On the arc index of knots and links

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Intelligence of **L**ow-**D**imensional **T**opology

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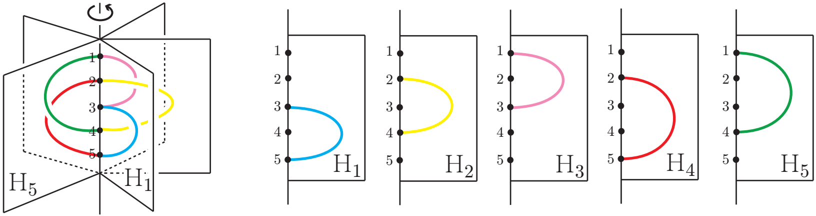
Arc presentation and arc index

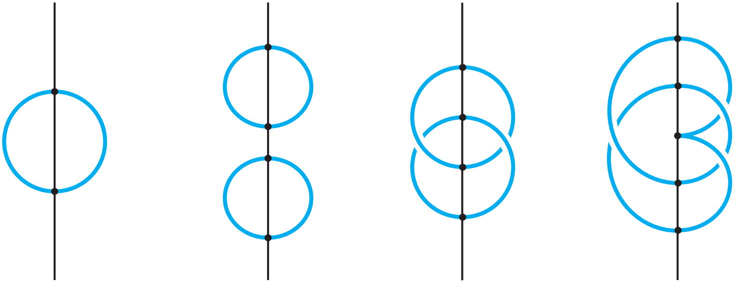
An *arc presentation* of a knot or a link L is an embedding of L contained in the union of finitely many half planes, called *pages*, with a common boundary line, called *binding axis*, in such a way that each half plane contains a properly embedded single simple arc.



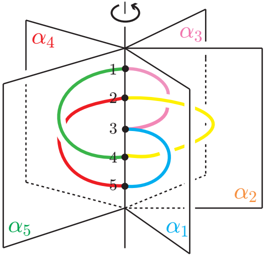
The minimum number of pages among all arc presentations of a link L is called the *arc index* of L and is denoted by $\alpha(L)$.

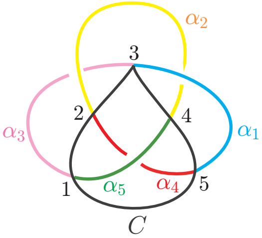
Links with arc index up to 5

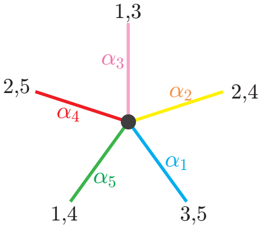
$\alpha(L)$	2	3	4	5
L	unknot	none	2-component unlink, Hopf link	trefoil

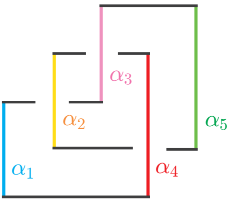


Representations of an arc presentation









History

- ⋆ Brunn(1897) proved that any link has a diagram with only one multiple point (not necessarily double).
- ⋆ Birman-Menasco(1994) used arc presentations of companion knots to study the braid index of their satellites.
- ⋆ Cromwell(1995) used the term "arc index" and established some of its basic properties.
- ⋆ Dynnikov(2006) proved that any arc-presentation of the unknot admits a monotonic simplification by elementary moves : this yields a simple algorithm for recognizing the unknot.

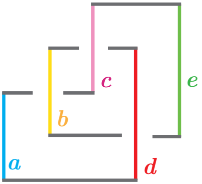
Grid diagram

Cromwell, 1995

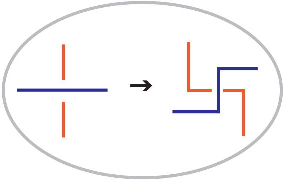
1. Every link admits an arc presentation.
2. If a nonsplit link L is a connected sum of two links L_1 and L_2, then

$$\alpha(L_1 \sharp L_2) = \alpha(L_1) + \alpha(L_2) - 2.$$

- Every link admits a grid diagram.
- A grid diagram gives rise to an arc presentation and vice versa.



grid diagram



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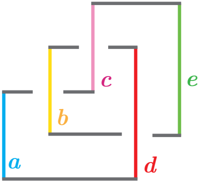

