Intelligence of Low-dimensional Topology
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A table of coherent band-Gordian distances between knots

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Abstract

- A coherent band surgery is a local move on an oriented link, which is equivalent to a smoothing a crossing.

- The coherent band-Gordian distance between two links is the least number of coherent band surgeries needed to transform one link into the other.

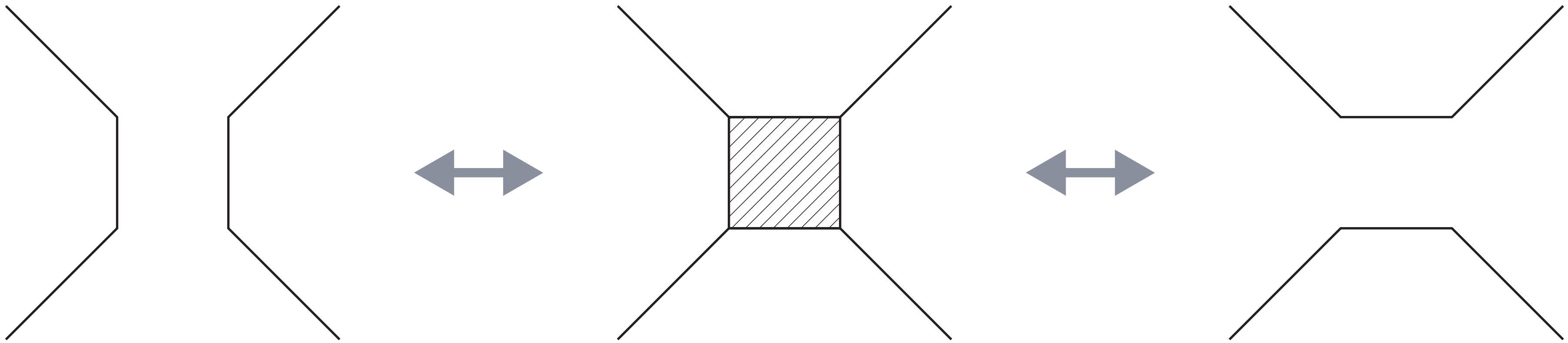
- We introduce some criteria for two links which are related by a coherent band surgery.

- We give a table of coherent band-Gordian distances between two knots with up to 7 crossings.

- This is a joint work with Taizo Kanenobu.

Band surgery

Let L be a link, and $b : I \times I \to S^3$ an embedding such that $b(I \times I) \cap L = b(I \times \partial I)$, where I is a closed interval. We obtain another link $L' = (L - b(I \times \partial I)) \cup b(\partial I \times I)$, which is called the link obtained from L by the band surgery along the band b.

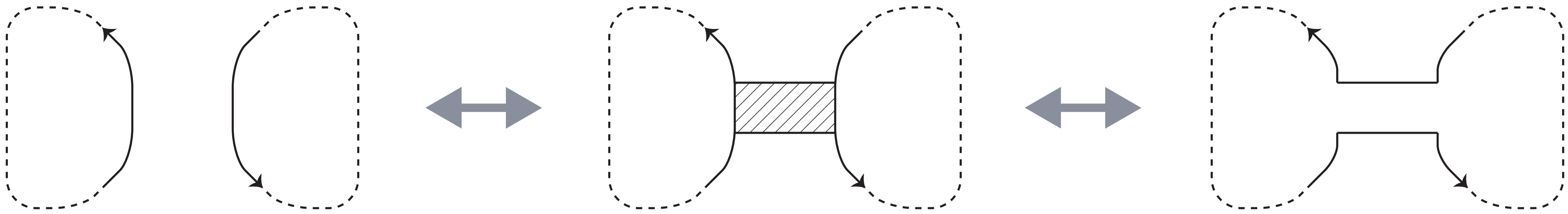


L $\qquad$ $L \cup b(I \times I)$ $\qquad$ L'

Coherent band surgery

If L , L' are oriented links and L' has the orientation compatible with the orientation of $L - b(I \times I) \cap L$ and $b(\partial I \times I)$, then L' is called the link obtained from L by the coherent band surgery along the band b. Then, $c(L) = c(L') \pm 1$, where $c(L)$ is the number of the components of L.

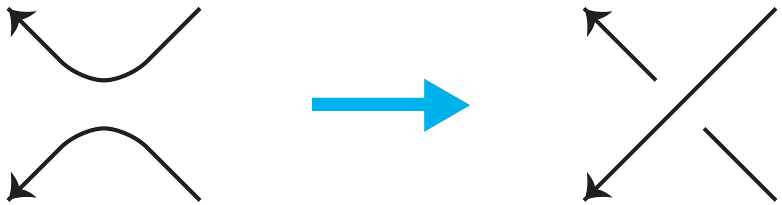
The case $c(L) > c(L')$:



L $\qquad$ $L \cup b(I \times I)$ $\qquad$ L'

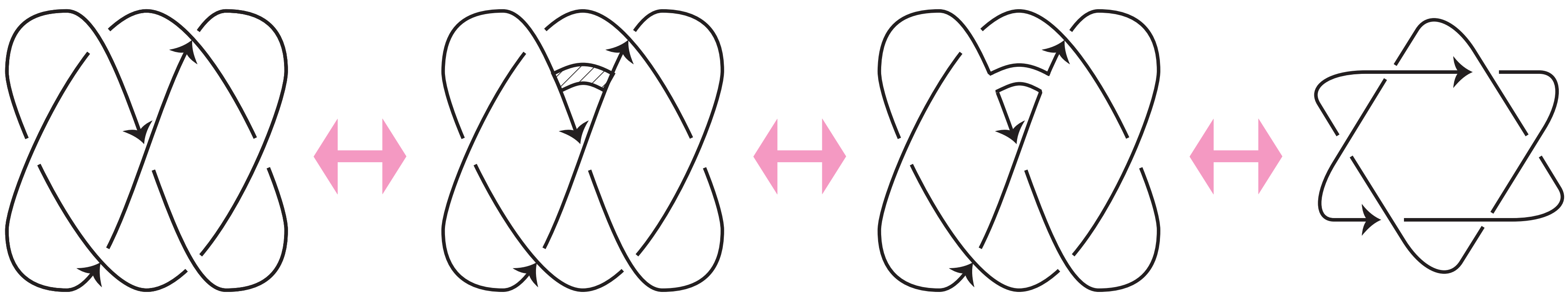
Motivation for the study of a band surgery

The Xer site-specific recombination system acts as



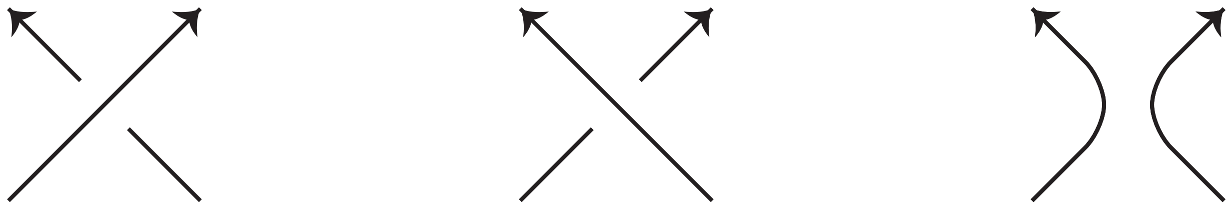
We may consider this action as band surgery.

Example. The following shows coherent band surgery between the knot 7_4 and the link 6_1^2 (torus link of type $(2, 6)$).



Skein triple

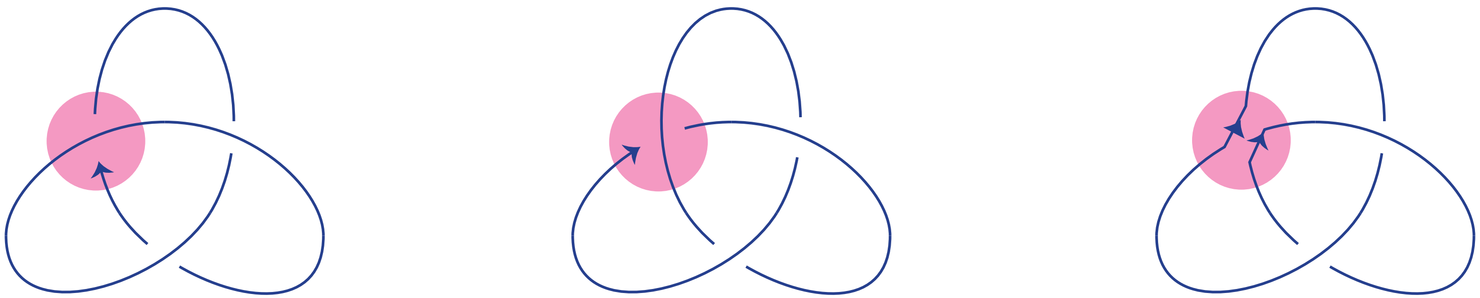
Let L_+, L_-, L_0 be three links that are identical except near one point where they are as in



L_+ $\qquad$ L_- $\qquad$ L_0.

We call (L_+, L_-, L_0) a skein triple.

