X), X) \simeq \mathcal{D}(I, FU(X \multimap X) \multimap X)$. In terms of the linear lambda calculus corresponding

---

[¶] In (Hasegawa 1997; Hasegawa 1999), only Kleisli adjunctions of commutative monads were considered, and this theorem was stated with an additional condition that F is an identity-on-object, strict monoidal functor. But this restriction is not essential, as we demonstrate here.

to intuitionistic linear logic (Barber and Plotkin 1997), this amounts to a *linear* fixed-point combinator $\mathsf{Y}_X :!(X \multimap X) \multimap X$ such that

$$\mathsf{Y}_X \,(!M) \;=\; M\,(\mathsf{Y}_X \,(!M))$$

holds for any term $M : X \multimap X$ with no free linear variable. Note that this is different from the usual fixed-point combinator with type $!(!X \multimap X) \multimap X$ which returns a fixed point of a non-linear map of type $!X \multimap X$. As demonstrated in (Hasegawa 1997; Hasegawa 1999), linear fixed-point operators can exist even in the settings where such a standard non-linear fixed-point operator is not available.[‖]

Proposition 5.1. Suppose that $\mathcal{D}$ is a traced symmetric monoidal closed category with a linear exponential comonad !. Then there exists a family of arrows $\mathsf{fix}_X :!(X \multimap X) \longrightarrow X$ such that, for any $f :!A \otimes X \longrightarrow X$, $f^\dagger = \mathsf{fix}_X \circ !(\Lambda(f)) \circ \delta_A :!A \rightarrow X$ (with $\Lambda(f) :!A \longrightarrow X \multimap X$ the adjoint mate of f) is a fixed-point of f, that is, $f^\dagger$ agrees with

$$!A \xrightarrow{d} !A \otimes !A \xrightarrow{id_{!A} \otimes f^\dagger} !A \otimes X \xrightarrow{f} X.$$

As we have seen, if $\mathcal{C}$ is a traced symmetric monoidal closed category with a linear exponential comonad ! (traced model of IMELL), then **Int** $\mathcal{C}$ is a compact closed category with a linear exponential comonad $!'(A^+, A^-) = (!(A^- \multimap A^+), I)$ (compact model of MELL). Therefore, both $\mathcal{C}$ and **Int** $\mathcal{C}$ admit interpretations of such linear fixed-points. The linear fixed-point on (X^+, X^-) in **Int** $\mathcal{C}$ is determined by the linear fixed-point on $X^- \multimap X^+$ in $\mathcal{C}$. Spelling out the detail, to give a linear fixed-point $\mathsf{Y}_{(X^+, X^-)} : !'((X^+, X^-) \multimap (X^+, X^-)) \multimap (X^+, X^-)$ in **Int** $\mathcal{C}$ is to give a map of

$$(X^- \otimes !((X^+ \otimes X^-) \multimap (X^+ \otimes X^-))) \multimap X^*$$

in $\mathcal{C}$, which is described as

$$\lambda(y^{X^1} \otimes !f^{((X^+ \otimes X^-) \multimap (X^+ \otimes X^-))}).\mathsf{Y}_{X^- \multimap X^+} \,(!(\mathcal{U}f))\,y.$$

Example 5.1. Let $\mathcal{C}$ be a traced cartesian closed category. Take a morphism $f = \langle f^+, f^- \rangle : (X^+, X^-) \rightarrow (X^+, X^-)$ in **Int** $\mathcal{C}$, thus $f^+ : X^+ \times X^- \rightarrow X^+$ and $f^- : X^+ \times X^- \rightarrow X^-$ in $\mathcal{C}$. Using the simply typed lambda calculus with μ-fixpoint operator as an internal language of $\mathcal{C}$, $\mathcal{U}f : X^{+^{X^-}} \rightarrow X^{+^{X^-}}$ can be described as

$$\lambda k^{X^- \rightarrow X^+}.\lambda y^{X^-}.f^+(\mu x^{X^+}.k\,(f^-\,(x,y)),y^-) : (X^- \rightarrow X^+) \rightarrow (X^- \rightarrow X^+)$$

where $\mu z.g(z)$ denotes the corresponding Conway fixed-point. The linear fixed-point $f^\dagger : (1,1) \rightarrow (X^+, X^-)$ is determined by a morphism from X^- to X^+ in $\mathcal{C}$, which is given by