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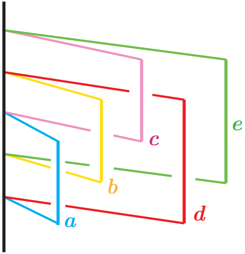

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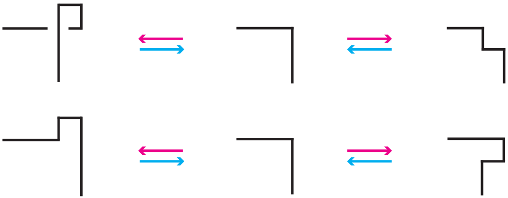
Elementary Moves on Grid Diagrams

Dynnikov, 2006

Two grid diagrams of the same link can be obtained from each other by a finite sequence of the following elementary moves.

- stabilization and destabilization;
- interchanging neighbouring edges if their pairs of endpoints do not interleave;
- cyclic permutation of vertical (horizontal) edges.



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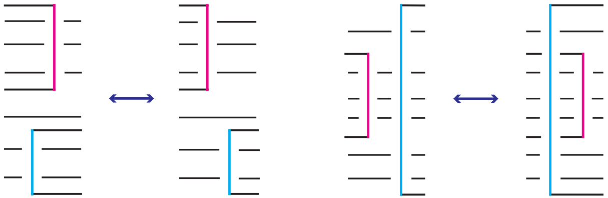

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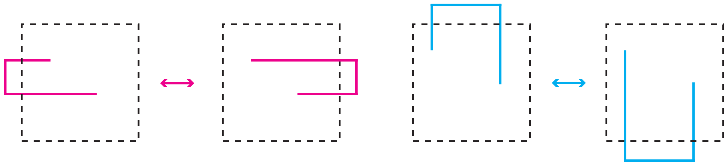

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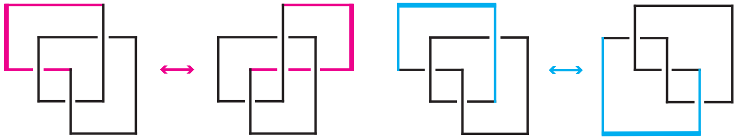


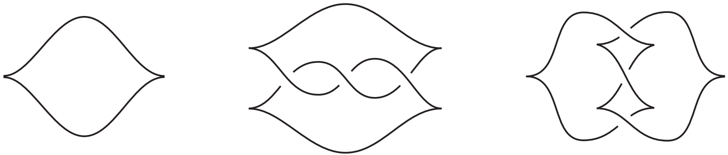
The front projection of a Legendrian knot

A knot diagram D represents *the front projection of a Legendrian knot* if

(1) D has no vertical tangencies,

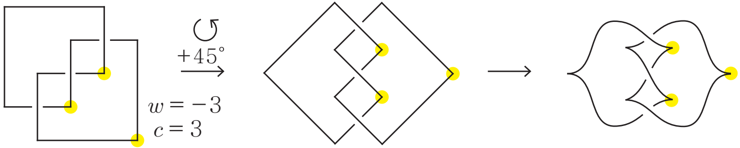
(2) the only non-smooth points are generalized cusps, and

(3) at each crossing the slope of the strand with the overcrossing is smaller than with the undercrossing.



Thurston-Bennequin number

★ A grid diagram gives rise to a Legendrian knot and vice versa.



For a grid diagram G, *Thurston-Bennequin number* is defined by

$$\mathrm{tb}(G) = w(G) - c(G)$$

where $w(G)$ and $c(G)$ are the writhe and the number of southeast corners of G, respectively.

The *maximal Thurston-Bennequin number* of a knot K, written $\overline{\mathrm{tb}}(K)$, is the maximal tb over all grid diagrams for K.