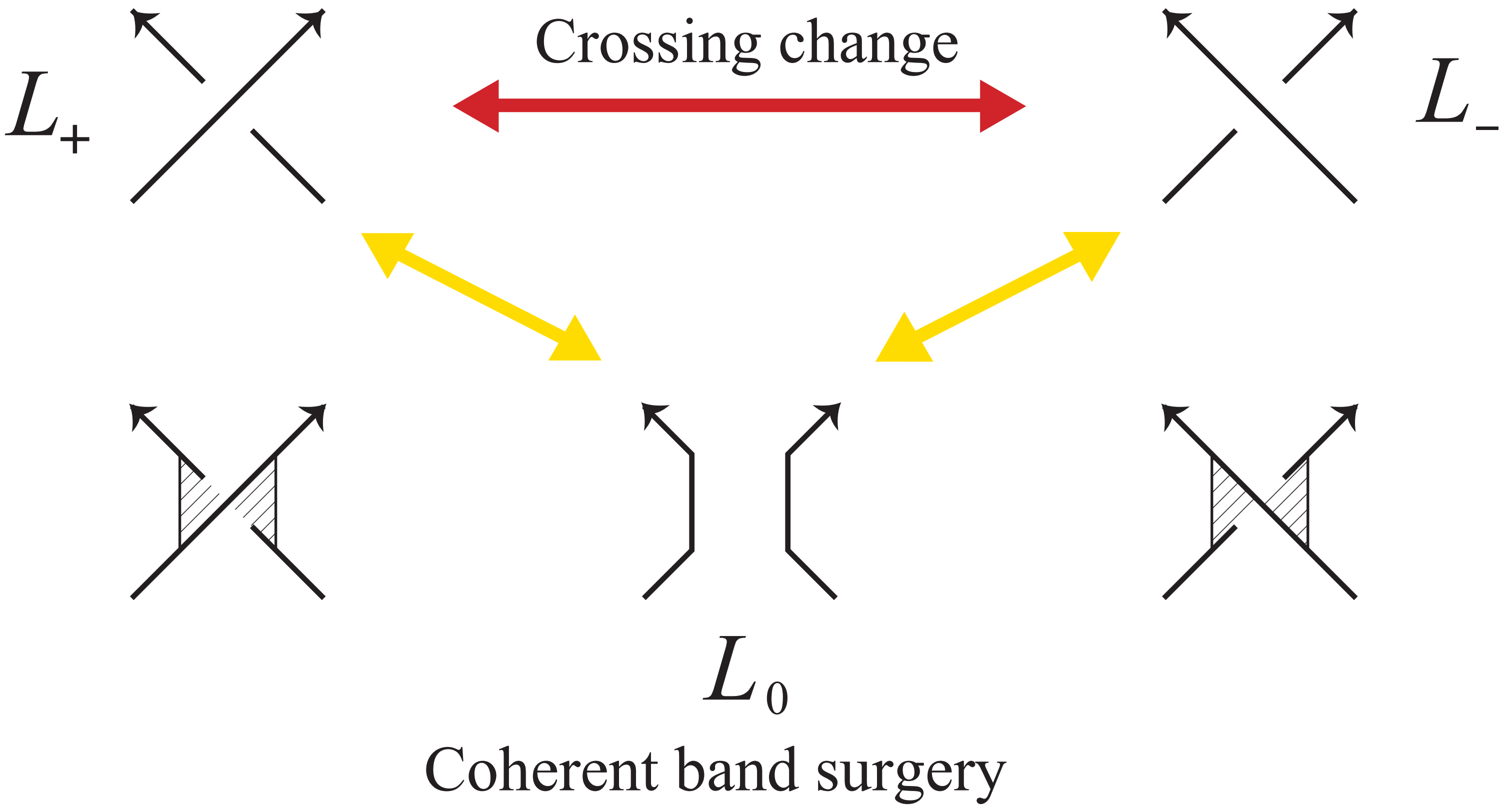
Example.



L_+ $\qquad$ L_- $\qquad$ L_0

Skein triple and coherent band surgery

Let (L_+, L_-, L_0) be a skein triple.
Then L_+ and L_- are related by a crossing change;
L_+ and L_0, L_- and L_0 are related by a coherent band surgery.



Nullification number

Any oriented link diagram can be deformed into a link diagram without any crossing by smoothing every crossing, which is considered to be a coherent band surgery.

The nullification number of an oriented link L, $\mathrm{n}(L)$, is the minimum number of the coherent band surgeries to deform L into some trivial link.

D. Sola, *Nullification number and flyping conjecture*, Rend. Sem. Mat. Univ. Padova, **86** (1991) 1–16.

Coherent band-Gordian distance

A trivial link is deformed into the trivial knot by a sequence of coherent band surgeries, and so we may define the following.

Let L and L' be oriented links.

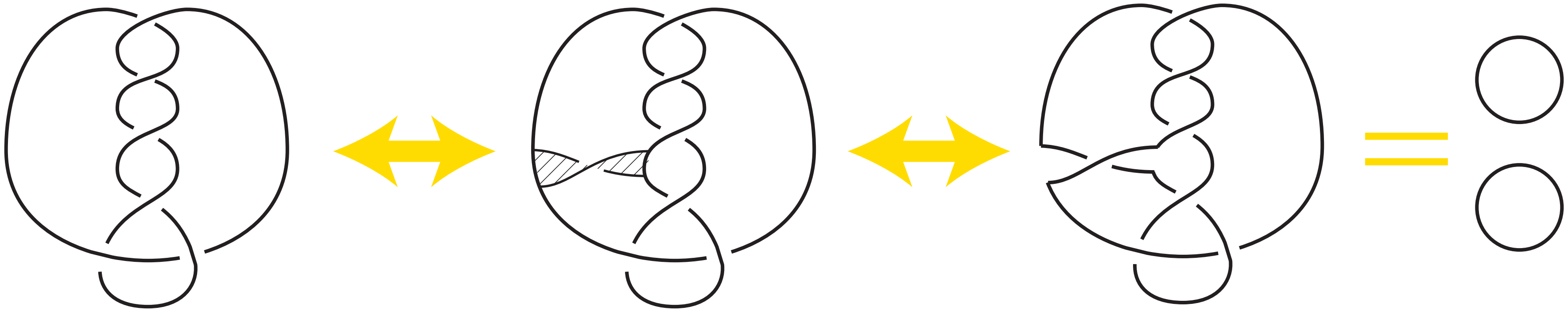
- The coherent band-Gordian distance from L to L', $\mathrm{d_{cb}}(L, L')$, is the minimum number of coherent band-surgeries that are necessary to deform L into L'.

- The coherent band-unknotting number of L, $\mathrm{u_{cb}}(L)$, is the coherent band-Gordian distance from L to the trivial knot; $\mathrm{u_{cb}}(L) = \mathrm{d_{cb}}(L, U)$.

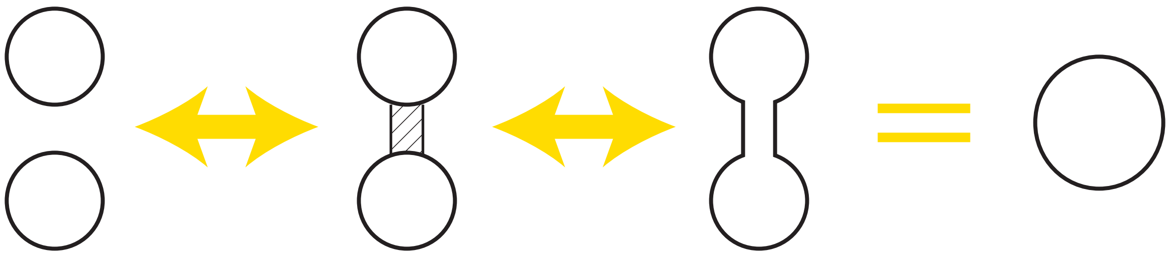
- $\mathrm{n}(L) \leq \mathrm{u_{cb}}(L)$. $\mathrm{d_{cb}}(L, L')$ =nullification distance.

Example. $\mathrm{n}(6_1) = 1$, $\mathrm{u_{cb}}(6_1) = 2$.

The knot 6_1 is deformed into the trivial 2-component link U^2 by a coherent band surgery; $\mathrm{n}(6_1) = \mathrm{d_{cb}}(6_1, U^r) = 1$ for some $r \geq 2$.



The trivial 2-component link is deformed into the trivial knot by a coherent band surgery, and so $\mathrm{u_{cb}}(6_1) = \mathrm{d_{cb}}(6_1, U) = 2$.



Tables of nullification numbers for knots and links

- C. Ernst, A. Montemayor, and A. Stasiak, *Nullification of small knots*, Progress of Theoretical Physics Supplement No. 191, (2011) 66–77.

  They settled the nullification number for all of the 84 prime knots with at most 9 crossings.

- A. Montemayor, *On nullification of knots and links*, Master's thesis, Western Kentucky Univ., 2012.

  He extended the table above up to knots with 10 crossings and links with 9 crossings.

- T. Kanenobu, *Band surgery on knots and links, II*, J. Knot Theory Ramifications 21 (2012), no. 9, 1250086.

  Band trivializability ($\mathrm{u}_{\mathrm{cb}}(L) = 1$ or > 0) for 2-component links with up to 9 crossings.

Tables of nullification distances (= d_{cb}) for knots and links

- K. Ishihara and D. Buck, *Nullification distance between links with small crossing numbers*, Proceedings of "Mathematics of Knots IV", 2011, 182–188.

  Tables of nullification distances between knots or links with up to 6 crossings.

- T. Kanenobu and M., Today's talk.

  We extend the table above for knots with 7 crossings.