---

[‖] One may define $\mathsf{Y}'_X :!(!X \multimap X) \multimap X$ from Y by (using the syntax of DILL (Barber and Plotkin 1997))

$$\lambda g^{!(!X \multimap X)}.\mathsf{let} \;!f^{!X \multimap X} \;\mathsf{be}\; g \;\mathsf{in}\; \mathsf{let}\; !z^X \;\mathsf{be}\; \mathsf{Y}_{!X}\,(\lambda y^{!X}.\mathsf{let}\; !x^X \;\mathsf{be}\; y \;\mathsf{in}\; !(f\,(!x))) \;\mathsf{in}\; z$$

but this does not satisfy the non-linear fixed-point equation $\mathsf{Y}'\,(!M) = M\,(!(\mathsf{Y}'\,(!M)))$. More concretely, consider the compact closed category **Rel** of sets and relations, with the powerset comonad — this has a linear fixed-point operator (Hasegawa 1997; Hasegawa 1999) but no non-linear one (cf. (Melliès 2006)).

the fixed-point of $\mathcal{U}f$, that is,

$$\mu k^{X^- \to X^+}.\lambda y^{X^-}.f^+(\mu x^{X^+}.k\,(f^-\,(x,y)),y^-) : X^- \to X^+.$$

6. Related Work and Discussions

6.1. *Program Transformations and Attribute Grammars*

Recently, Katsumata and Nishimura (Katsumata and Nishimura 2006) introduced a program transformation technique called *(semantic) higher-order removal*. Roughly, their technique transforms a higher-order map $g : (A^- \Rightarrow A^+) \Rightarrow (B^- \Rightarrow B^+)$ (created in the process of dealing with fusions of functions with accumulating parameters, which involves certain bi-directional information flow) to a less-expensive first-order map $f :$ $A^+ \times B^- \Rightarrow B^+ \times A^-$ such that $\mathcal{U}(f) = g$ holds, where $\mathcal{U}$ is right adjoint to $\mathcal{N}$. They give a syntactic condition which ensures that g is in the image of $\mathcal{U}$ in their semantic models, and presented a procedure for identifying f such that $\mathcal{U}(f) = g$.

More recently, Katsumata (Katsumata 2007) has shown that a substantial part of the theory of attribute grammars (Knuth 1968) can be carried out very cleanly in terms of traced monoidal categories and Int-construction. Very roughly, an attribute grammar assigns computation with bidirectional information flow to term trees of a context free grammar, which can be interpreted in traced monoidal categories just in the same manner as Geometry of Interaction:
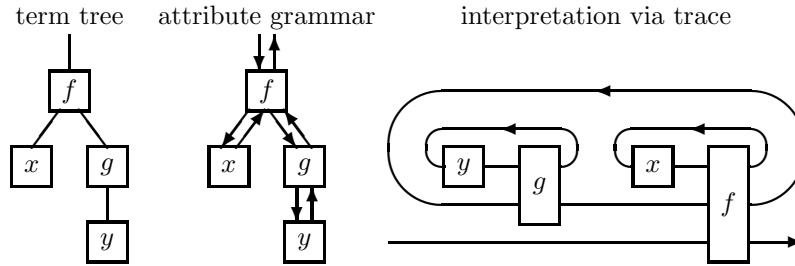
In Katsumata's work, the adjunction $\mathcal{N} \dashv \mathcal{U}$ of Theorem 4.1 provides the equivalence between attribute grammars and synthesized attribute grammars (attribute grammars with just bottom-up information flow). This generalizes the result on the equivalence between such attribute grammars formulated in a domain-theoretic setting citeCM79.

6.2. *Game Semantics*

An interesting (non)example is the category $\mathcal{Y}^-$ of negative Conway games (Melliès 2004). $\mathcal{Y}^-$ is a symmetric monoidal full subcategory of the compact closed category $\mathcal{Y}$ of Conway games (Joyal 1977). The inclusion from $\mathcal{Y}^-$ to $\mathcal{Y}$ has a right adjoint. Thus (by the folklore lemma) $\mathcal{Y}^-$ is a *traced symmetric monoidal closed category*. $\mathcal{Y}^-$ is one of very few interesting traced symmetric monoidal closed categories which are neither cartesian closed nor compact closed.

Int $\mathcal{Y}^-$ is *not* equivalent to $\mathcal{Y}$ — thus it does not really fit in our result. But the difference is subtle; more precisely, **Int** $\mathcal{Y}^-$ is equivalent to a compact closed full subcategory of $\mathcal{Y}$,

whose objects are the tensor products of positive and negative games. We expect that similar situation should be found in many categories of games, which would deserve further study. A recent relevant example is the category of multi-bracketed games of Melliès and Tabareau (Melliès and Tabareau 2007).