Thurston-Bennequin number

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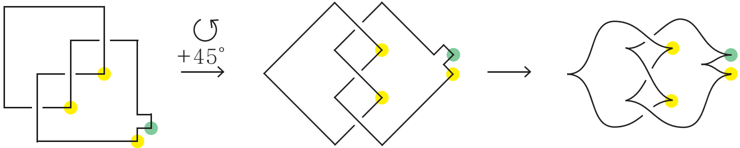

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A relation between $\alpha(K)$ and $\overline{tb}(K)$

Matsuda, 2006

$$-\alpha(K) \leq \overline{\mathrm{tb}}(K) + \overline{\mathrm{tb}}(K^*),$$

where K^* is the mirror image of a knot K.

Question(Ng), 2012

Does a grid diagram realizing $\alpha(K)$ of a knot K necessarily realize $\overline{tb}(K)$?
An equivalent statement is that

$$-\alpha(K) = \overline{tb}(K) + \overline{tb}(K^*)$$

for any knot K.

Known Results I

- [Beltrami, 2002] Arc index for prime knots up to 10 crossings are determined.
- [Ng, 2006] Arc index for prime knots up to 11 crossings are determined.

★ [Nutt, 1999] All knots up to arc index 9 are identified.

★ [Jin et al., 2006] All prime knots up to arc index 10 are identified.

★ [Jin-Park, 2010] All prime knots up to arc index 11 are identified.

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Kauffman polynomial $F_L(a,z)$

The *Kauffman polynomial* of an oriented knot or link L is defined by

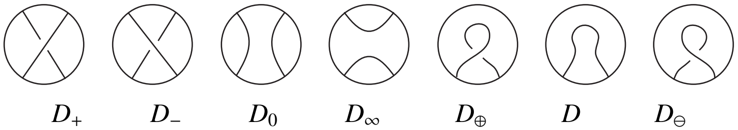
$$F_L(a,z) = a^{-w(D)} \Lambda_D(a,z)$$

where D is a diagram of L, $w(D)$ the writhe of D and $\Lambda_D(a,z)$ the polynomial determined by the rules (K1), (K2) and (K3).

(K1) $\Lambda_O(a,z) = 1$ where O is the trivial knot diagram.

(K2) $\Lambda_{D_+}(a,z) + \Lambda_{D_-}(a,z) = z(\Lambda_{D_0}(a,z) + \Lambda_{D_\infty}(a,z))$.

(K3) $a\,\Lambda_{D_\oplus}(a,z) = \Lambda_D(a,z) = a^{-1}\Lambda_{D_\ominus}(a,z)$.



$\alpha(L) \geq \text{spread}_a(F_L) + 2.$

The a-spread of the Kauffman polynomial $F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$ is denoted by $\text{spread}_a(F_L)$ and defined by the formula

$$\text{spread}_a(F_L) = \text{max-deg}_a(F_L) - \text{min-deg}_a(F_L).$$

Morton-Beltrami, 1998

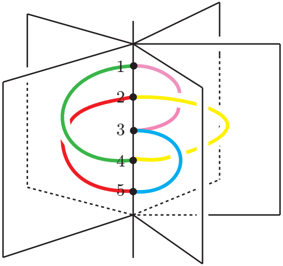
Let L be a link. Then

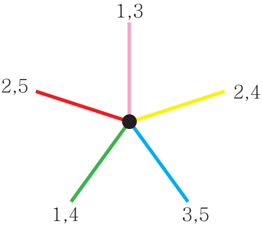
$$\alpha(L) \geq \text{spread}_a(F_L) + 2.$$

Notice : $\text{spread}_a(F_L) = \text{spread}_a(\Lambda_D)$ for any diagram D of L.

Wheel diagram

- Bae-Park described an algorithm which transforms a diagram of a non-split link with n crossings into a wheel diagram with at most $n + 2$ spokes.





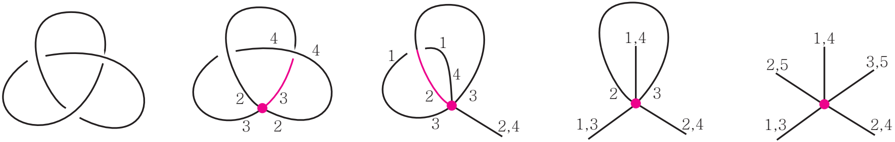
Bae-Park, 2000

If L is a non-split link, then $\alpha(L) \leq c(L) + 2$.

The idea of Bae-Park Theorem

Bae-Park, 2000

If L is a non-split link, then $\alpha(L) \leq c(L) + 2$.