Table of $\mathsf{d}_{\mathsf{cb}}(K, K')$ [Ishihara and Buck]

	U	3_1	$3_1!$	4_1	5_1	$5_1!$	5_2	$5_2!$	6_1	$6_1!$	6_2	$6_2!$	6_3	$3_1\#3_1$	$3_1!\#3_1!$	$3_1!\#3_1$
U	0	2	2	2	4	4	2	2	2	2	2	2	2	4	4	2
3_1		0	4	2	2	6	2	4	2	2	2	4	2	2	6	2
$3_1!$			0	2	6	2	4	2	2	2	4	2	2	6	2	2
4_1				0	4	4	2	2	2	2	2	2	2	4	4	4
5_1					0	8	2	6	4	4	2	6	4	2	8	4
$5_1!$						0	6	2	4	4	6	2	4	8	2	4
5_2							0	4	2	2	2	4	2	2	6	4
$5_2!$								0	2	2	4	2	2	6	2	4
6_1									0	2	2	2	2	4	4	2
$6_1!$										0	2	2	2	4	4	2
6_2											0	4	2	4	6	2
$6_2!$												0	2	6	4	2
6_3													0	4	4	2
$3_1\#3_1$														0	8	4
$3_1!\#3_1!$															0	4
$3_1!\#3_1$																0

Table of $\mathrm{d}_{\mathrm{cb}}(K, L)$ [Ishihara and Buck]

	U	3_1	$3_1!$	4_1	5_1	$5_1!$	5_2	$5_2!$	6_1	$6_1!$	6_2	$6_2!$	6_3	$3_1\#3_1$	$3_1!\#3_1!$	$3_1!\#3_1$
U^2	1	3	3	3	5	5	3	3	1	1	3	3	3	5	5	1
2_1^2	1	1	3	1	3	5	1	3	3	1	1	3	1	3	5	3
4_1^2	3	1	5	3	1	7	3	5	3	3	1	5	3	3	7	3
$4_1^{2'}$	1	3	3	3	5	3	3	1	3	1	3	3	3	5	3	3
5_1^2	3	1	3	1	3	5	3	3	3	3	1	3	1	3	5	3
$3_1\#2_1^2$	3	1	5	3	1	7	1	5	3	3	3	5	3	1	7	3
$3_1\#2_1^{2'}$	1	1	3	3	3	5	3	3	1/3	1/3	1	3	1	3	5	1
6_1^2	5	3	7	5	1	9	3	7	5	5	3	7	5	1	9	5
$6_1^{2'}$	1	5	1	3	5	3	3	3	1/3	1/3	3	3	1/3	5	3	1/3
6_2^2	3	1	5	3	1/3	7	1	5	3	3	3	5	3	3	7	3
6_3^2	3	3	5	3	3	7	1	5	3	3	3	5	3	1	7	3/5
$6_3^{2'}$	3	3	1	1	5	3	3	3	3	3	3	3	3	5	3	3
$4_1\#2_1^2$	1	3	3	1	3	5	3	3	1	3	1	3	3	3	5	3

New criteria for a coherent band surgery

We give necessary conditions for two links which are related by a band surgery or crossing change, using the determinant, the Jones polynomial, and the HOMFLYPT polynomial. They can improve these tables.

Improved table of $\mathsf{d}_{\mathsf{cb}}(K, K')$

	U	3_1	$3_1!$	4_1	5_1	$5_1!$	5_2	$5_2!$	6_1	$6_1!$	6_2	$6_2!$	6_3	$3_1\#3_1$	$3_1!\#3_1!$	$3_1!\#3_1$
U	0	2	2	2	4	4	2	2	2	2	2	2	2	4	4	2
3_1		0	4	2	2	6	2	4	4	2	2	4	2	2	6	2
$3_1!$			0	2	6	2	4	2	2	4	4	2	2	6	2	2
4_1				0	4	4	2	2	2	2	2	2	2	4	4	4
5_1					0	8	2	6	4	4	2	6	4	2	8	4
$5_1!$						0	6	2	4	4	6	2	4	8	2	4
5_2							0	4	2	2	2	4	2	2	6	4
$5_2!$								0	2	2	4	2	2	6	2	4
6_1									0	2	2	2	2	4	4	2
$6_1!$										0	2	2	2	4	4	2
6_2											0	4	2	4	6	2
$6_2!$												0	2	6	4	2
6_3													0	4	4	2
$3_1\#3_1$														0	8	4
$3_1!\#3_1!$															0	4
$3_1!\#3_1$																0

Improved table of $\mathrm{d_{cb}}(K, L)$

	U	3_1	$3_1!$	4_1	5_1	$5_1!$	5_2	$5_2!$	6_1	$6_1!$	6_2	$6_2!$	6_3	$3_1\#3_1$	$3_1!\#3_1!$	$3_1!\#3_1$
U^2	1	3	3	3	5	5	3	3	1	1	3	3	3	5	5	1
2_1^2	1	1	3	1	3	5	1	3	3	1	1	3	1	3	5	3
4_1^2	3	1	5	3	1	7	3	5	3	3	1	5	3	3	7	3
$4_1^{2'}$	1	3	3	3	5	3	3	1	3	1	3	3	3	5	3	3
5_1^2	3	1	3	1	3	5	3	3	3	3	1	3	1	3	5	3
$3_1\#2_1^2$	3	1	5	3	1	7	1	5	3	3	3	5	3	1	7	3
$3_1\#2_1^{2'}$	1	1	3	3	3	5	3	3	3	3	1	3	1	3	5	1
6_1^2	5	3	7	5	1	9	3	7	5	5	3	7	5	1	9	5
$6_1^{2'}$	1	5	1	3	5	3	3	3	3	3	3	1/3	1/3	5	3	3
6_2^2	3	1	5	3	1	7	1	5	3	3	3	5	3	3	7	3
6_3^2	3	3	5	3	3	7	1	5	3	3	3	5	3	1	7	3/5
$6_3^{2'}$	3	3	1	1	5	3	3	3	3	3	3	3	3	5	3	3
$4_1\#2_1^2$	1	3	3	1	3	5	3	3	1	3	1	3	3	3	5	3

Known methods to estimate $\mathrm{d_{cb}}(L, L')$

Signature: For links L, L', $|\sigma(L) - \sigma(L')| \leq \mathrm{d_{cb}}(L, L')$.

Let L, L' be links with $\mathrm{d_{cb}}(L, L') = 1$.

Arf invariant: If L is a 2-component link with $\mathrm{lk}(L)$ even and L' is a knot, then $\mathrm{Arf}(L) = \mathrm{Arf}(L')$.

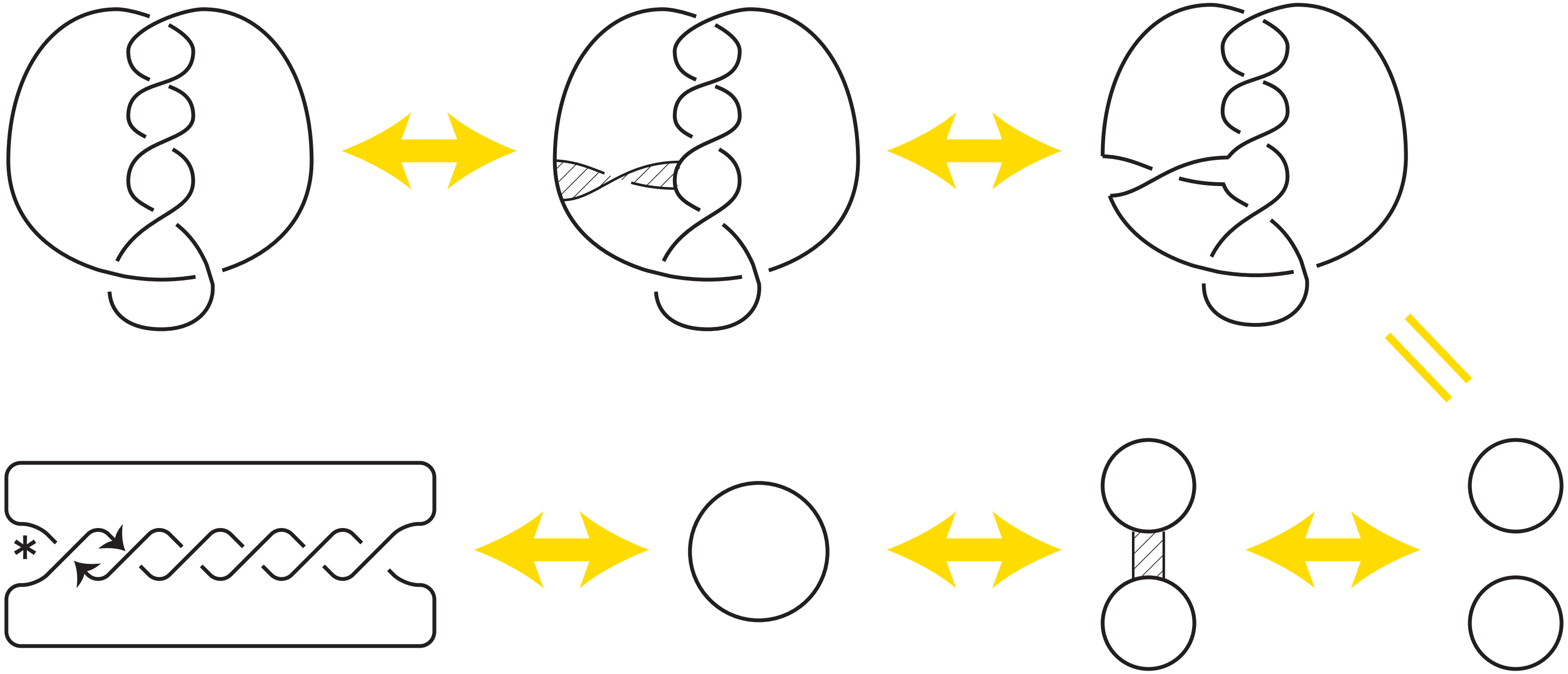
Jones polynomial: $V(L;\omega)/V(L';\omega) \in \left\{\pm i, -\sqrt{3}^{\pm 1}\right\}$, where $\omega = e^{i\pi/3}$.

Q polynomial: $\rho(L)/\rho(L') \in \left\{ \pm 1, \sqrt{5}^{\pm 1} \right\}$, where $\rho(L) = Q\left(L; \frac{\sqrt{5}-1}{2}\right)$.

HOMFLYPT polynomial: $P(L; i, i)/P(L'; i, i) \in \left\{1, (-2)^{\pm 1}\right\}$.

Question. $\mathrm{d_{cb}}(6_1, 6_1^{2\prime}) = 1$ or 3 ?