6.3. *Concluding Remarks*

We have observed that closedness for a traced monoidal category is equivalent to an adjointness associated to the Int-construction. This simple result has a number of applications, on models of linear logic and fixed-point computation. We hope that these provide some good motivations to study traced monoidal *closed* categories. In particular, we expect that it is possible to develop "Geometry of Higher-Order Interaction" along the line of this work.

We shall conclude this paper by giving some further research directions. Firstly, to obtain a more solid understanding of traced monoidal closed categories, it is desirable to find a good concrete description of *free* traced monoidal closed categories. Intuitively, it should be a sort of "higher-order tangles" — in the same sense that a free traced monoidal category can be given as a category of usual (framed) tangles. Alternatively, it might be useful to consider free tortile monoidal categories with idempotent balanced monoidal comonads.

Secondly, related to the previous direction, and less ambitiously, it should be useful to develop a syntax for traced monoidal closed categories. Perhaps this can be done by extending existing term calculi or proof nets for linear logic.

Thirdly, useful ways of constructing traced monoidal closed categories should be studied. We are yet to see if the constructions based on uniformity (Hasegawa 2004) can be used for the closed setting as well.

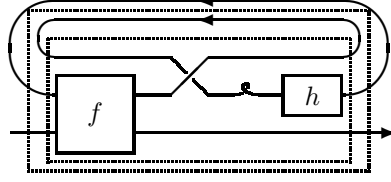### Appendix A. Derivation of Sliding and Vanishing

Proposition A.1. *Sliding* $Tr_{A,B}^X((id_B \otimes h) \circ f) = Tr_{A,B}^Y(f \circ (id_A \otimes h))$ is derivable.
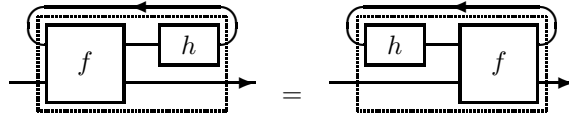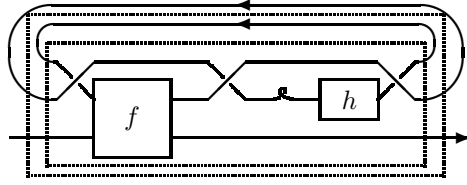

Proof. By *Yanking*, the left hand side is equal to
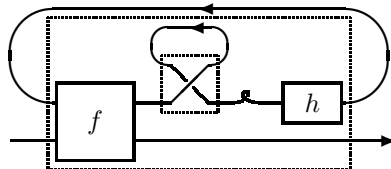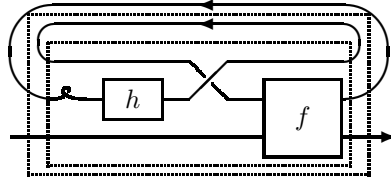

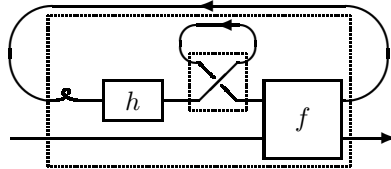Using *Superposing* and *Tightening*, we have
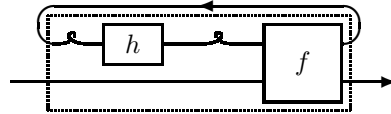
Then we apply *Exchange*:

Thanks to the naturality of braidings, this is equal to the following.

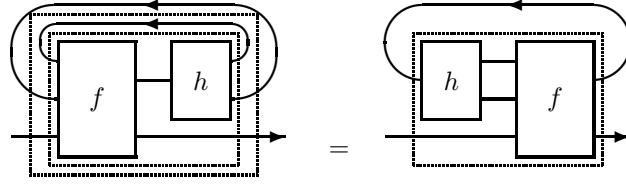Applying *Tightening* and *Superposing*, we obtain

By *Yanking*

This is equal to the right hand side of the equation, by the naturality of twisting. $\qquad\square$