Idea : The sum of number of regions and spokes is unchanged.

A relation between $\alpha(L)$ and $c(L)$

- L : **non-split alternating link** $\Longrightarrow \alpha(L) = c(L) + 2$.
    - [Morton-Beltrami, 1998] For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a,z)) + 2$.
    - [Thistlethwaite, 1988] If L is an alternating link, $\text{spread}_a(F_L(a,z)) \geq c(L)$.
    - [Bae-Park, 2000] If L is a non-split link, then

$$\alpha(L) \leq c(L) + 2.$$

- L : **nonalternating prime** $\Longrightarrow \text{spread}_a(F_L(a,z)) + 2 \leq \alpha(L) \leq c(L)$.

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  - [M-B] For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a,z)) + 2$.
  - [Jin-Park, 2010] A prime link L is nonalternating *if and only if*

$$\alpha(L) \leq c(L).$$

Known Results III

- ⋆ [Etnyre-Honda, 2001] $\alpha(T_{p,q}) = |p| + |q|$

- ⋆ [L-Jin] Arc index of pretzel knots of type $(-p, q, r)$ (submitted)
- ⋆ [L] Arc index of Montesinos links of type $(-r_1, r_2, r_3)$ (preprint)

- ⋆ [L-Takioka] On the arc index of Kanenobu knots
- ⋆ [L-Takioka] On the arc index of cable links

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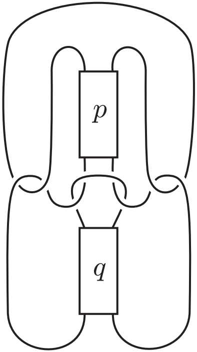
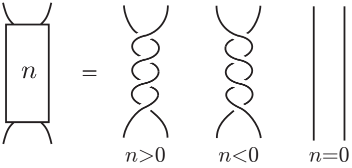
Arc index of Kanenobu knots

What are Kanenobu Knots?



$K(p,\ q)$



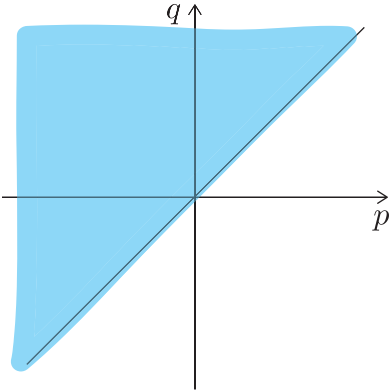
◇ T. Kanenobu, *Infinitely many knots with the same polynomial invariant*, Proc. Amer. Math. Soc. 97 (1986) 158–162.

◇ T. Kanenobu, *Examples on polynomial invariants of knots and links*, Math. Ann. **275** (1986) 555–572.

Kanenobu, 1986

$K(p,q) = K(q,p)$ and $K(p,q)^* = K(-p,-q)$.

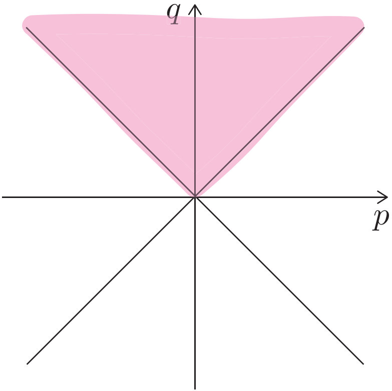
$K(p,q)$ with $|p| \leq q$



It is sufficient to consider $K(p,q)$ with $|p| \leq q$ in order to determine the arc index of $K(p,q)$.

- ★ $K(p,q) = K(q,p)$.
- ★ $K(p,q)^* = K(-p,-q) = K(-q,-p)$.
- ★ $\alpha(L) = \alpha(L^*)$.

$K(p, q)$ with $|p| \leq q$



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- ⋆ $K(p, q) = K(q, p)$.
- ⋆ $K(p, q)^* = K(-p, -q) = K(-q, -p)$.
- ⋆ $\alpha(L) = \alpha(L^*)$.

