Bialgebras in Rel

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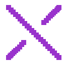
Background

Now we have lots of nice examples of traced monoidal categories and ribbon categories (tortile monoidal categories) in computer science and mathematics. Wonderful!

However, the CS examples and Math examples look quite different and almost unrelated.

In CS, examples include models of recursion/cyclic data structure, as well as Geometry of Interaction/bi-directional data flow, and quantum information protocols. They are mostly symmetric monoidal.

In Math, most interesting examples are categories arising from quantum groups (quasi-triangular Hopf algebras) which are closely related to areas like mathematical physics and low-dimensional topology. Many of them are braided, in which the braiding $c =$  is distinguished from its inverse $c^{-1} =$  .

Goal of this talk

In the large perspective:
Fill this gap between CS and Math, by providing some cases which can be interesting from both mathematical and computational points of view. Promote technology transfer between CS (semantics) and Math (quantum groups).

More concrete goal:
Demonstrate that a popular construction in quantum groups theory (the quantum double construction) can be carried out in categories familiar to computer scientists, and that it gives rise to a ribbon category with non-symmetric braiding ( $\neq$ ).

cf. Earlier work by Blute on topological vector spaces

In this talk we focus on the category $\mathbf{Rel}$ of sets and binary relations.

A rough sketch of the development

Step 1: Pick up your favorite compact closed category.

Step 2: Find a Hopf algebra H in your category.

Step 3: Consider the category of H-modules, i.e., algebras of the monad $H \otimes (-)$. If you are lucky, you have a nice monoidal category with non-symmetric braiding. Congratulations!

Step 3': Even if Step 3 does not work, do not give up.
If H satisfies some modest conditions, then you can perform the quantum double construction of Drinfel'd, which gives rise to a nice (quasi-triangular) Hopf algebra $D(H)$ with a universal R-matrix.
The category of $D(H)$-modules features a braiding.

While the default category for Step 1 is that of finite dimensional vector spaces and linear maps, in this talk we choose $\mathbf{Rel}$ as our base category.

Part I: Monoidal categories

Part II: Monoids, comonoids and Hopf algebras

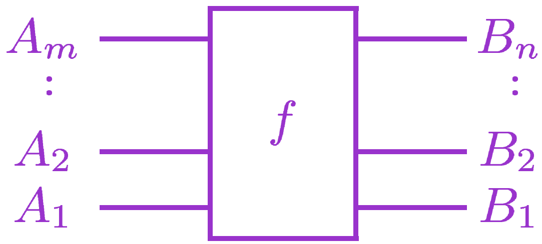
Part III: A quantum double construction in $\mathbf{Rel}$

Part IV: Concluding remarks

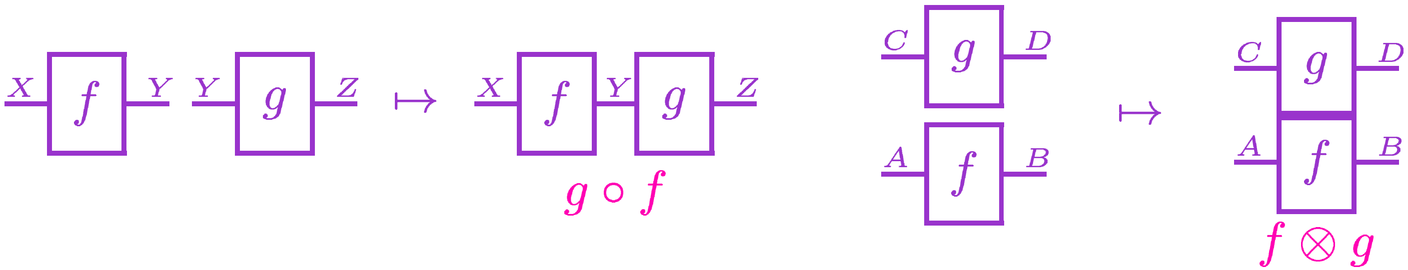
Geometry of monoidal categories

(Geometry of tensor calculus, Joyal and Street / string diagrams, Penrose)

In this talk, a morphism $f : A_1 \otimes A_2 \otimes \ldots \otimes A_m \to B_1 \otimes B_2 \otimes \ldots \otimes B_n$ in a monoidal category will be drawn as (to be read from left to right)