- $\mathrm{d_{cb}}(6_1, 6_1^{2\prime}) \leq 3$; $6_1 \leftrightarrow U^2 \leftrightarrow U \leftrightarrow 6_1^{2\prime}$.



- $\sigma(6_1) = 0$, $\sigma(6_1^{2\prime}) = -1 \Rightarrow \mathrm{d_{cb}}(6_1, 6_1^{2\prime}) \geq 1$
- $\mathrm{lk}(6_1^{2\prime}) = 3 \Rightarrow 6_1^{2\prime}$ does not have the Arf invariant.

Proof of $\mathrm{d_{cb}}(6_1, 6_1^{2'}) > 1$ (1)

Suppose that $\mathrm{d_{cb}}(6_1, 6_1^{2'}) = 1$.
$\Rightarrow \exists$ a knot K such that $(6_1, K, 6_1^{2'})$ is a skein triple.
$\Rightarrow$ The knot K and 6_1 are related by a crossing change.
Since 6_1 is a 2-bridge knot of Schubert's normal form $S_{9,2}$ and $\det(K) = 21$,
we may use a result of Hitoshi Murakami:

Proposition 1. *Suppose that a knot K is obtained from a 2-bridge knot $S_{p,q}$ by a crossing change. Then there exists an integer s such that:*

$$|\det(K) - p|/2 \equiv \pm q s^2 \pmod{p}.$$

Corollary 2.8 in H. Murakami, *Some metrics on classical knots*,
Math. Ann. **270** (1985) 35–45.

Then we obtain: $\exists s \in \boldsymbol{Z}$ such that $6 = |21 - 9|/2 \equiv \pm 2s^2 \pmod{9}$.
We can put $s = 3n$, $n \in \boldsymbol{Z}$.
Then $6 \equiv \pm 18n^2 \equiv 0 \pmod{9}$; a contradiction.

Determinant of a knot

$\det(K) = |H_1(\Sigma(K); \boldsymbol{Z})|$, where $\Sigma(K)$ is the double cover of S^3 branched over K.

$\det(K) = |\nabla(K; 2i)| = |V(K; -1)|$,
where $\nabla(K; 2i)$ is the Conway polynomial of K, $\nabla(K; z)$, at $z = 2i$,
and $V(K; -1)$ is the Jones polynomial of K, $V(K; t)$, at $t = -1$.

For a skein triple $(6_1, K, 6_1^{2'})$, we have

$$\nabla(6_1) - \nabla(K) = z\nabla(6_1^{2'}).$$

Then, $\nabla(K) = \nabla(6_1) - z\nabla(6_1^{2'}) = 1 - 2z^2 - z(3z) = 1 - 5z^2$.

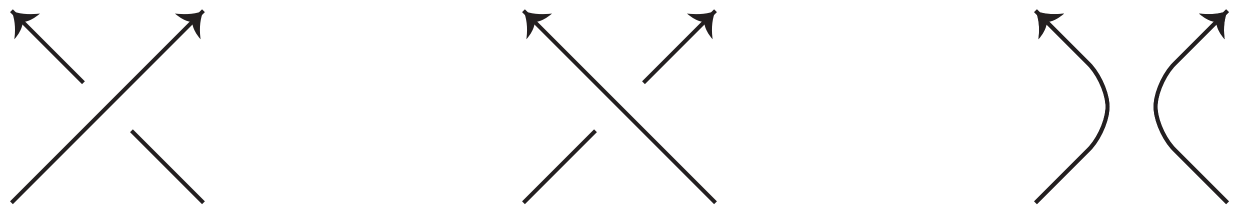
$\Rightarrow \det(K) = |\nabla(K; 2i)| = 21$.

Conway polynomial $\nabla(L) = \nabla(L; z) \in \boldsymbol{Z}[z]$

An invariant of the isotopy type of an oriented link L defined by:

- $\nabla(U) = 1$;
- $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$,

where U is the unknot and (L_+, L_-, L_0) is a skein triple.



L_+ L_- L_0

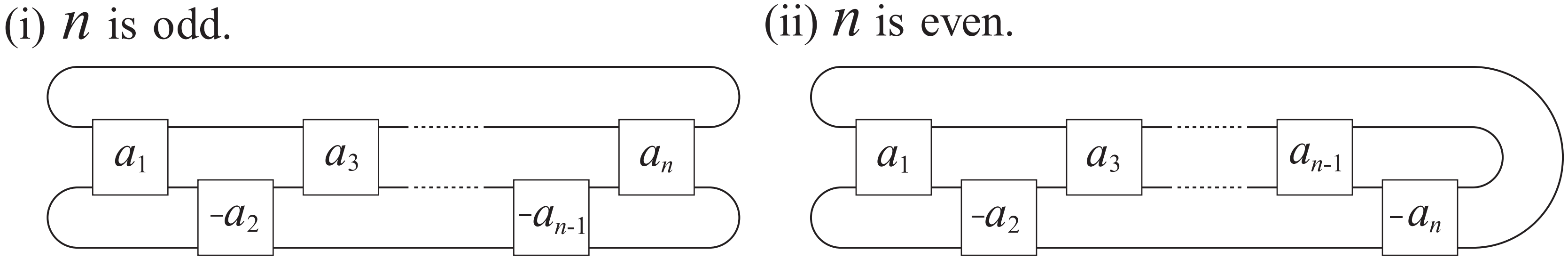
A criterion for $\mathrm{d}_{\mathrm{cb}}(K, L)$ with $K = S_{p,q}$

Theorem 1. *Let L be a 2-component link and K a 2-bridge knot $S_{p,q}$. If $\mathrm{d}_{\mathrm{cb}}(K, L) = 1$, then $\exists s \in \boldsymbol{Z}$ such that*

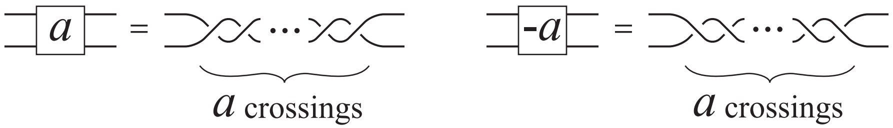
$$\det(L) \equiv \pm q s^2 \pmod{p}.$$

Here, $S_{p,q}$ denotes the 2-bridge knot for which the lens space of type (p, q) is the 2-fold branched cover of S^3, where p, q are relatively prime integers with $p > q > 0$ and p odd.

A 2-bridge knot in Conway's normal form $C(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$:



Here,



Let a_1, a_2, $a_3, \ldots, a_n$ be positive integers obtained from the continued fraction

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

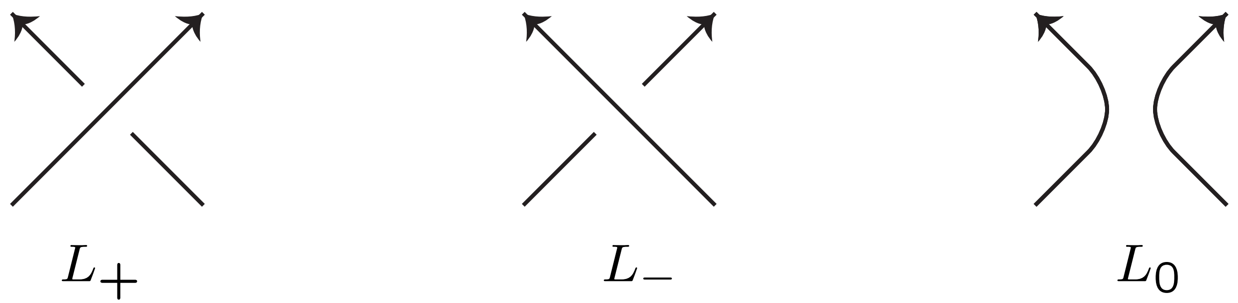
Then $S_{p,q}$ is isotopic to $C(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.
Also, $S_{p,-q}$ presents the mirror image of $S_{p,q}$.

Jones polynomial $V(L) = V(L;t) \in \boldsymbol{Z}[t^{\pm 1/2}]$

An invariant of the isotopy type of an oriented link L defined by:

- $V(U) = 1$;
- $t^{-1}V(L_+) - tV(L_-) = \left(t^{1/2} - t^{-1/2}\right)V(L_0)$,

where U is the unknot and (L_+, L_-, L_0) is a skein triple:



The value of the Jones polynomial at $t = e^{i\pi/3}$

For a c-component link L, Lickorish and Millett have shown:

$$V(L; e^{i\pi/3}) = \pm i^{c-1}(i\sqrt{3})^{\delta},$$

where $\delta = \dim H_1(\Sigma(L); \boldsymbol{Z}_3)$ with $\Sigma(L)$ the double cover of S^3 branched over L.

$V(L; e^{i\pi/3})$ means $V(L; t)$ at $t^{1/2} = e^{i\pi/6}$.

W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helv. **61** (1986), 349–359.

The value of the Jones polynomial at $t = e^{i\pi/3}$

Put $\omega = e^{i\pi/3}$.