Main results

Theorem K1

Let $1 \leq p \leq q$ and $pq \geq 3$. Then

$$\alpha(K(p,q)) = p + q + 6.$$

Theorem K3

Let $p = -1$ and $q \geq 3$. Then

$$q + 5 \leq \alpha(K(-1,q)) \leq q + 7.$$

Theorem K2

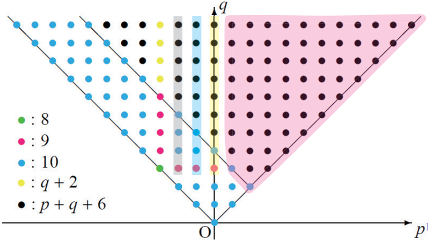
Let $p = 0$ and $q \geq 3$. Then

$$q + 6 \leq \alpha(K(0,q)) \leq q + 7.$$

Theorem K4

Let $p = -2$ and $q \geq 3$. Then

$$q + 4 \leq \alpha(K(-2,q)) \leq q + 7.$$



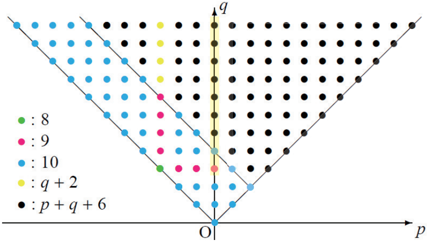
Theorem K2. Let $p = 0$, $q \geq 3$. Then $q + 6 \leq \alpha(K(0, q)) \leq q + 7$.

K	DT Name	$b + 6$	$\alpha(K)$	$b + 7$
$K(0,3) \doteq K(0,-3)$	$11n50$	9	$10^{\dagger}$	10
$K(0,4) \doteq K(0,-4)$	$12n145$	10	$11^{\ddagger}$	11
$K(0,5) \doteq K(0,-5)$	$13n579$	11	$11^{\ddagger}$	12
$K(0,6) \doteq K(0,-6)$	$14n2459$	12	$12^{\natural}$	13

† Jin et al., *Prime knots with arc index up to 10*, Series on Knots and Everything Book vol. 40, World Scientific Publishing Co., 6574, 2006.

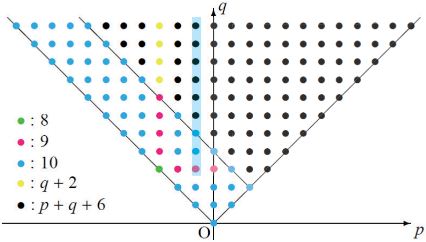
‡ Jin-Park, *A tabulation of prime knots up to arc index 11*, JKTR vol. 20, No. 11, pp. 1537–1635.

♮ Jin-Kim, *Prime knots with arc index 12 up to 16 crossings*, preprint.



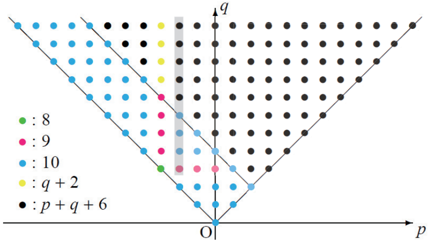
Theorem K3. Let $p = -1, q \geq 3$. Then $q + 5 \leq \alpha(K(-1, q)) \leq q + 7$.

K	DT Name	$b + 5$	$\alpha(K)$	$b + 7$
$K(-1,3) \doteq K(1,-3)$	$11n37$	8	10	10
$K(-1,4) \doteq K(1,-4)$	$12n414$	9	11	11
$K(-1,5) \doteq K(1,-5)$	$13n2036$	10	11	12
$K(-1,6) \doteq K(1,-6)$	$14n9271$	11	12	13
$K(-1,7) \doteq K(1,-7)$	$15n46855$	12	12	14



Theorem K4. Let $p = -2, q \geq 3$. Then $q + 4 \leq \alpha(K(-2, q)) \leq q + 7$.