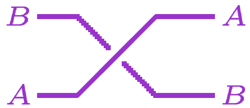


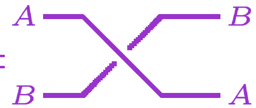
Morphisms can be composed, either sequentially or in parallel:



Braiding and symmetry on monoidal categories

A *braiding* is a natural isomorphism $c_{A,B} : A \otimes B \to B \otimes A$ such that,

using the notations $c_{A,B} =$ 

and $c^{-1}_{A,B} =$ 

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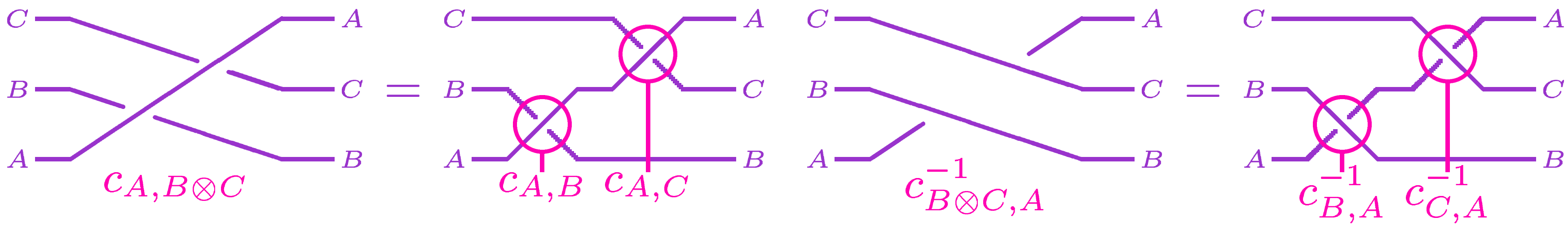
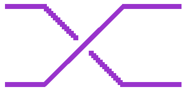
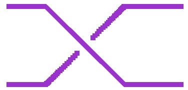
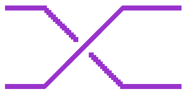
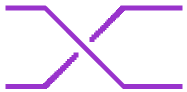
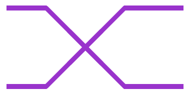


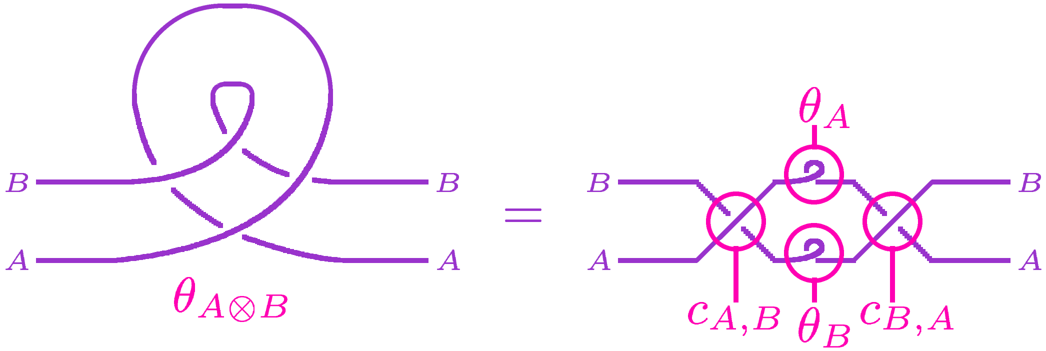
In general $c_{A,B} \neq c^{-1}_{B,A}$, i.e.,  $\neq$  .

A *symmetry* is a braiding such that  $=$  $=$  .

A *braided/symmetric monoidal category* is a monoidal category equipped with a braiding/symmetry.

Twists on braided monoidal categories

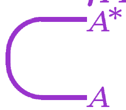
Often a braided monoidal category is equipped with a *twist*, which is a natural isomorphism $\theta_A : A \to A$, drawn as $\theta_A =$  and $\theta_A^{-1} =$  , such that $\theta_I = id_I$ and

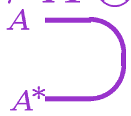


balanced monoidal category = braided monoidal category with a twist

- $\forall A \;\; \theta_A = id_A \implies \forall A, B \; c_{A,B} = c_{B,A}^{-1}$
- In a balanced monoidal category, lines in our pictures are better understood to be framed strings or ribbons, rather than simple strings.