Cases	$V(L_+;\omega)/V(L_-;\omega)$	$V(L_0;\omega)/V(L_-;\omega)$	$V(L_0;\omega)/V(L_+;\omega)$
(a)	1	$-\sqrt{3}$	$-\sqrt{3}$
(b)	-1	i	$-i$
(c)	$i\sqrt{3}$	$-i$	$-\sqrt{3}^{-1}$
(d)	$i\sqrt{3}^{-1}$	$-\sqrt{3}^{-1}$	i

Proposition 2. *If two links L and L' are related by a band surgery, then*

$$V(L;\omega)/V(L';\omega) \in \left\{\pm i,\ -\sqrt{3}^{\pm 1}\right\}.$$

T. Kanenobu, *Band surgery on knots and links*,
J. Knot Theory Ramifications **19** (2010) 1535–1547.

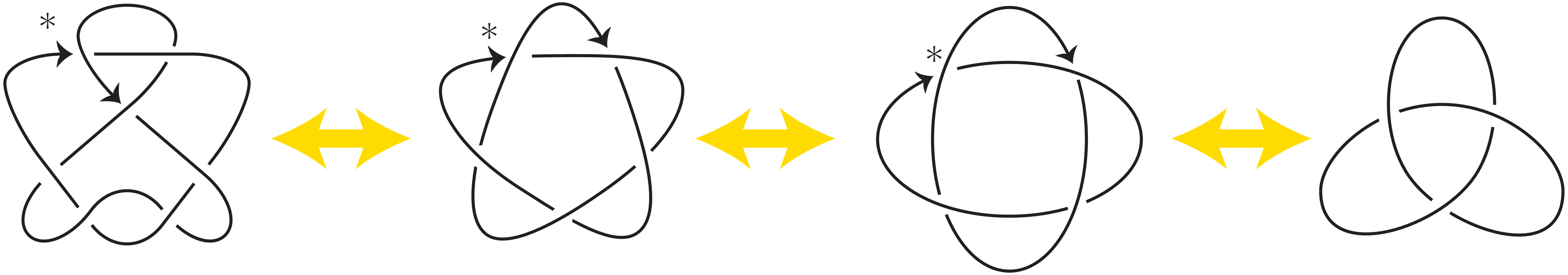
Example. $\mathsf{d_{cb}}(3_1, 7^2_1) = 3.$

- $\sigma(3_1) = 2$, $\sigma(7^2_1) = 3 \Rightarrow \mathsf{d_{cb}}(3_1, 7^2_1) \geq 1.$

- $V(3_1; \omega) = -i\sqrt{3}, \quad V(7^2_1; \omega) = i,$

  $V(3_1; \omega)/V(7^2_1; \omega) = \sqrt{3} \Rightarrow \mathsf{d_{cb}}(3_1, 7^2_1) > 1.$

- $\mathsf{d_{cb}}(3_1, 7^2_1) \leq 3; \qquad 7^2_1 \leftrightarrow 5_1 \leftrightarrow 4^2_1 \leftrightarrow 3_1.$



Proof of $\mathsf{d}_{\mathsf{cb}}(6_1, 6_1^{2'}) > 1$ **(2)**

Suppose that $\mathsf{d}_{\mathsf{cb}}(6_1, 6_1^{2'}) = 1$.
$\Rightarrow$ $\exists$ a knot K such that $(6_1, K, 6_1^{2'})$ is a skein triple (L_+, L_-, L_0).

Cases	$V(L_+;\omega)/V(L_-;\omega)$	$V(L_0;\omega)/V(L_-;\omega)$	$V(L_0;\omega)/V(L_+;\omega)$
(a)	1	$-\sqrt{3}$	$-\sqrt{3}$
(b)	-1	i	$-i$
(c)	$i\sqrt{3}$	$-i$	$-\sqrt{3}^{-1}$
(d)	$i\sqrt{3}^{-1}$	$-\sqrt{3}^{-1}$	i

$V(6_1;\omega) = i\sqrt{3}$, $V(6_1^{2'};\omega) = -\sqrt{3}$. $V(6_1^{2'};\omega)/V(6_1;\omega) = i$.
$\Rightarrow$ $V(6_1^{2'};\omega)/V(K;\omega) = -\sqrt{3}^{-1}$. $V(K;\omega) = (-\sqrt{3})^2 = 3$.
$\Rightarrow$ $\dim H_1(\Sigma; \boldsymbol{Z}_3) = 2$, Σ is the double cover of S^3 branched over K.
$\Rightarrow$ $\det(K) \equiv 0 \pmod{3^2}$.

On the other hand,
$\nabla(K) = \nabla(6_1) - z\nabla(6_1^{2'}) = 1 - 2z^2 - z(3z) = 1 - 5z^2$.
$\Rightarrow$ $\det(K) = |\nabla(K; 2i)| = 21 \not\equiv 0 \pmod{3^2}$; a contradiction.

A criterion for $\mathrm{d_{cb}}(K, L)$

Note that $\det(K) = |\nabla(K; 2i)| = |V(K; -1)|$.

Theorem 2. *Suppose that a 2-component link L is obtained from a knot K by a coherent band surgery.*

$V(L;\omega) = \eta i V(K;\omega) = \pm i(i\sqrt{3})^{\delta}$, $\eta = \pm 1$

$\Rightarrow iV(L;-1) \equiv \eta V(K;-1) \pmod{3^{\delta+1}}$.

Using either Theorem 1 or Theorem 2, we may prove:

$$\mathrm{d_{cb}}(6_1!, 6_1^{2'}) = \mathrm{d_{cb}}(6_1, 3_1\#2_1^2) = \mathrm{d_{cb}}(6_1!, 3_1\#2_1^{2'}) = 3,$$

where 2_1^2, $2_1^{2'}$ are negative, positive Hopf links.

Note. $V(6_1!;\omega) = i\sqrt{3}$, $V(6_1!;-1) = 9$.

$V(6_1^{2'};\omega) = -\sqrt{3}$, $V(6_1^{2'};-1) = -6i$.

$V(3_1\#2_1^2;\omega) = \sqrt{3}$, $V(3_1\#2_1^2;-1) = -6i$.

$V(3_1\#2_1^{2'};\omega) = -\sqrt{3}$, $V(3_1\#2_1^{2'};-1) = 6i$.

Moreover, using Theorem 2, we may prove $d_{cb}(6_1, 3_1) = d_{cb}(6_1!, 3_1!) = 4$.

Suppose that $d_{cb}(6_1, 3_1) = 2$.
$\Rightarrow$ $\exists$ a link L such that either $(6_1, 3_1, L)$ or $(3_1, 6_1, L)$ is a skein triple (L_+, L_-, L_0).

Cases	$V(L_+;\omega)/V(L_-;\omega)$	$V(L_0;\omega)/V(L_-;\omega)$	$V(L_0;\omega)/V(L_+;\omega)$
(a)	1	$-\sqrt{3}$	$-\sqrt{3}$
(b)	-1	i	$-i$
(c)	$i\sqrt{3}$	$-i$	$-\sqrt{3}^{-1}$
(d)	$i\sqrt{3}^{-1}$	$-\sqrt{3}^{-1}$	i

$V(3_1;\omega) = -i\sqrt{3}$, $V(6_1;\omega) = i\sqrt{3}$. $\quad V(6_1;\omega)/V(3_1;\omega) = -1$.
$\Rightarrow V(L;\omega)/V(3_1;\omega) = \pm i$. $\Rightarrow V(L;\omega) = \pm\sqrt{3}$. $\Rightarrow \delta = 1$.

Here, $V(3_1;-1) = 3$, $V(6_1;-1) = 9$.

$$\begin{cases} iV(L;-1) \equiv \pm V(3_1;-1) \equiv 3 \text{ or } 6 \pmod{3^2}, \\ iV(L;-1) \equiv \pm V(6_1;-1) \equiv 0 \pmod{3^2}. \end{cases} \quad \text{; a contradiction.}$$

- $d_{cb}(6_1, 3_1) \leq 4$; $\quad 6_1 \leftrightarrow U^2 \leftrightarrow U \leftrightarrow 2_1^2 \leftrightarrow 3_1$.

Question. $\mathrm{d_{cb}}(5_1, 6^2_2) = 1$ or 3 ?

$\sigma(5_1) = 4$, $\sigma(6^2_2) = 3$.
Suppose that $\mathrm{d_{cb}}(5_1, 6^2_2) = 1$.
$\Rightarrow$ (i) $\exists$ a knot K such that $(5_1, K, 6^2_2)$ is a skein triple;
and (ii) $\exists$ a knot J such that $(J, 5_1, 6^2_2)$ is a skein triple.

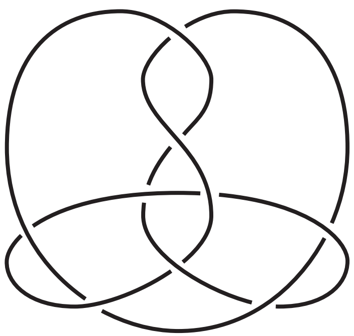
$\Rightarrow$ (i) $\nabla(5_1) - \nabla(K) = z\nabla(6^2_2)$.

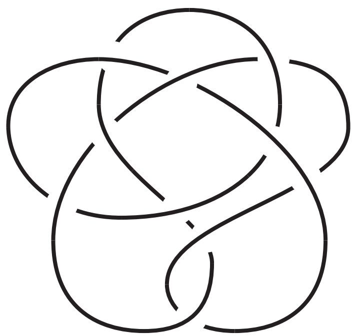
$\nabla(5_1) = 1 + 3z^2 + z^4$, $\nabla(6^2_2) = -3z - 2z^3$.

$\nabla(K) = 1 + 3z^2 + z^4 - z(-3z - 2z^3) = 1 + 6z^2 + 3z^4 = \nabla(9_{49})$.

(ii) $\nabla(J) = 1 + 3z^2 + z^4 + z(-3z - 2z^3) = 1 - z^4 = \nabla(10_{136})$.