K	DT Name	$b + 4$	$\alpha(K)$	$b + 7$
$K(-2,3) \doteq K(2,-3)$	$13n1836$	7	10	10
$K(-2,4) \doteq K(2,-4)$	$14n11995$	8	11	11
$K(-2,5) \doteq K(2,-5)$	$15n54616$	9	11	12
$K(-2,6) \doteq K(2,-6)$	$16n331702$	10	12	13

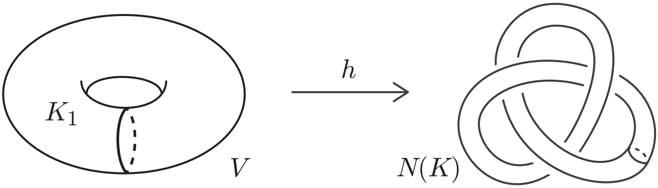


Arc index of cable links

What is a satellite knot?

- $V = S^1 \times D^2 \subset S^3$: a standard solid torus.
- K_1 : a knot embedded in V s.t. every meridinal disk of V intersects K_1.
- $N(K)$: a tubular neighborhood of a knot K.

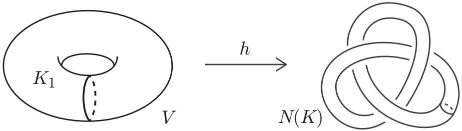
Let h be a faithful homeomorphism from V onto $N(K)$.

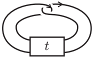
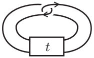


The image $h(K_1)$ is called a *satellite knot* with *companion knot K*.

Cable links and Whitehead doubles

Let p, q be integers with $p > 0$.

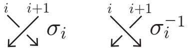


K_1	$h(K_1)$	denoted by
(p, q)-torus link $T_{p,q}$	*(p, q)-cable link*	$K^{(p,q)}$
	*positive Whitehead double*	$K^{(+,t)}$
	*negative Whitehead double*	$K^{(-,t)}$

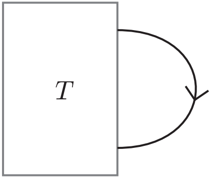
 $t=2$,  $t=-2$

A standard diagram of cable links

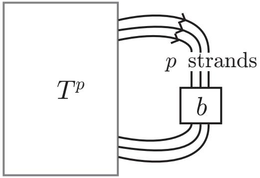
- D : a diagram of a knot K with an $(1,1)$-tangle T.
- $w(D)$: the writhe of D.
- $\beta_p := \sigma_1\sigma_2\cdots\sigma_{p-1}$ and $b := \beta_p^{q-pw(D)}$.



Then $S_D^{(p,q)}$ is a diagram of $K^{(p,q)}$ where T^p is a p-strand parallel tangle of T as shown in the figure below. We call $S_D^{(p,q)}$ the *standard diagram* of $K^{(p,q)}$ *obtained from D* and the braid b *$(p, q - pw(D))$-twist*.



D



$S_D^{(p,q)}$

Canonical arc index

Let G be a grid diagram of a knot K and p, q integers with $p > 0$.

The grid diagram obtained by the canonical (p, q)-cabling algorithm of G is called the *canonical grid diagram* of $K^{(p,q)}$ *obtained from G* and denoted by $G^{(p,q)}$.

Let $\alpha(G^{(p,q)})$ denote the number of vertical line segments of $G^{(p,q)}$. The *canonical arc index* of $K^{(p,q)}$, denoted by $\alpha_c(K^{(p,q)})$, is defined as follows:

$$\alpha_c(K^{(p,q)}) = \min\{\alpha(G^{(p,q)}) \mid G \text{ is a grid diagram of } K\}.$$

Note : $\alpha(K^{(p,q)}) \leq \alpha_c(K^{(p,q)})$.