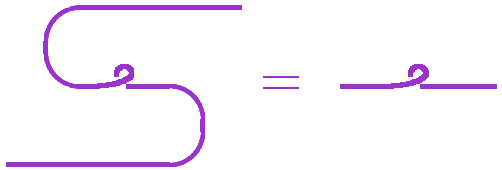
Duality in monoidal categories

A *(left) dual* of an object A in a monoidal category is an object A^* equipped with a unit morphism $\eta_A : I \to A \otimes A^*$ and a counit morphism $\varepsilon_A : A^* \otimes A \to I$, drawn as 

and 

respectively, such that

A *ribbon category (tortile monoidal category)* is a balanced monoidal category in which every object has a left dual and moreover satisfies



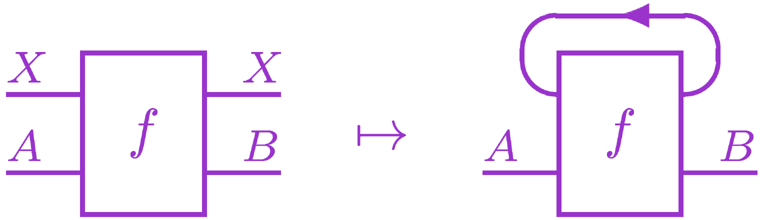
compact closed category $=$ ribbon category with $\theta = id$ (& $c_{A,B} = c_{B,A}^{-1}$)

Theorem (Shum). The ribbon category freely generated by a single object is equivalent to the category of framed tangles.

Traced monoidal categories

A *traced monoidal category* is a balanced monoidal category $\mathcal{C}$ equipped with a family of functions, called *trace* operator

$$Tr^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \longrightarrow \mathcal{C}(A, B)$$



subject to a few coherence axioms.

The structure theorem by Joyal, Street and Verity tells us that traced monoidal categories can be characterized as monoidal full subcategories of ribbon categories: every traced monoidal category $\mathcal{C}$ fully faithfully embeds in a ribbon category $\mathbf{Int}\,\mathcal{C}$ (more about this later).

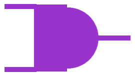
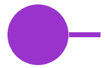
Part I: Monoidal categories

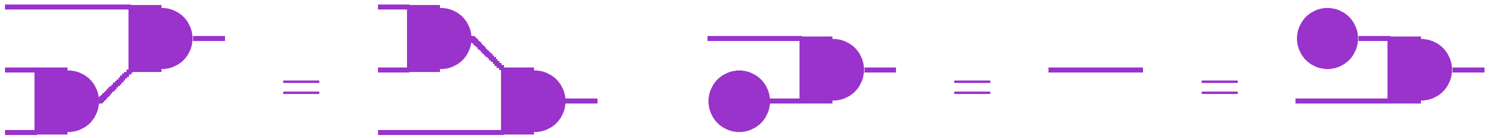
Part II: Monoids, comonoids and Hopf algebras

Part III: A quantum double construction in $\mathbf{Rel}$

Part IV: Concluding remarks

Monoids in a monoidal category

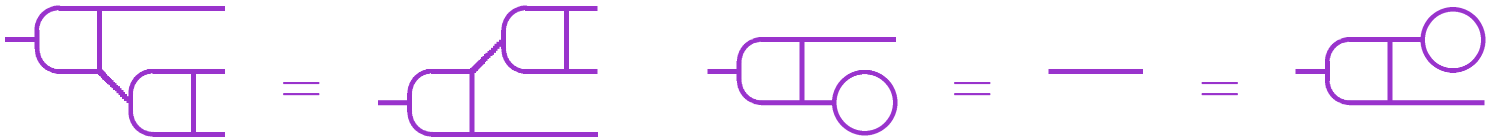
A *monoid* in a monoidal category is an object A equipped with the multiplication $m : A \otimes A \to A$ and unit $1 : I \to A$ such that, with notations $m =$  and $1 =$  ,

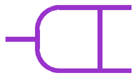
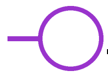


A monoid $A = (A, m, 1)$ in a symmetric monoidal category is *commutative* if  holds.