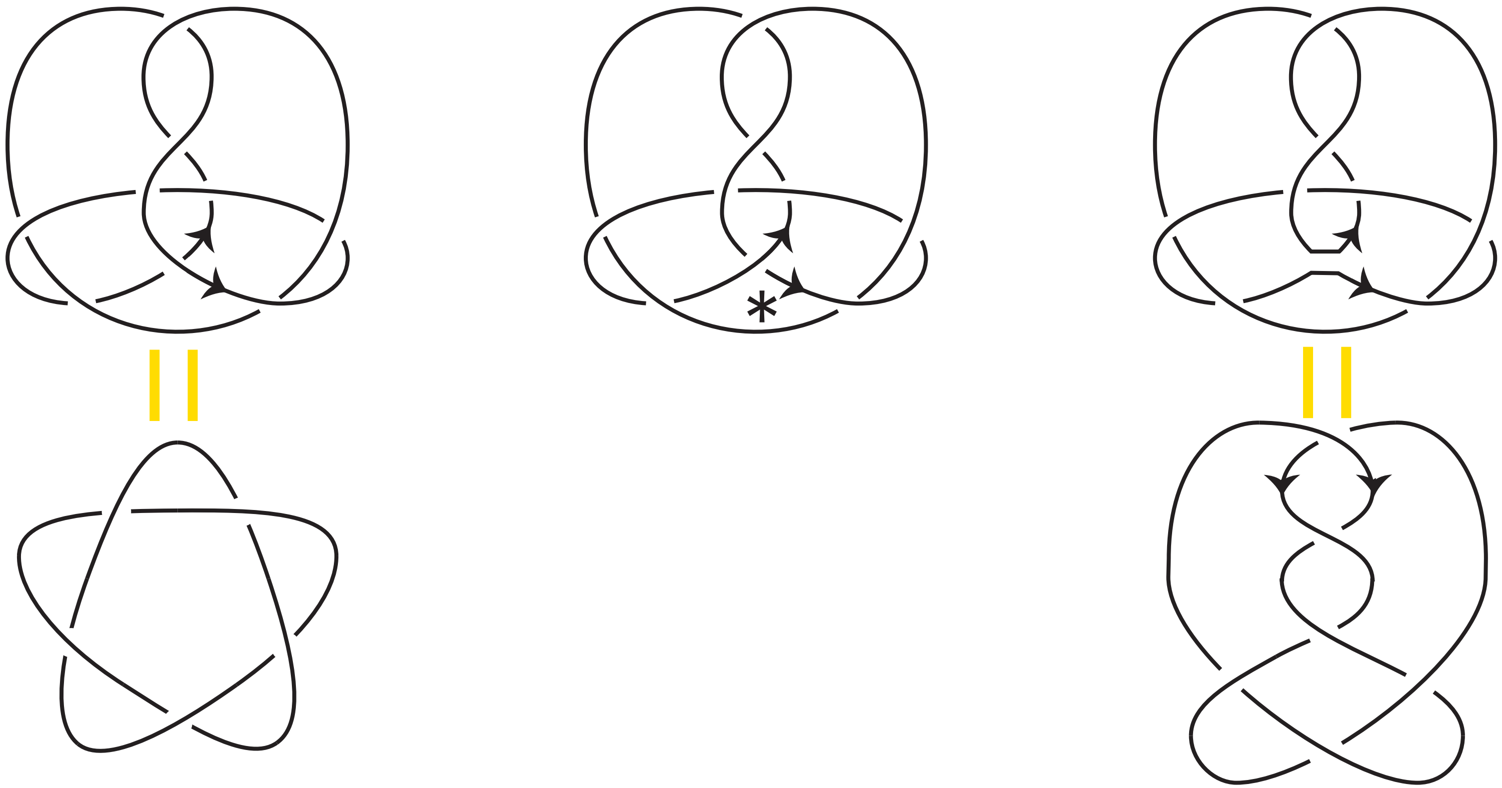
Examine the knots 9_{49}, 10_{136}.





Proof of $\mathrm{d}_{\mathrm{cb}}(5_1, 6^2_2) = 1$

In fact, $\exists$ a skein triple $(5_1, 9_{49}!, 6^2_2)$.



Therefore, $\mathrm{d}_{\mathrm{cb}}(5_1, 6^2_2) = \boxed{1}$.

HOMFLYPT polynomial $P(L) = P(L; v, z) \in \boldsymbol{Z}[v^{\pm 1}, z^{\pm 1}]$

An invariant of the isotopy type of an oriented link L defined by:

- $P(U) = 1$;
- $v^{-1}P(L_+) - vP(L_-) = zP(L_0)$,

where U is the unknot and (L_+, L_-, L_0) is a skein triple.

The value of the HOMFLYPT polynomial at $v = z = i$

For a link L, Lickorish and Millett have shown:

$$P(L; i, i) = (i\sqrt{2})^{\tau},$$

where $\tau = \dim H_1(\Sigma_3(L); \boldsymbol{Z}_2)$ with $\Sigma_3(L)$ the 3-fold cyclic cover of S^3 branched over L.

W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helv. **61** (1986), 349–359.

Put $v = z = i$, we obtain $-iP(L_+; i, i) - iP(L_-; i, i) = iP(L_0; i, i)$.

Therefore, $P(L_+; i, i) + P(L_-; i, i) + P(L_0; i, i) = 0$.

Cases	$P(L_+; i, i)/P(L_-; i, i)$	$P(L_0; i, i)/P(L_-; i, i)$	$P(L_0; i, i)/P(L_+; i, i)$
(a)	1	-2	$-1/2$
(b)	-2	1	$-1/2$
(c)	$-1/2$	$-1/2$	1

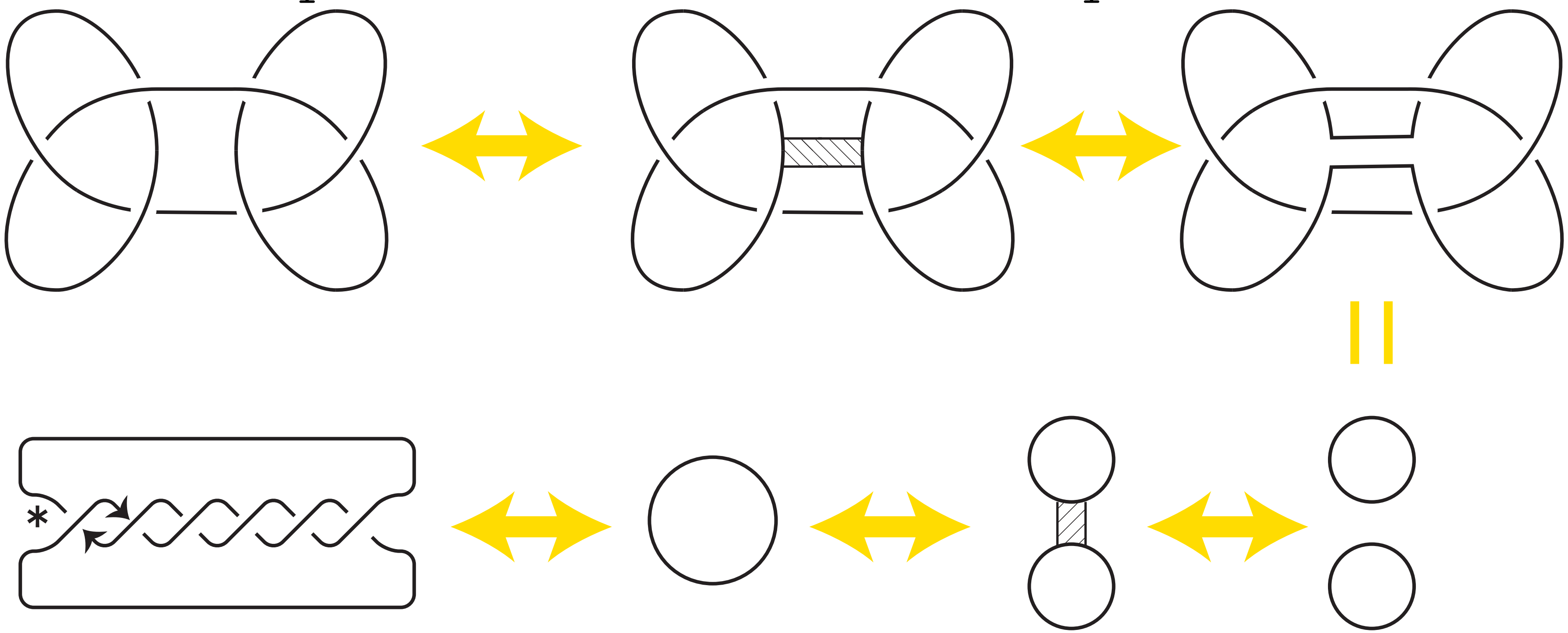
Proposition 3. *If two links L and L' are related by a coherent band surgery, then*