Question 1

$$\alpha(K^{(p,q)}) = \alpha_c(K^{(p,q)})?$$

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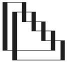
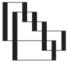
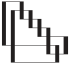
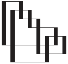
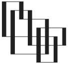
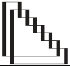
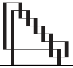
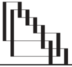
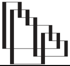
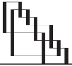
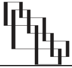
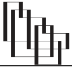
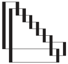
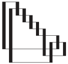
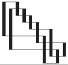
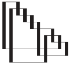
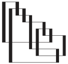
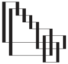
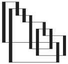
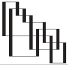
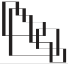
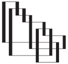
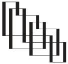
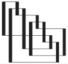
K	$\alpha(K^{(2,q)})$	$\alpha(K^{(+,t)})$	$\alpha(K^{(-,t)})$
3_1^*	$-q+12$ *if* $q \leq 1$	$-2t+13$ *if* $t \leq 0$	$-2t+14$ *if* $t \leq 1$
	10 *if* $2 \leq q \leq 12$	11 *if* $1 \leq t \leq 5$	11 *if* $2 \leq t \leq 6$
	$q-2$ *if* $q \geq 13$	$2t$ *if* $t \geq 6$	$2t-1$ *if* $t \geq 7$
4_1	$-q+6$ *if* $q \leq -7$	$-2t+7$ *if* $t \leq -4$	$-2t+8$ *if* $t \leq -3$
	12 *if* $-6 \leq q \leq 6$	13 *if* $-3 \leq t \leq 2$	13 *if* $-2 \leq t \leq 3$
	$q+6$ *if* $q \geq 7$	$2t+8$ *if* $t \geq 3$	$2t+7$ *if* $t \geq 4$
5_1^*	$-q+20$ *if* $q \leq 5$	$-2t+21$ *if* $t \leq 2$	$-2t+22$ *if* $t \leq 3$
	14 *if* $6 \leq q \leq 20$	15 *if* $3 \leq t \leq 9$	15 *if* $4 \leq t \leq 10$
	$q-6$ *if* $q \geq 21$	$2t-4$ *if* $t \geq 10$	$2t-5$ *if* $t \geq 11$
5_2	$-q-2$ *if* $q \leq -17$	$-2t-1$ *if* $t \leq -9$	$-2t$ *if* $t \leq -8$
	14 *if* $-16 \leq q \leq -2$	15 *if* $-8 \leq t \leq -2$	15 *if* $-7 \leq t \leq -1$
	$q+16$ *if* $q \geq -1$	$2t+18$ *if* $t \geq -1$	$2t+17$ *if* $t \geq 0$
6_1	$-q+6$ *if* $q \leq -11$	$-2t+7$ *if* $t \leq -6$	$-2t+8$ *if* $t \leq -5$
	16 *if* $-10 \leq q \leq 6$	17 *if* $-5 \leq t \leq 2$	17 *if* $-4 \leq t \leq 3$
	$q+10$ *if* $q \geq 7$	$2t+12$ *if* $t \geq 3$	$2t+11$ *if* $t \geq 4$

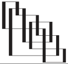
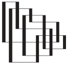
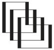
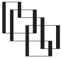
K	$\alpha(K^{(2,q)})$	$\alpha(K^{(+,t)})$	$\alpha(K^{(-,t)})$
6_2^* 	$-q+14$ *if* $q \le -3$	$-2t+15$ *if* $t \le -2$	$-2t+16$ *if* $t \le -1$
	16 *if* $-2 \le q \le 14$	17 *if* $-1 \le t \le 6$	17 *if* $0 \le t \le 7$
	$q+2$ *if* $q \ge 15$	$2t+4$ *if* $t \ge 7$	$2t+3$ *if* $t \ge 8$
6_3 	$-q+8$ *if* $q \le -9$	$-2t+9$ *if* $t \le -5$	$-2t+10$ *if* $t \le -4$
	16 *if* $-8 \le q \le 8$	17 *if* $-4 \le t \le 3$	17 *if* $-3 \le t \le 4$
	$q+8$ *if* $q \ge 9$	$2t+10$ *if* $t \ge 4$	$2t+9$ *if* $t \ge 5$
7_1^* 	$-q+28$ *if* $q \le 9$	$-2t+29$ *if* $t \le 4$	$-2t+30$ *if* $t \le 5$
	18 *if* $10 \le q \le 28$	19 *if* $5 \le t \le 13$	19 *if* $6 \le t \le 14$
	$q-10$ *if* $q \ge 29$	$2t-8$ *if* $t \ge 14$	$2t-9$ *if* $t \ge 15$
7_2 	$-q-2$ *if* $q \le -21$	$-2t-1$ *if* $t \le -11$	$-2t$ *if* $t \le -10$
	18 *if* $-20 \le q \le -2$	19 *if* $-10 \le t \le -2$	19 *if* $-9 \le t \le -1$
	$q+20$ *if* $q \ge -1$	$2t+22$ *if* $t \ge -1$	$2t+21$ *if* $t \ge 0$
7_3 	$-q+24$ *if* $q \le 5$	$-2t+25$ *if* $t \le 2$	$-2t+26$ *if* $t \le 3$
	18 *if* $6 \le q \le 24$	19 *if* $3 \le t \le 11$	19 *if* $4 \le t \le 12$
	$q-6$ *if* $q \ge 25$	$2t-4$ *if* $t \ge 12$	$2t-5$ *if* $t \ge 13$