Comonoids in a monoidal category

Dually, a *comonoid* in a monoidal category is an object A equipped with the *comultiplication* $\Delta : A \to A \otimes A$ and *counit* $\epsilon : A \to I$ satisfying

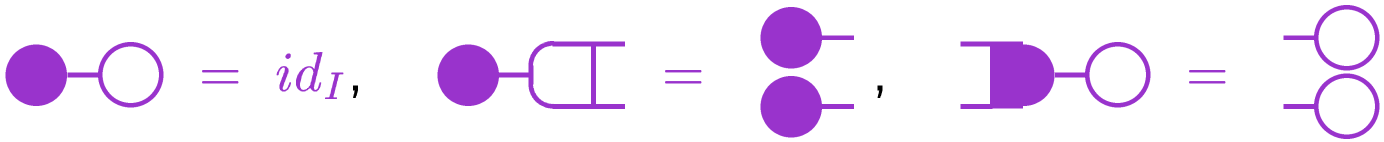


where $\Delta =$  and $\epsilon =$ .

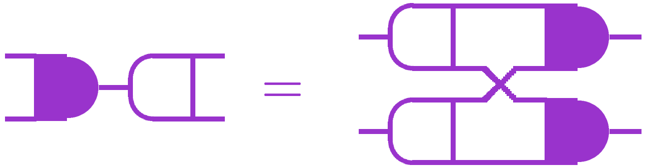
A comonoid $A = (A, \Delta, \epsilon)$ in a symmetric monoidal category is *co-commutative* if  holds.

Bialgebras and Hopf algebras in a symmetric monoidal category

A *bialgebra* in a symmetric monoidal category is an object A with a monoid structure $(A, m, 1)$ and a comonoid structure (A, Δ, ϵ) such that



and



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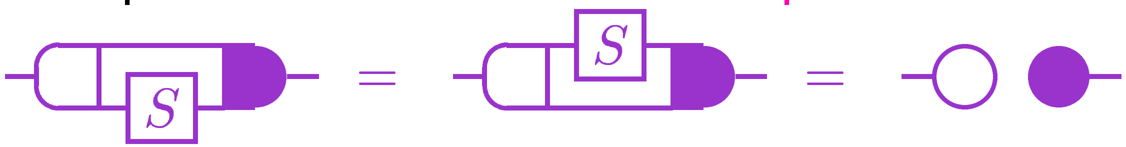
A *Hopf algebra* is a bialgebra $A = (A, m, 1, \Delta, \epsilon)$ equipped with a morphism $S : A \to A$ called antipode which satisfies



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Example: **Set**

The category **Set** of sets and functions is a symmetric monoidal category, with $A \otimes B = A \times B$ and $I = 1 = \{*\}$.

A monoid with respect to this monoidal structure on **Set** is just a monoid in the usual sense.

A comonoid is any set A with $\Delta_A = \lambda x^A.(x, x) : A \to A \times A$ and $\epsilon_A = \lambda x^A.* : A \to 1$, which is always co-commutative.

A bialgebra is just a monoid $(A, \cdot, e)$ with Δ_A and ϵ_A.

A Hopf algebra is just a group $(A, \cdot, e, (-)^{-1})$ with Δ_A and ϵ_A, where the antipode $S : A \to A$ is $S(x) = x^{-1}$.

Modules

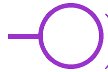
Given a monoid $A = (A, m, 1)$ in a monoidal category $\mathcal{C}$,
an algebra of the monad $A \otimes (-)$ is called an A-*module*.
Explicitly, an A-module is an object X equipped with an A-action
$\alpha : A \otimes X \to X$ which satisfies

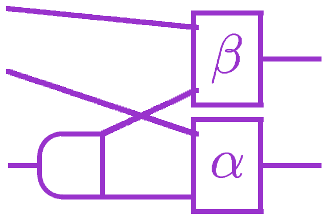
A morphism of A-modules from (X, α) to (Y, β) is a morphism
$f : X \to Y$ in $\mathcal{C}$ which is compatible with the A-actions.

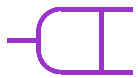
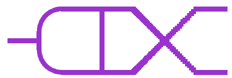
The category of A-modules will be denoted by $\mathbf{Mod}_{\mathcal{C}}(A)$.

From now on, we focus on the structure of the category $\mathbf{Mod}_{\mathcal{C}}(A)$
for a monoid/bialgebra/Hopf algebra A.

A bialgebra induces a monoidal category

When A is not just a monoid but a bialgebra in a symmetric monoidal category $\mathcal{C}$, $\mathbf{Mod}_{\mathcal{C}}(A)$ is a monoidal category, with $I = (I,$ $)$ and

$(X, \alpha) \otimes (Y, \beta) = \Big(X \otimes Y,$  $\Big)$.

