Lifting traced monoidal structure to the categories of algebras

(work in progress; joint work with J.S. Lemay)

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Monads and algebras

Given a monad T on a category $\mathcal{C}$, we have the notion of algebras over T. These algebras and homomorphisms form a category (the Eilenberg-Moore category) $\mathcal{C}^T$.

When the base category $\mathcal{C}$ has a nice structure/property, it is natural to ask if the structure/property can be lifted to the category of algebras $\mathcal{C}^T$ so that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ preserves the structure.
Such situations are ubiquitous and of interest in various areas of mathematics, physics and computer science.

Lifting monoidal structures

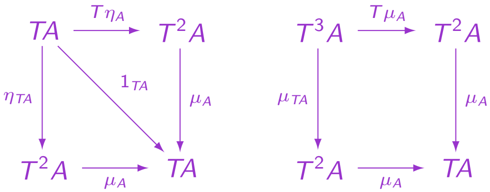
In this talk I will discuss conditions on monads for lifting the structure of monoidal categories (tensor categories), as well as several additional structures including symmetry/braiding, duality, closed structure, $*$-autonomy, and trace.

In most cases, opmonoidal (= oplax monoidal) monads and Hopf monads provide satisfactory answers. However, the case of trace is much subtler.

The case of $*$-autonomy is a joint work with J.S. Lemay (2018).
The case of trace is also joint with J.S., though still in progress.

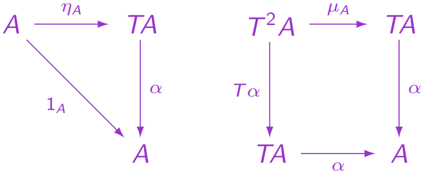
Monads

A monad on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \stackrel{\cdot}{\to} T$ (unit) and $\mu : T^2 \stackrel{\cdot}{\to} T$ (multiplication) such that $\mu \circ \eta T = \mu \circ T\eta = 1_T$ (the unit law) and $\mu \circ T\mu = \mu \circ \mu T$ (associativity) hold.

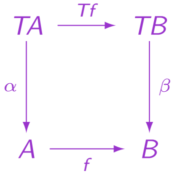


Algebras over a monad

An algebra over a monad T on a category $\mathcal{C}$ consists of an object A of $\mathcal{C}$ and a morphism $\alpha : TA \to A$ satisfying $\alpha \circ \eta_A = 1_A$ and $\alpha \circ \mu_A = \alpha \circ T\alpha$.



A homomorphism from an algebra (A, α) to an algebra (B, β) is a morphism $f : A \to B$ such that $f \circ \alpha = \beta \circ Tf$ holds:



The category of algebras and homomorphisms, called Eilenberg-Moore category, will be denoted by $\mathcal{C}^T$. There is an obvious forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ sending (A, α) to A.

Example

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Consider the functor $T = M \times (-) : \mathsf{Set} \to \mathsf{Set}$.

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T is a monad with unit $\eta_A : A \to M \times A$ and multiplication $\mu_A : M \times (M \times A) \to M \times A$ given by $\eta_A(a) = (e, a)$ and $\mu_A(m, (n, a)) = (m \cdot n, a)$.

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An algebra over T is a set A equipped with an M-action $\bullet : M \times A \to A$ satisfying $e \bullet a = a$ and $m \bullet (n \bullet a) = (m \cdot n) \bullet a$.
A homomorphism from $(A, \bullet)$ to $(B, \bullet)$ is a map $f : A \to B$ such that $f(m \bullet a) = m \bullet f(a)$ holds.

Thus the Eilenberg-Moore category Set^T is just the category of M-sets (which is equivalent to the presheaf category $[M^{\mathrm{op}}, \mathrm{Set}]$ where M is regarded as a one-object category).

Example

Set is cartesian closed. How about Set^T for $T = M \times (-)$?
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- Set^T has finite products.
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- Set^T is cartesian closed, with exponential

$$(A, \bullet) \Rightarrow (B, \bullet) = \{f : M \times A \to B \mid m \bullet f(n, x) = f(m \cdot n, m \bullet x)\}$$

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  and M-action $(m \bullet f)(n, x) = m \bullet f(n, x)$.
- However, $U : \mathrm{Set}^T \to \mathrm{Set}$ may *not* preserve the exponential.
- U preserves the cartesian closed structure exactly when M is a group.
  When M is a group, $(A, \bullet) \Rightarrow (B, \bullet) \cong B^A$, since $f \in (A, \bullet) \Rightarrow (B, \bullet)$ is determined by $f(e, -) \in B^A$ as $f(m, a) = m \bullet f(e, m^{-1} \bullet a)$.