Proposition A.2. *Vanishing for tensor* $Tr_{A,B}^{X}(Tr_{A\otimes X,B\otimes X}^{Y}(f)) = Tr_{A,B}^{X\otimes Y}(f)$ *is derivable.*
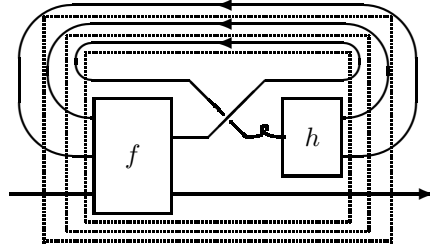
    Proof. We first show a variant of sliding

$$Tr_{A,B}^{X}(Tr_{A\otimes X,B\otimes X}^{Y}((id_B \otimes h) \circ f)) = Tr_{A,B}^{Z}(f \circ (id_A \otimes h))$$

for $f : A \otimes X \otimes Y \to B \otimes Z$ and $h : Z \to X \otimes Y$

by the same technique used for deriving the sliding axiom above. From the left hand side of the equation, by yanking, superposing and tightening, we get

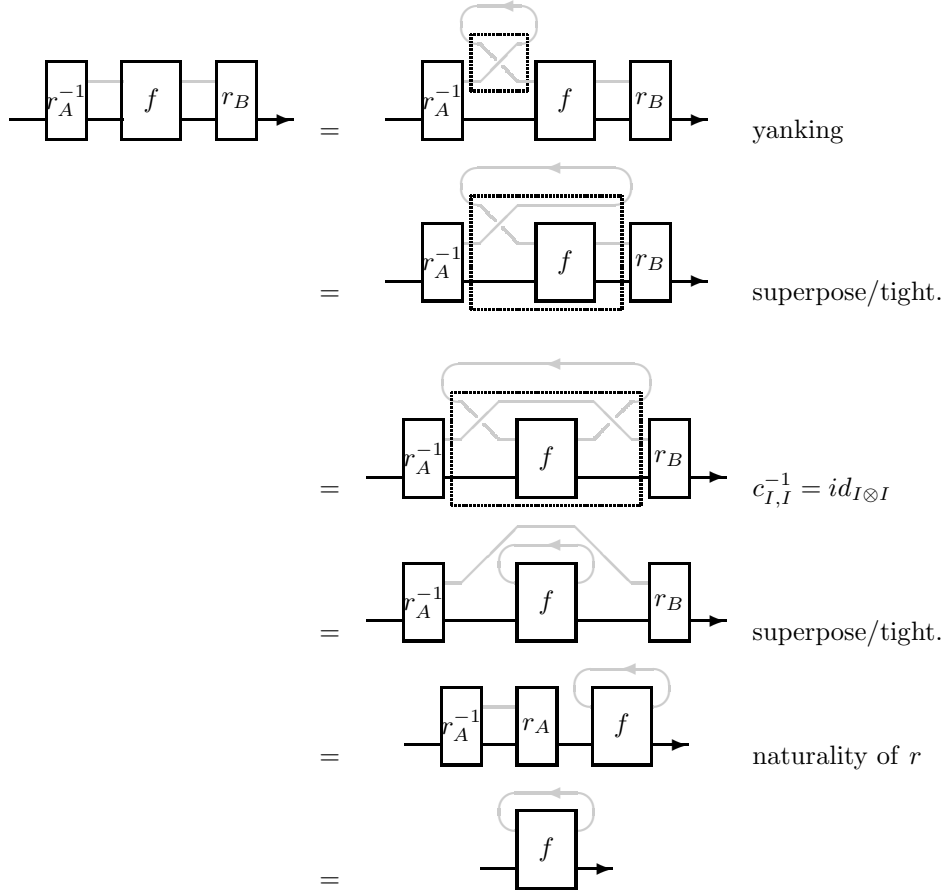to which we apply the exchange axiom twice:
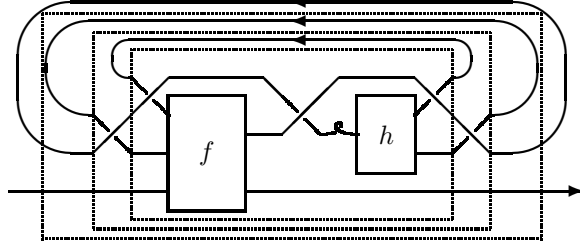
By the naturality and bilinearity of braidings, and by tightening, superposing, we have
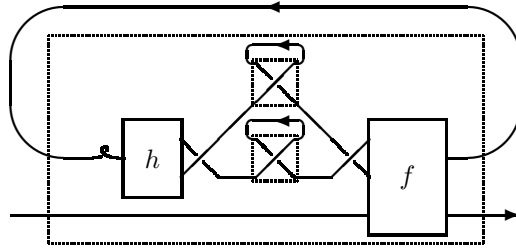
The rest of proof is now easy, by yanking and the naturality of twisting; here we should recall the axiom $\theta_{A \otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}$ of twists. The vanishing axiom for tensor follows from this equation, by letting Z be $X \otimes Y$ and h be $id_{X \otimes Y}$. $\qquad \square$