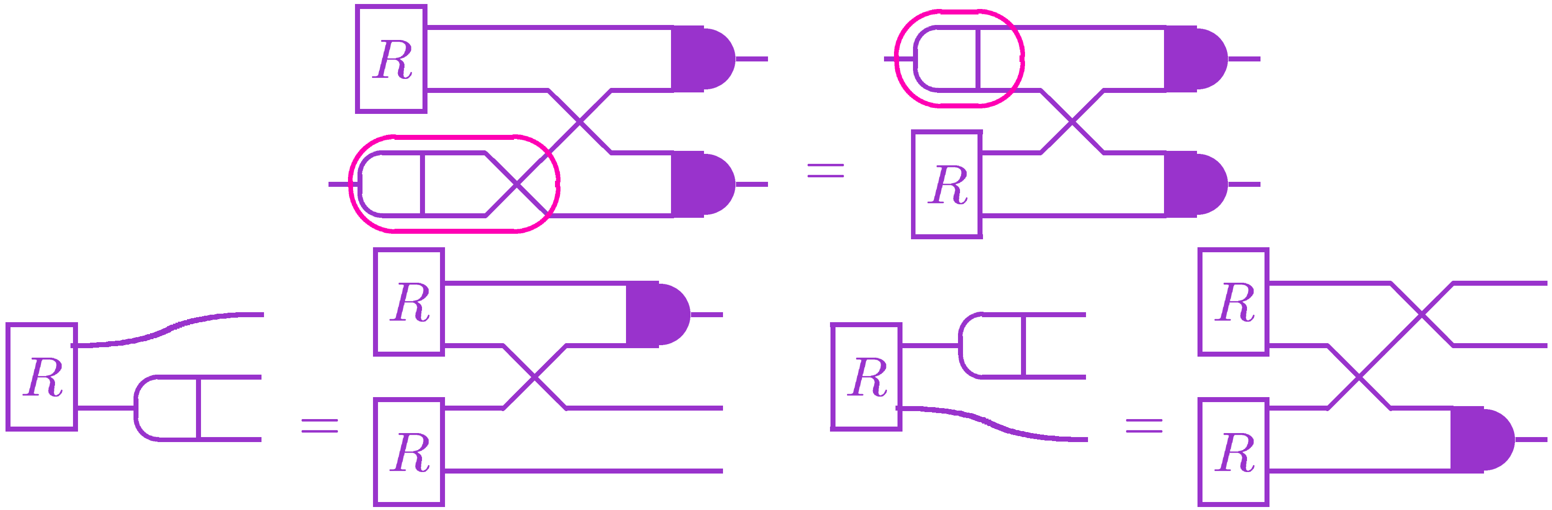
If A is co-commutative ( $=$ ), $\mathbf{Mod}_{\mathcal{C}}(A)$ has a symmetry inherited from $\mathcal{C}$.

For giving a non-symmetric braiding on $\mathbf{Mod}_{\mathcal{C}}(A)$, we need a sort of relaxed version of co-commutativity. This is addressed in the next slide.

A quasi-triangular bialgebra induces a braided monoidal category

When A is a bialgebra, there is a bijective correspondence between natural transformations of $X \otimes Y \to Y \otimes X$ on $\mathbf{Mod}_{\mathcal{C}}(A)$ and morphisms of $I \to A \otimes A$ in $\mathcal{C}$.

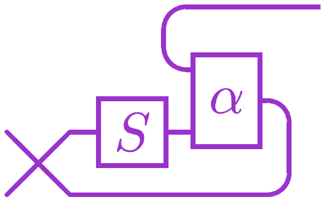
In particular, there is a bijective correspondence between braidings on $\mathbf{Mod}_{\mathcal{C}}(A)$ and morphisms $R : I \to A \otimes A$ in $\mathcal{C}$ called *universal R-matrices*, which are convolution-invertible and satisfy



A *quasi-triangular bialgebra* is a bialgebra with a universal R-matrix.

A Hopf algebra induces a monoidal category with left duals

When $\mathcal{C}$ is a compact closed category and A is a Hopf algebra in $\mathcal{C}$, $\mathbf{Mod}_{\mathcal{C}}(A)$ is a monoidal category in which every A-module $X = (X, \alpha)$

has a (left) dual $(X, \alpha)^* = \Big(X^*,$  $\Big)$.