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The category $(\mathrm{Vec}_k^{\mathrm{fin}})^T$ of its algebras is the category of B-modules (= representations of B).

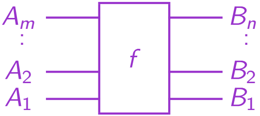
$(\mathrm{Vec}_k^{\mathrm{fin}})^T$ is a symmetric monoidal category and $U : (\mathrm{Vec}_k^{\mathrm{fin}})^T \to \mathrm{Vec}_k^{\mathrm{fin}}$ preserves the symmetric monoidal structure.

$(\mathrm{Vec}_k^{\mathrm{fin}})^T$ is compact closed and U preserves the structure exactly when B is a Hopf algebra (= bialgebra with antipode).

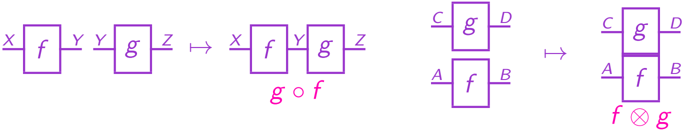
Monoidal categories

Monoidal categories (= tensor categories) are categories equipped with monoidal product $\otimes$ and its unit I as well as suitable isomorphisms for associativity $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ and unit law $A \otimes I \simeq A \simeq I \otimes A$.

In this talk, a morphism $f : A_1 \otimes A_2 \otimes \cdots \otimes A_m \to B_1 \otimes B_2 \otimes \cdots \otimes B_n$ in a monoidal category will be drawn as (to be read from left to right)



Morphisms can be composed, either sequentially or in parallel:



Opmonoidal monads

A monad (T, η, μ) on a monoidal category $\mathcal{C}$ is opmonoidal (= oplax monoidal = comonoidal) when

- T is an opmonoidal functor with a natural transformation $m_{A,B} : T(A \otimes B) \to TA \otimes TB$ and a morphism $m_I : TI \to I$, meaning that $m_{A,B}$ and m_I satisfies coherence conditions for associativity and unit law
- η and μ are opmonoidal natural transformation, meaning that they are compatible with $m_{A,B}$ and m_I

Examples:

- For any monoid M, the monad $M \times (-)$ on $\mathbf{Set}$ is opmonoidal.
- In fact, any monad T is opmonoidal when the tensor is cartesian, with $m_1 = !_{T1} : T1 \to 1$ and $m_{A,B} = \langle T\pi_1, T\pi_2 \rangle : T(A \times B) \to TA \times TB$.
- For any bialgebra B, the monad $B \otimes (-)$ on $\mathrm{Vec}_k^{\mathrm{fin}}$ is opmonoidal.

[Warning: Moerdijk (2002) called an opmonoidal monad a Hopf monad. His terminology is no longer standard.]

Lifting monoidal structure

Theorem (folklore/Moerdijk 2002)
A monad on a monoidal category lifts the monoidal structure if and only if it is an opmonoidal monad.

Proof: We have a bijective correspondence:

- An opmonoidal monad T determines a monoidal structure on algebras

$$(A, \alpha) \otimes (B, \beta) = (A \otimes B, T(A \otimes B) \stackrel{m_{A,B}}{\to} TA \otimes TB \stackrel{\alpha \otimes \beta}{\to} A \otimes B)$$

  with tensor unit $I = (I, m_I)$, and the forgetful functor preserves the monoidal structure.
- Conversely, when the algebras form a monoidal category and the forgetful functor preserves the structure, let

$$(TA \otimes TB, \gamma_{A,B} : T(TA \otimes TB) \to TA \otimes TB)$$

  be the tensor product of free algebras (TA, μ_A) and (TB, μ_B).
  With $m_{A,B} = \gamma_{A,B} \circ T(\eta_A \otimes \eta_B) : T(A \otimes B) \to TA \otimes TB$ and $m_I : TI \to I$ the unit algebra, T is opmonoidal.

Lifting symmetry (and braiding)

A monoidal category is symmetric if it is equipped with a natural isomorphism (called symmetry) $\sigma_{A,B} : A \otimes B \cong B \otimes A$ satisfying the hexagon axiom and $\sigma_{A,B}^{-1} = \sigma_{B,A}$.