$$P(L; i, i)/P(L'; i, i) \in \left\{1,\ (-2)^{\pm 1}\right\}.$$

T. Kanenobu, *Band surgery on knots and links, II*, J. Knot Theory Ramifications 21 (2012), no. 9, 1250086.

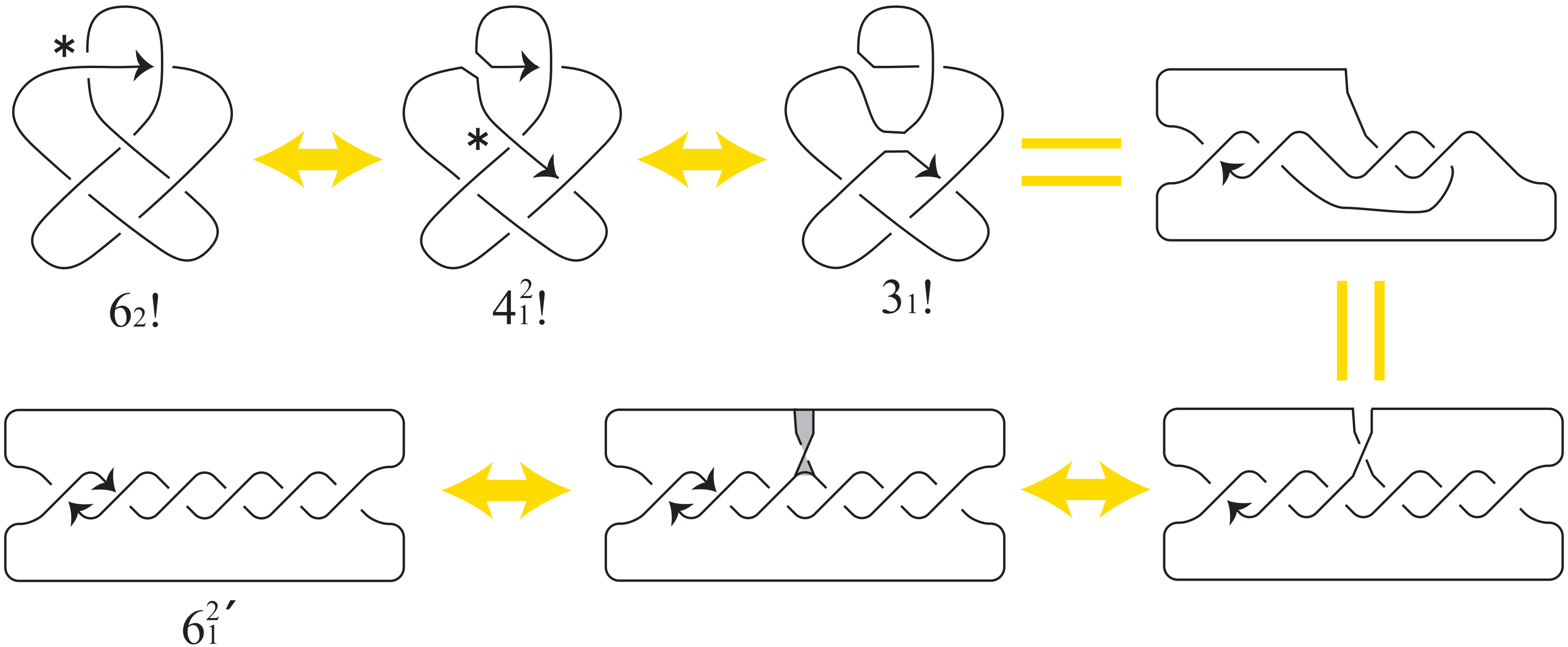
Example. $\mathsf{d_{cb}}(3_1!\#3_1, 6_1^{2'}) = 3$.

- $\sigma(3_1!\#3_1) = 0$, $\sigma(6_1^{2'}) = -1 \Rightarrow \mathsf{d_{cb}}(3_1!\#3_1, 6_1^{2'}) \geq 1$.
- $P(3_1!\#3_1; i, i) = 4$, $\quad P(6_1^{2'}; i, i) = 1$,
  $P(3_1!\#3_1; i, i)/P(6_1^{2'}; i, i) = 4 \Rightarrow \mathsf{d_{cb}}(3_1, 7_1^2) > 1$.
- $\mathsf{d_{cb}}(3_1!\#3_1, 6_1^{2'}) \leq 3$; $\qquad 3_1!\#3_1 \leftrightarrow U^2 \leftrightarrow U \leftrightarrow 6_1^{2'}$



Question. $d_{cb}(6_2!, 6_1^{2\prime}) =$ 1 or 3 ?

- $d_{cb}(6_2!, 6_1^{2\prime}) \leq 3$; $6_2! \leftrightarrow 4_1^2! \leftrightarrow 3_1! \leftrightarrow 6_1^{2\prime}$



A. Kawauchi showed the following:

Proposition 4. *Let* ${T_{2k}}'$ *be anti-parallel* $(2,2k)$*-torus link.*
If $\mathrm{d_{cb}}(K, {T_{2k}}') = 1$ *for a knot* K*, then* $\exists$ *polynomial* $f(t)$ *such that*

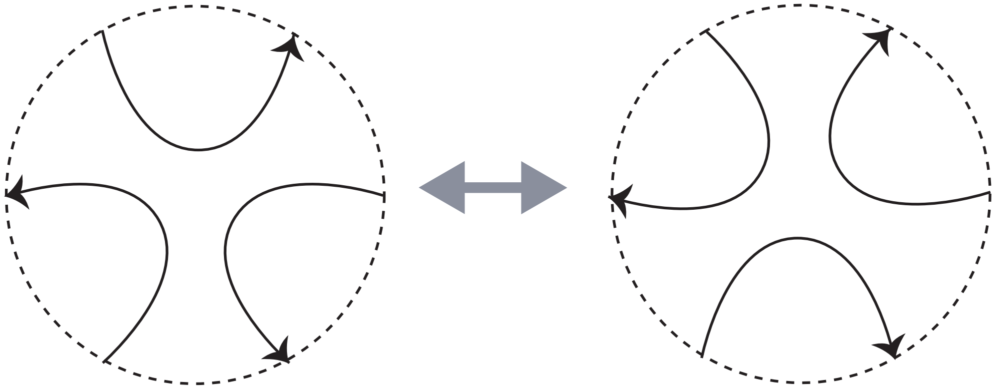
$$\Delta_K(t) \equiv \pm t^r f(t) f(t^{-1}) \pmod{k},$$

where $\Delta_K(t)$ *is the Alexander polynomial of* K*,* r *is some integer.*

By using Proposition 4, we $\mathrm{d_{cb}}(7_6, {T_6}') > 1$. However, we can not detect $\mathrm{d_{cb}}(6_2, {T_6}') > 1$.

$SH(3)$-**move** [Hoste, Nakanishi and Taniyama]

An $SH(3)$-move is a local change for an oriented link diagram which preserves the number of components as follows:



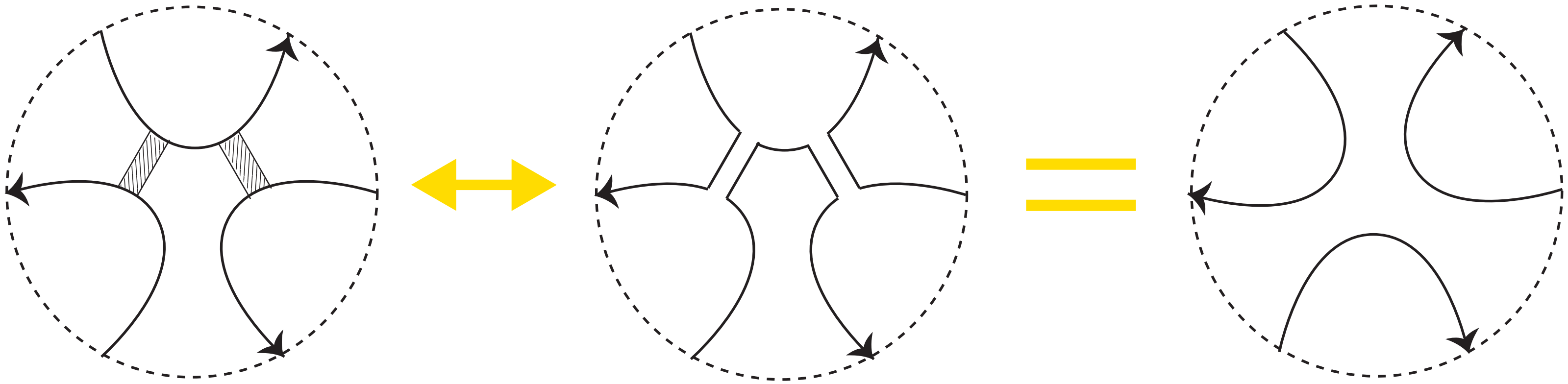
Remark. This move is unknotting operation.

- The $SH(3)$-Gordian distance from L to L', $\mathrm{sd}_3(L, L')$, is the minimum number of $SH(3)$-moves that are necessary to deform L into L'.

- The $SH(3)$-unknotting number of K, $\mathrm{su}_3(K)$, is the $SH(3)$-Gordian distance from K to the trivial knot; $\mathrm{su}_3(K) = \mathrm{sd}_3(K, U)$.

$SH(3)$-move and coherent band surgery

An $SH(3)$-move is realized by a sequence of two coherent band surgeries.



Proposition 5. *For two oriented knots K and K',*

$$\mathrm{d}_{\mathrm{cb}}(K, K') = 2\mathrm{sd}_3(K, K').$$

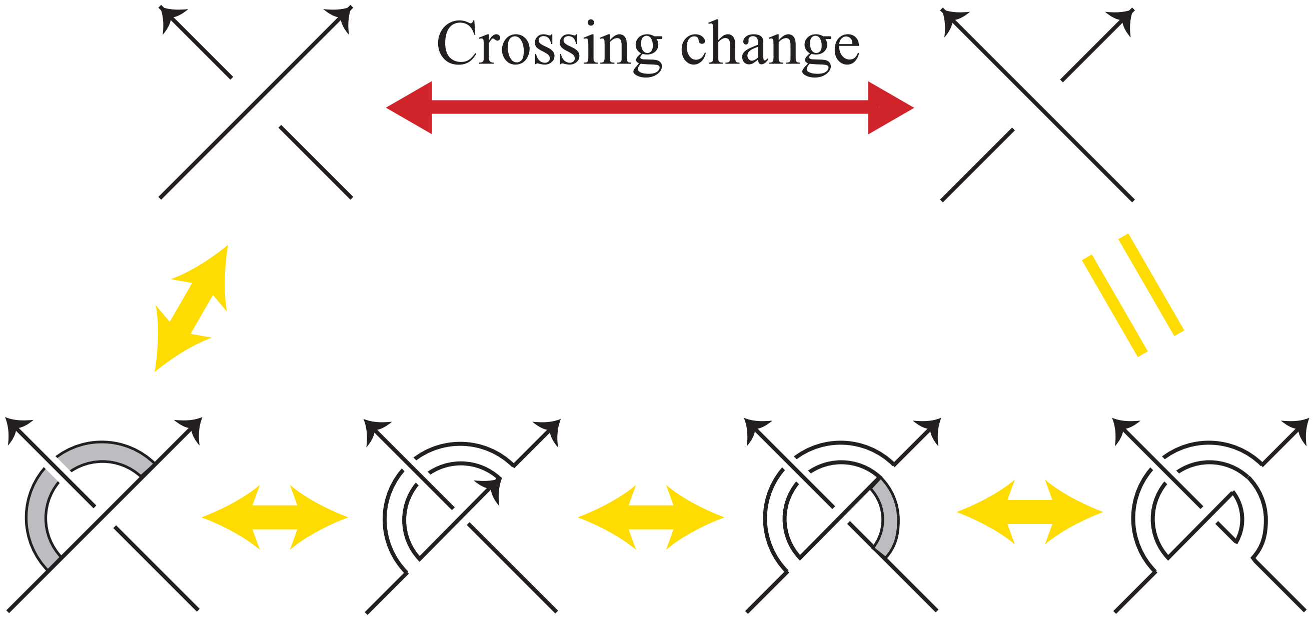
In particular,

$$\mathrm{u}_{\mathrm{cb}}(K) = 2\mathrm{su}_3(K).$$

T. Kanenobu, *SH(3)-move and other local moves on knots*,
http://www.sci.osaka-cu.ac.jp/math/OCAMI/.
Table for $\mathrm{su}_3(K)$ of knots with up to 9 crossings.