Robbon Hopf algebra induces a ribbon category

Putting these results together,
when A is a quasi-triangular Hopf algebra in a compact closed category $\mathcal{C}$, $\mathbf{Mod}_{\mathcal{C}}(A)$ is a braided monoidal category with duals.

Moreover, there is a bijective correspondence between twists on $\mathbf{Mod}(A)$ and certain morphisms of $I \to A$ in $\mathcal{C}$ called *universal twists*.
A quasi-triangular Hopf algebra equipped with a universal twist is called a *ribbon Hopf algebra*.

Theorem. If A is a ribbon Hopf algebra in a compact closed category $\mathcal{C}$, the category $\mathbf{Mod}_{\mathcal{C}}(A)$ of A-modules is a ribbon category.

In the rest of this talk, we construct a ribbon Hopf algebra A in the compact closed category $\mathbf{Rel}$ of sets and binary relations.

Part I: Monoidal categories

Part II: Monoids, comonoids and Hopf algebras

Part III: A quantum double construction in $\mathbf{Rel}$

Part IV: Concluding remarks

The compact closed category $\mathbf{Rel}$

Recall that objects of $\mathbf{Rel}$ are sets, and a morphism from A to B is a binary relation $r \subseteq A \times B$.

$\mathbf{Rel}$ is a compact closed category, with

- tensor $A \otimes B = A \times B$, $I = 1 = \{*\}$,
- duals $A^* = A$,
  $\eta_A = \{(*, (x, x)) \mid x \in A\} : I \to A \otimes A^*$ and
  $\varepsilon_A = \{((x, x), *) \mid x \in A\} : A^* \otimes A \to I$, and
- symmetry
  $c_{A,B} = \{((x, y), (y, x)) \mid x \in A, y \in B\} : A \otimes B \to B \otimes A$.

Our task is to find a ribbon Hopf algebra in $\mathbf{Rel}$ whose category of modules is a ribbon category with non-symmetric braiding.

A group gives rise to a Hopf algebra in $\mathbf{Rel}$

Recall that any group gives rise to a Hopf algebra in $\mathbf{Set}$.

Since the inclusion functor from $\mathbf{Set}$ to $\mathbf{Rel}$ is strong symmetric monoidal, any Hopf algebra in $\mathbf{Set}$ gives rise to a Hopf algebra in $\mathbf{Rel}$.

Explicitly, a group $G = (G, \cdot, e, (-)^{-1})$ gives rise to a Hopf algebra $\overline{G} = (G, m, 1, \Delta, \epsilon, S)$ in $\mathbf{Rel}$ where

$$
\begin{array}{rcl}
m & = & \{((a_1, a_2), a_1 \cdot a_2) \mid a_1, a_2 \in G\} : G \times G \to G \\
1 & = & \{(*, e)\} : I \to G \\
\Delta & = & \{(a, (a, a)) \mid a \in G\} : G \to G \times G \\
\epsilon & = & \{(a, *) \mid a \in G\} : G \to I \\
S & = & \{(a, a^{-1}) \mid a \in G\} : G \to G
\end{array}
$$

$\overline{G}$ is co-commutative, and $\mathbf{Mod}_{\mathbf{Rel}}(\overline{G})$ is compact closed. $\mathbf{Mod}_{\mathbf{Rel}}(\overline{G})$ does not feature a non-symmetric braiding.

Turning $\overline{G}$ to a quasi-triangular Hopf algebra via quantum double

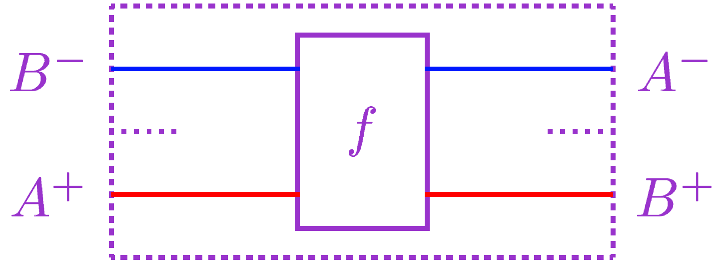
While $\overline{G}$ itself is not quite what we want, there is a wonderful construction in the quantum group theory which turns any Hopf algebra with invertible antipode to a quasi-triangular Hopf algebra.
This is the *quantum double* of Drinfel'd.