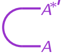
An opmonoidal monad T on a symmetric monoidal category is symmetric when $m_{A,B}$ is compatible with the symmetry.

Theorem (folklore/Moerdijk)
A monad on a symmetric monoidal category lifts the symmetric monoidal structure if and only if it is a symmetric opmonoidal monad.

The similar result holds for braiding (which may not satisfy $\sigma_{A,B}^{-1} = \sigma_{B,A}$) :

Theorem (folklore)
A monad on a braided monoidal category lifts the braided monoidal structure if and only if it is a braided opmonoidal monad.

Duality in monoidal categories

A *(left) dual* of an object A in a monoidal category is an object A^* equipped with a unit morphism $\eta_A : I \to A \otimes A^*$ and a counit morphism $\varepsilon_A : A^* \otimes A \to I$, drawn as 

and 

respectively, such that

A compact closed category is a symmetric monoidal category in which every object has a dual.

Monoidal closed categories

A monoidal category $\mathcal{C}$ is (bi-)closed when both $A \otimes (-) : \mathcal{C} \to \mathcal{C}$ and $(-) \otimes A : \mathcal{C} \to \mathcal{C}$ are left adjoint for every A.

$$\begin{array}{rcl} \mathcal{C}(A \otimes X, Y) & \cong & \mathcal{C}(X, A \multimap Y) \\ \mathcal{C}(X \otimes A, Y) & \cong & \mathcal{C}(X, Y \multimapinv A) \end{array}$$

When $\mathcal{C}$ is symmetric or braided, it suffices to ask just one of them.

A compact closed category is (bi)closed, with $A \multimap B = A^* \otimes B$.

When the tensor is cartesian, we say $\mathcal{C}$ is cartesian closed.

$$\mathcal{C}(A \times X, Y) \cong \mathcal{C}(X, Y^A)$$

Hopf monads

An opmonoidal monad T on a monoidal category is a Hopf monad (Bruguières, Lack and Virelizier 2011) when the fusion maps

$$H^l_{A,B} \;=\; T(A \otimes TB) \stackrel{m_{A,TB}}{\rightarrow} TA \otimes T^2B \stackrel{1_{TA} \otimes \mu_B}{\rightarrow} TA \otimes TB$$

$$H^r_{A,B} \;=\; T(TA \otimes B) \stackrel{m_{TA,B}}{\rightarrow} T^2A \otimes TB \stackrel{\mu_A \otimes 1_{TB}}{\rightarrow} TA \otimes TB$$

are invertible.

Examples:

- For a monoid M, the opmonoidal monad $M \otimes (-)$ on Set is a Hopf monad when M is a group.
- For a bialgebra B, the opmonoidal monad $B \otimes (-)$ on $\mathsf{Vec}_k^{\mathrm{fin}}$ is a Hopf monad when B is a Hopf algebra.
- More generally, for a bialgebra B in a symmetric monoidal category, the opmonoidal monad $B \otimes (-)$ is Hopf when B is a Hopf algebra with invertible antipode.

Example

For a monoid M, $T = M \times (-)$ on $\mathbf{Set}$ is an opmonoidal monad with

$$
\begin{array}{llrcl}
\eta_A : A \to TA & & \eta_A(a) & = & (e, a) \\
\mu_A : T^2 A \to TA & & \mu_A(m, (n, a)) & = & (m \cdot n, a) \\
m_{A,B} : T(A \times B) \to TA \times TB & & m_{A,B}(m, (a, b)) & = & ((m, a), (m, b)) \\
m_1 : T1 \to 1 & & m_1(m, *) & = & *
\end{array}
$$

The fusion maps are

$$
\begin{array}{llrcl}
H^l : T(A \times TB) \to TA \times TB & & H^l_{A,B}(m, (a, (n, b))) & = & ((m, a), (m \cdot n, b)) \\
H^r : T(TA \times B) \to TA \times TB & & H^r_{A,B}(m, ((n, a), b)) & = & ((m \cdot n, a), (m, b))
\end{array}
$$

which are invertible when M is a group:

$$
\begin{array}{rcl}
{H^l_{A,B}}^{-1}((m, a), (n, b)) & = & (m, (a, (m^{-1} \cdot n, b))) \\
{H^r_{A,B}}^{-1}((m, a), (n, b))) & = & (m, ((m^{-1} \cdot n, a), b))
\end{array}
$$

Lifting duality and monoidal closure

Surprisingly, opmonoidal monads lifting duality / monoidal closed structure are neatly characterized as Hopf monads:

Theorem (Bruguières and Virelizier / Bruguières, Lack and Virelizier)
An opmonoidal monad on a monoidal category with duals (autonomous category) lifts the structure if and only if it is a Hopf monad.