Remark A.1. *Vanishing for unit* $Tr^I_{A,B} f = r_B \circ f \circ r^{-1}_A$ ($f : A \otimes I \to B \otimes I$) is redundant

in original axioms in (Joyal *et al.* 1996). For completeness, here is a direct demonstration:
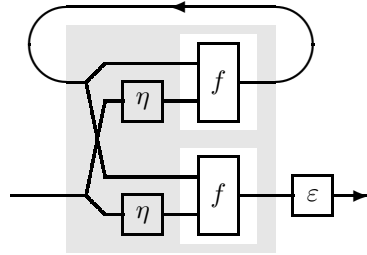
$$
\begin{array}{ll}
\boxed{r_A^{-1}} \; \boxed{f} \; \boxed{r_B} \to & = \quad \boxed{r_A^{-1}} \; \boxtimes \; \boxed{f} \; \boxed{r_B} \to & \text{yanking} \\
\end{array}
$$

yanking

superpose/tight.

$c_{I,I}^{-1} = id_{I \otimes I}$

superpose/tight.

naturality of r

Appendix B. Linear Fixed-points Graphically

We show that, for $f : FA \otimes X \to X$,

$$f^\dagger = \varepsilon \circ Tr^{FUX}(m^{F^{-1}} \circ F(\Delta \circ Uf \circ m^U \circ (\eta \times id)) \circ m^F) : FA \to X$$
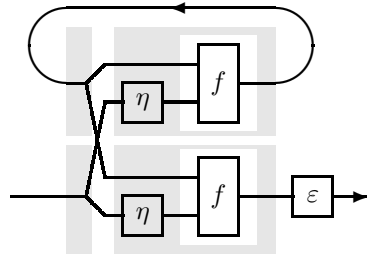
satisfies the (linear) fixed-point equation, using the box notation for monoidal functors (Cockett and Seely 1999; Melliès 2006).
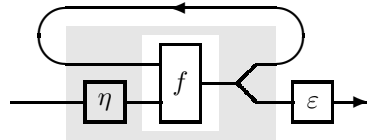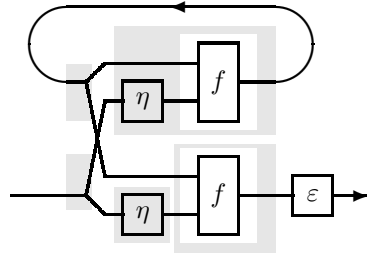

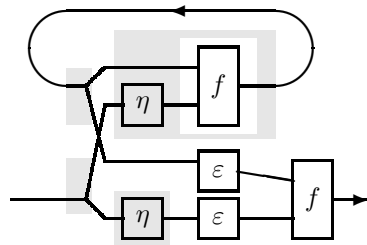First, duplicate $Uf \circ m^U \circ (\eta \times id)$:

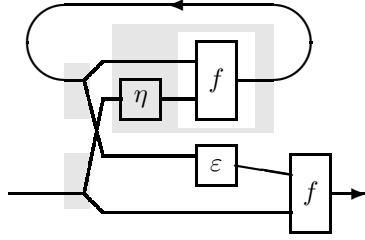Since F is strong monoidal, the gray box can be decomposed —
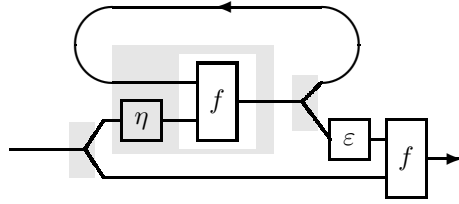

and further:

Since ε is a monoidal natural transformation from FU to the identity functor, we have
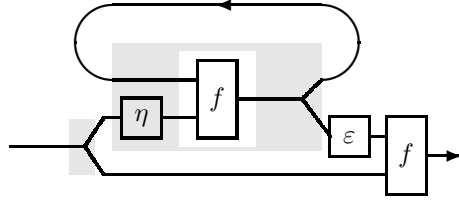

By adjointness, $\varepsilon \circ U\eta = id$:

By axioms of trace:

Now we have done: $f^\dagger = f \circ (id \otimes f) \circ m^{-1} \circ F\Delta$.

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