In short, the quantum double of a Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$ (with S invertible) is a bicrossed product of the dual Hopf algebra $A^{\mathrm{op}*} = (A^*, \Delta^*, \epsilon^*, (m \circ c)^*, 1^*, (S^{-1})^*)$ and A. It is very much like a tensor product of $A^{\mathrm{op}*}$ and A, except that there are some adjustment on the multiplication and antipode.

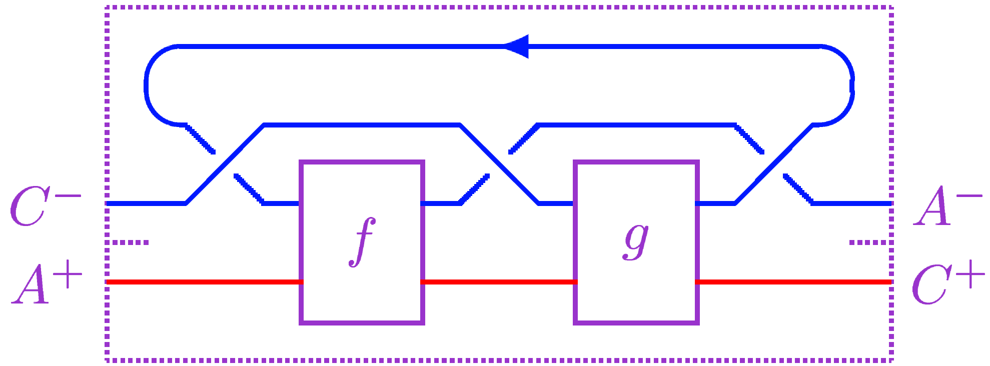
A direct description of quantum double in general compact closed categories is rather complicated. We give a simpler and managable description in terms of the Int-construction (this is not given in the paper).

Int-construction

Suppose that $\mathcal{C}$ is a traced monoidal category. Recall that a morphism $f : (A^+, A^-) \to (B^+, B^-)$ in $\mathbf{Int}(\mathcal{C})$ is drawn as



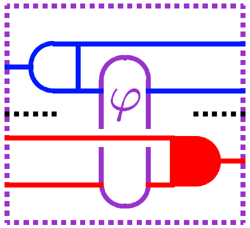
The composition of $f : (A^+, A^-) \to (B^+, B^-)$ and $g : (B^+, B^-) \to (C^+, C^-)$ is



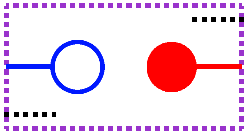
Quantum double in $\mathbf{Int}(\mathcal{C})$

Theorem. Given a Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$ *with an invertible antipode in a traced symmetric monoidal category* $\mathcal{C}$*, there is a quasi-triangular Hopf algebra* $((A, A), m^{\mathsf{d}}, 1^{\mathsf{d}}, \Delta^{\mathsf{d}}, \epsilon^{\mathsf{d}}, S^{\mathsf{d}}, R)$ *in* $\mathbf{Int}(\mathcal{C})$ *with the following data.*

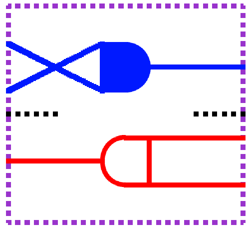
m^{d}



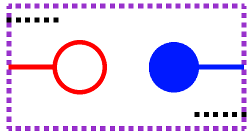
1^{d}



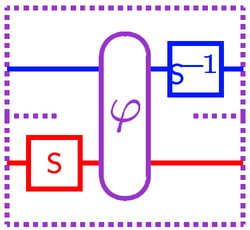
Δ^{d}



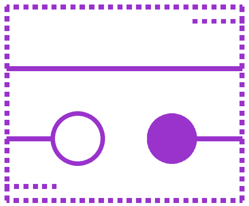
ϵ^{d}



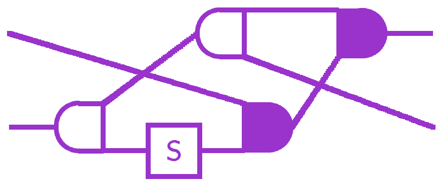
S^{d}



R



where φ is



(The red components are from $A = (A, m, 1, \Delta, \epsilon, S)$ while blue ones are from $A^{\mathrm{op}*} = (A^*, \Delta^*, \epsilon^*, (m \circ c)^*, 1^*, (S^{-1})^*)$.)