Theorem (Bruguières, Lack and Virelizier)
An opmonoidal monad on a monoidal bi-closed category lifts the structure if and only if it is a Hopf monad.

Proof Idea (for the second theorem): For algebras (A, α) and (B, β), we can define an algebra on $A \multimap B$ as the transpose of

$$A \otimes T(A \multimap B) \xrightarrow{\eta_A \otimes T(\alpha \multimap B)} TA \otimes T(TA \multimap B) \xrightarrow{(H^r)^{-1}} T(TA \otimes (TA \multimap B)) \xrightarrow{T ev} TB \xrightarrow{\beta} B.$$

These theorems cover the case of $M \times (-)$ on Set and $B \otimes (-)$ on $\mathrm{Vec}_k^{\mathrm{fin}}$.

Lifting $*$-autonomous categories

A $*$-autonomous category (Barr) is a monoidal bi-closed category equipped with a dualizing object $\perp$ making the canonical "double negation" map $A \longrightarrow \perp \circ\!\!- (A \multimap \perp)$ invertible.
They give models of classical linear logic, and also give an abstract account for Grothendieck-Verdier duality in the study of constructible sheaves.

Monads lifting $*$-autonomous structure can be characterized in terms of Hopf monads:

Theorem (Pastro and Street / Hasegawa and Lemay)
A opmonoidal monad on a $*$-autonomous category lifts the structure if and only if it is Hopf and there is an algebra structure on the dualizing object.

(Pastro and Street studied comonads lifting $*$-autonomous structure to the category of coalgebras, though the relation to Hopf (co)monads was not obvious.)

Example

Consider the real numbers $\mathbb{R}$ with the usual order $\leq$. Fix a real number $r \in \mathbb{R}$.

With $I = 0$, $x \otimes y = x + y$, $x \multimap y = y - x$ and $\perp = r$, the linear order (regarded as a category) $(\mathbb{R}, \leq)$ is $*$-autonomous.

The ceiling function (regarded as a functor) $T = \lceil - \rceil : \mathbb{R} \to \mathbb{R}$ sending x to the least integer y such that $x \leq y$ is a Hopf monad on $\mathbb{R}$. Note that $\mathbb{R}^T \simeq \mathbb{Z}$.

T lifts *-autonomous structure iff r has an algebra structure (i.e. r is an integer).

Traced monoidal categories

A *traced symmetric monoidal category* (Joyal, Street and Verity) is a symmetric monoidal category $\mathcal{C}$ equipped with a *trace*

$$Tr^{X}_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \longrightarrow \mathcal{C}(A, B)$$

subject to a few coherence axioms. (Traces can be defined on braided monoidal categories, but today I will discuss just the symmetric case.)

Alternatively, traced monoidal categories are characterized as monoidal full subcategories of compact closed categories: every traced symmetric monoidal category $\mathcal{C}$ fully faithfully embeds in a compact closed category $\mathrm{Int}\,\mathcal{C}$.

In particular, any compact closed category is a traced monoidal category (in a unique way).

Traced monoidal categories: examples

In computer science:

- the traced cartesian closed category of pointed ω-cpo's and continuous maps
- cartesian categories with Conway fixed-point operators
- the compact closed category of Conway games
- the compact closed category of sets and binary relations
- models of Geometry of Interaction via Int-construction
- dagger compact closed categories in categorical quantum mechanics

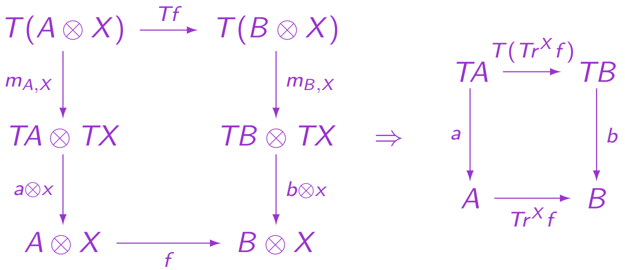
In mathematics:

- the compact closed category of finite dimensional vector spaces
- ribbon categories of linear representations of quantum groups
- the ribbon category of framed tangles
- modular tensor categories for TQFT

(Many of them are braided rather than symmetric)

Traced monads

We say a symmetric opmonoidal monad T on a traced symmetric monoidal category $\mathcal{C}$ is traced when it lifts trace, i.e. $\mathcal{C}^T$ is traced and $U : \mathcal{C}^T \to \mathcal{C}$ preserves trace.
That is, T is traced if



for any $f : A \otimes X \to B \otimes X$ and algebras (A, a), (B, b) and (X, x).

Is there a simple characterization of traced monads (with no mention to algebras)?

Traced monads vs Hopf monads

Since traced monoidal categories are full subcategories of compact closed categories, it is natural to expect that a Hopf monad lifts trace as well. However, it is not quite the case. In the positive side, we have

Theorem (Hasegawa and Lemay).
A symmetric Hopf monad T on a traced symmetric monoidal category is traced if and only if the following trace-coherence condition holds:

$$T(Tr^{TX}(f)) = Tr^{TX}(H^{l}_{B,X} \circ Tf \circ {H^{l}}^{-1}_{A,X}) : TA \to TB$$

holds for any $f : A \otimes TX \to B \otimes TX$.

For instance, for any co-commutative Hopf algebra H in a traced category, the monad $H \otimes (-)$ is symmetric Hopf and trace-coherent, hence traced.

Non-Hopf traced monads

There are several traced monads which are not Hopf.
Many of them are from basic domain theory:

- Let Cppo be the traced cartesian closed category of ω-cpo's with bottom (pointed cpo's = cppo's) and continuous maps. Then the lifting monad $TX = X_\perp$ is traced but not Hopf.
- Consider the Sierpinski space $\Sigma = \{\perp \leq \top\}$ in Cppo with the monoid structure with $(\top, \wedge)$. $T = \Sigma \times (-)$ is a traced monad, but not Hopf (as Σ is not a group).
- Even on a compact closed category, a traced monad may not be symmetric Hopf (a counterexample can be given by traced symmetric monoidal closed categories and the monad induced by the Int-construction).

A Hopf monad via distributive coproducts New

There exists a symmetric Hopf monad which is not traced.
Let $\mathcal{C}$ be a symmetric monoidal category with distributive binary coproducts and two distinct traces Tr and Tr'.

- $\mathcal{C}^2$ is a traced symmetric monoidal category with

$$Tr^{(X,X')}_{(A,A'),(B,B')}(f,f') = (Tr^{X}_{A,B} f, Tr'^{X'}_{A',B'} f') : (A,A') \to (B,B')$$

  where $f : A \otimes X \to B \otimes X$ and $f' : A' \otimes X' \to B' \otimes X'$.
- $T(X,Y) = (X+Y, X+Y)$ is a Hopf monad on $\mathcal{C}^2$; its fusion map is given by the canonical map

$$(X_1+X_2)\otimes Y_1 + (X_1+X_2)\otimes Y_2 \to (X_1+X_2)\otimes(Y_1+Y_2)$$

  hence distributivity means T is Hopf.
- $(\mathcal{C}^2)^T$ is equivalent to C — the diagonal functor $\mathrm{C} \to \mathrm{C}^2$ is monadic. Alas, the forgetful functor (or the diagonal functor) cannot be traced unless Tr and Tr' are the same trace.

Summary

- A monad lifts monoidal structure exactly when it is opmonoidal.
- An opmonoidal monad lifts duality/closed structure exactly when it is a Hopf monad.
- An opmonoidal monad lifts $*$-autonomous structure exactly when it is Hopf and there is an algebra on the dualizing object.
- When an opmonoidal monad lifts trace, we call it a traced monad. However, we do not have a good characterization of traced monad.
- Although we identified when a Hopf monad is a traced monad (the trace-coherence condition), there are several non-Hopf traced monads and non-traced Hopf monads.

The gap between traced monads and Hopf monads reflects the fact that trace is a structure on symmetric monoidal categories while being compact closed is a property of symmetric monoidal categories.
At least, a characterization of traced monads needs to explicitly specify the trace to be lifted (e.g. the trace-coherence condition).

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