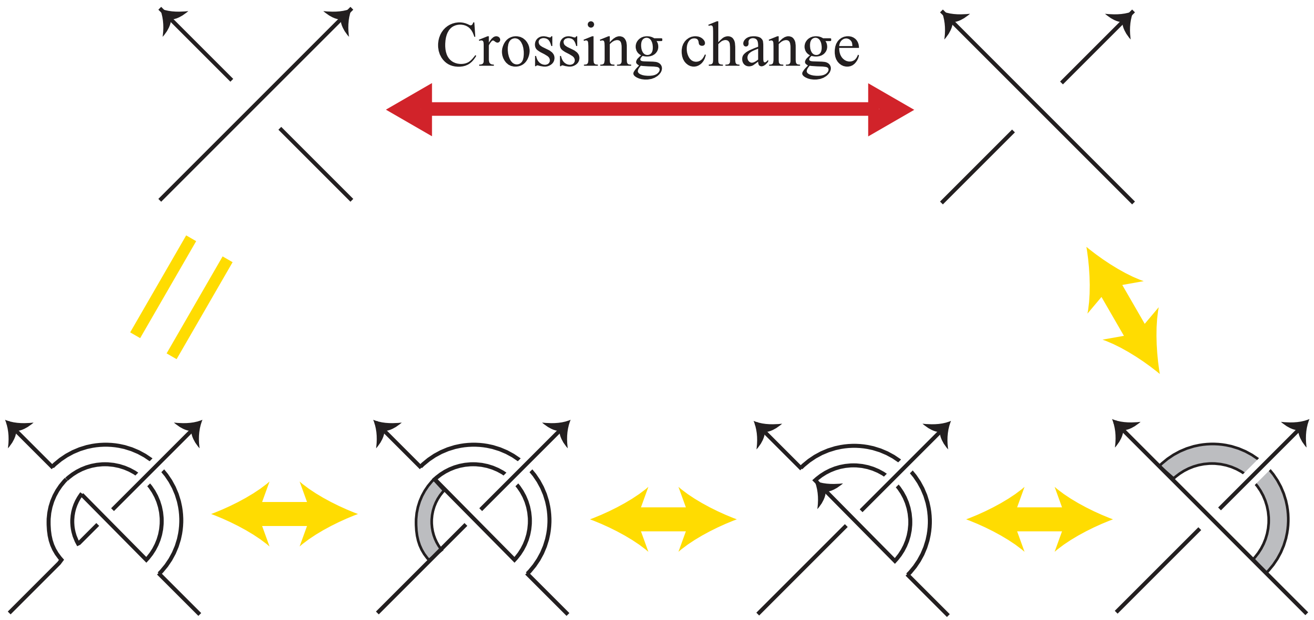
Crossing change and coherent band surgery

An crossing change is realized by a sequence of two coherent band surgeries.



Crossing change and coherent band surgery

An crossing change is realized by a sequence of two coherent band surg-eries.



How to make a table of $\mathrm{d_{cb}}(K, K')$

I. $|\sigma(K) - \sigma(K')| \leq \mathrm{d_{cb}}(K, K') \leq 2\mathrm{d}_G(K, K')$ (←Gordian distance)
or
$|\sigma(K) - \sigma(K')| \leq \mathrm{d_{cb}}(K, K') \leq 2\mathrm{su}_3(K) + 2\mathrm{su}_3(K')$.

II. Find a link L with $\mathrm{d_{cb}}(K, L) = \mathrm{d_{cb}}(K', L) = 1$ by applying crossing change / smoothing.

III. If $V(K;\omega)/V(K';\omega) \notin \{\pm 1, \pm i\sqrt{3}^{\pm 1}, 3^{\pm 1}\}$, then $\mathrm{d_{cb}}(K, K') > 2$.
($\because$ Proposition 2.)

IV. If $\mathrm{d_{cb}}(K, K') = 2$ and $V(K, \omega) = -V(K'; \omega) = \pm(i\sqrt{3})^{\delta}$, then

$$V(K; -1) \equiv -V(K'; -1) \pmod{3^{\delta+1}}.$$

($\because$ Theorem 2.)

Pairs of knots K and K' with $|\sigma(K) - \sigma(K')| \leq 2$ and $\mathrm{d_{cb}}(K, K') > 2$

K	K'	$\sigma(K)$	$\sigma(K')$	$V(K;\omega)$	$V(K';\omega)$
4_1	$3_1!\#3_1$	0	0	-1	3
5_2	$3_1!\#3_1$	2	0	-1	3
7_6	$3_1!\#3_1$	2	0	-1	3
6_2	$3_1\#3_1$	2	4	1	-3
7_2	$3_1\#3_1$	2	4	1	-3
$7_3!$	$3_1\#3_1$	4	4	1	-3

K	K'	$\sigma(K)$	$\sigma(K')$	$V(K;\omega)$	$V(K';\omega)$	$V(K;-1)$	$V(K';-1)$
6_1	3_1	0	2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-3
6_1	7_4	0	-2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-15
6_1	7_7	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	9	21
6_1	$3_1!\#4_1$	0	-2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-15
$7_4!$	7_7	2	0	$i\sqrt{3}$	$-i\sqrt{3}$	-15	21
$7_7!$	7_7	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	21	21
$3_1\#4_1$	7_7	2	0	$i\sqrt{3}$	$-i\sqrt{3}$	-15	21

New table of $\mathrm{d_{cb}}(K, K')$

	7_1	$7_1!$	7_2	$7_2!$	7_3	$7_3!$	7_4	$7_4!$	7_5	$7_5!$	7_6	$7_6!$	7_7	$7_7!$	$3_1\#4_1$	$3_1!\#4_1$
U	6	6	2	2	4	4	2	2	4	4	2	2	2	2	2	2
3_1	4	8	2	4	6	2	4	2	2	6	2	4	2	2	2	4
$3_1!$	8	4	4	2	2	6	2	4	6	2	4	2	2	2	4	2
4_1	6	6	2	2	4	4	2/4	2/4	4	4	2	2	2	2	2	2
5_1	2	10	2	6	8	2	6	2	2	8	2	6	4	4	2	6
$5_1!$	10	2	6	2	2	8	2	6	8	2	6	2	4	4	6	2
5_2	4	8	2	4	6	2	4	2	2	6	2	4	2	2/4	2	4
$5_2!$	8	4	4	2	2	6	2	4	6	2	4	2	2/4	2	4	2
6_1	6	6	2/4	2	4	4	4	2	4	4	2/4	2	4	2	2	4
$6_1!$	6	6	2	2/4	4	4	2	4	4	4	2	2/4	2	4	4	2
6_2	4	8	2	4	6	2	4	2/4	2/4	6	2	4	2	2	2/4	4
$6_2!$	8	4	4	2	2	6	2/4	4	6	2/4	4	2	2	2	4	2/4
6_3	6	6	2	2	4	4	2/4	2/4	4	4	2	2	2	2	2	2
$3_1\#3_1$	2	10	4	6	8	4	6	2	2	8	2	6	4	4	2	6
$3_1!\#3_1!$	10	2	6	4	4	8	2	6	8	2	6	2	4	4	6	2
$3_1!\#3_1$	6	6	2/4	2/4	4	4	2/4	2/4	4	4	4	4	2	2	2	2

New table of $\mathrm{d}_{\mathrm{cb}}(K, K')$ (cont'd)

	7_1	$7_1!$	7_2	$7_2!$	7_3	$7_3!$	7_4	$7_4!$	7_5	$7_5!$	7_6	$7_6!$	7_7	$7_7!$	$3_1\#4_1$	$3_1!\#4_1$
7_1	0	12	4	8	10	2	8	4	2	10	4	8	6	6	4	8
$7_1!$		0	8	4	2	10	4	8	10	2	8	4	6	6	8	4
7_2			0	4	6	2	4	2	2	6	2	4	2	2/4	2/4	4
$7_2!$				0	2	6	2	4	6	2	4	2	2/4	2	4	2/4
7_3					0	8	2	6	8	2	6	2	4	4	6	2/4
$7_3!$						0	6	2	2	8	2	6	4	4	2/4	6
7_4							0	4	6	2	4	2	2/4	4	4	2
$7_4!$								0	2	6	2	4	4	2/4	2	4
7_5									0	8	2	6	4	4	2	6
$7_5!$										0	6	2	4	4	6	2
7_6											0	4	2	2	2	4
$7_6!$												0	2	2	4	2
7_7													0	4	4	2
$7_7!$														0	2	4
$3_1\#4_1$															0	4
$3_1!\#4_1$																0