The quantum double of $\overline{G}$

Since the antipode of $\overline{G}$ is invertible, we can apply the quantum double construction to $\overline{G}$. The resulting quasi-triangular Hopf algebra $D(\overline{G}) = (G \times G, m^{\mathrm{d}}, 1^{\mathrm{d}}, \Delta^{\mathrm{d}}, \epsilon^{\mathrm{d}}, S^{\mathrm{d}}, R)$ in $\mathbf{Rel}$ is explicitly described as follows (imported from $\mathbf{Int\,Rel}$ via the equivalence $\mathbf{Rel} \simeq \mathbf{Int\,Rel}$).

$$\begin{aligned}
m^{\mathrm{d}} &= \{(((g, h_1), (h_1^{-1} g h_1, h_2)), (g, h_1 h_2)) \mid g, h_1, h_2 \in G\} \\
1^{\mathrm{d}} &= \{(*, (g, e)) \mid g \in G\} \\
\Delta^{\mathrm{d}} &= \{((g_1 g_2, h), ((g_1, h), (g_2, h))) \mid g_1, g_2, h \in G\} \\
\epsilon^{\mathrm{d}} &= \{((e, g), *) \mid g \in G\} \\
S^{\mathrm{d}} &= \{((g, h), (h^{-1} g^{-1} h, h^{-1})) \mid g, h \in G\} \\
R &= \{(*, ((g, e), (h, g))) \mid g, h \in G\}
\end{aligned}$$

Fortunately, $D(\overline{G})$ also has a universal twist $v = \{(*, (g, g)) \mid g \in G\}$, hence it is a ribbon Hopf algebra.

$D(\overline{G})$-modules are the crossed G-sets

For a group G, a *crossed G-set* is a G-set X equipped with a map $|\_| : X \to G$ such that $|gx| = g|x|g^{-1}$ holds.
It turns out that $D(\overline{G})$-modules are in bijective correspondence to crossed G-sets, and

Theorem. $\mathbf{Mod}_{\mathbf{Rel}}(D(\overline{G}))$ *is isomorphic to the category* $\mathbf{XRel}(G)$ *of crossed G-sets whose morphisms from* $(X, \cdot, |\_|)$ *to* $(Y, \cdot, |\_|)$ *are binary relations* $r : X \to Y$ *such that* $(x, y) \in r$ *implies* $(gx, gy) \in r$ *as well as* $|x| = |y|$*.*

Hence we obtain a ribbon category $\mathbf{XRel}(G)$ of crossed G-sets with non-symmetric braiding. Also in the paper:

- explicit descriptions of the braiding, twist and trace in $\mathbf{XRel}(G)$
- an outline of the invariant of tangles derived from $\mathbf{XRel}(G)$ via racks
- an outline of modelling a braided variant of linear logic in $\mathbf{XRel}(G)$

Part I: Monoidal categories

Part II: Monoids, comonoids and Hopf algebras

Part III: A quantum double construction in $\mathbf{Rel}$

Part IV: Concluding remarks

Summary

We have demonstrated that there is a non-trivial ribbon Hopf algebra in **Rel**, which gives rise to a new ribbon category of crossed G-sets.

The main technical tool is Drinfel'd's quantum double, which is a standard machinery of the quantum group theory.

It seems that many results on quantum groups are directly applicable to many categories used in the semantics of logic and computation. While its computational significance is yet to be examined (topological quantum computation?), this certainly is a fun!

Many of the ideas are applicable to $*$-autonomous categories as well. For now I am studying Hopf algebras in some $*$-autonomous categories arising from the Chu-construction; so far results look encouraging.

Why not your favorite compact closed categories or $*$-autonomous categories — there might be some nice Hopf algebras awaiting for you!

Some personal remarks

Let me conclude this talk with a rather personal recollection.

In the spring of 1995 (15 years ago), I was a PhD student in Edinburgh. Following John Power's suggestion and Robin Milner's encouragement, I was just starting to study the semantics of Robin's (reflexive) action calculi (a precursor of his bigraphs) using traced monoidal categories. (At that time the preprint of the paper by Joyal, Street and Verity was already circulated among specialists; I obtained a copy from John.)

Since then, I have been working on traced monoidal categories and related structures, and wrote some papers, including this one. I do not know how Robin felt about my work, but in any case I love this area, and I am very grateful to my superb teachers for their guidance. Let me thank you again, Robin.

End of slides — thank you very much!