K	$\alpha(K^{(2,q)})$		$\alpha(K^{(+,t)})$		$\alpha(K^{(-,t)})$	
7_4^* 	$-q-2$	*if* $q \leq -21$	$-2t-1$	*if* $t \leq -11$	$-2t$	*if* $t \leq -10$
	18	*if* $-20 \leq q \leq -2$	19	*if* $-10 \leq t \leq -2$	19	*if* $-9 \leq t \leq -1$
	$q+20$	*if* $q \geq -1$	$2t+22$	*if* $t \geq -1$	$2t+21$	*if* $t \geq 0$
7_5^* 	$-q+24$	*if* $q \leq 5$	$-2t+25$	*if* $t \leq 2$	$-2t+26$	*if* $t \leq 3$
	18	*if* $6 \leq q \leq 24$	19	*if* $3 \leq t \leq 11$	19	*if* $4 \leq t \leq 12$
	$q-6$	*if* $q \geq 25$	$2t-4$	*if* $t \geq 12$	$2t-5$	*if* $t \geq 13$
7_6 	$-q+2$	*if* $q \leq -17$	$-2t+3$	*if* $t \leq -9$	$-2t+4$	*if* $t \leq -8$
	18	*if* $-16 \leq q \leq 2$	19	*if* $-8 \leq t \leq 0$	19	*if* $-7 \leq t \leq 1$
	$q+16$	*if* $q \geq 3$	$2t+18$	*if* $t \geq 1$	$2t+17$	*if* $t \geq 2$
7_7^* 	$-q+8$	*if* $q \leq -11$	$-2t+9$	*if* $t \leq -6$	$-2t+10$	*if* $t \leq -5$
	18	*if* $-10 \leq q \leq 8$	19	*if* $-5 \leq t \leq 3$	19	*if* $-4 \leq t \leq 4$
	$q+10$	*if* $q \geq 9$	$2t+12$	*if* $t \geq 4$	$2t+11$	*if* $t \geq 5$
8_1 	$-q+6$	*if* $q \leq -15$	$-2t+7$	*if* $t \leq -8$	$-2t+8$	*if* $t \leq -7$
	20	*if* $-14 \leq q \leq 6$	21	*if* $-7 \leq t \leq 2$	21	*if* $-6 \leq t \leq 3$
	$q+14$	*if* $q \geq 7$	$2t+16$	*if* $t \geq 3$	$2t+15$	*if* $t \geq 4$

K	$\alpha(K^{(2,q)})$		$\alpha(K^{(+,t)})$		$\alpha(K^{(-,t)})$	
8_2^* 	$-q+22$	*if* $q \le 1$	$-2t+23$	*if* $t \le 0$	$-2l+24$	*if* $t \le 1$
	20	*if* $2 \le q \le 22$	21	*if* $1 \le t \le 10$	21	*if* $2 \le t \le 11$
	$q-2$	*if* $q \ge 23$	$2t$	*if* $t \ge 11$	$2l-1$	*if* $t \ge 12$
8_3 	$-q+10$	*if* $q \le -11$	$-2t+11$	*if* $t \le -6$	$-2l+12$	*if* $t \le -5$
	20	*if* $-10 \le q \le 10$	21	*if* $-5 \le t \le 4$	21	*if* $-4 \le t \le 5$
	$q+10$	*if* $q \ge 11$	$2t+12$	*if* $t \ge 5$	$2l+11$	*if* $t \ge 6$
8_4^* 	$-q+14$	*if* $q \le -7$	$-2t+15$	*if* $t \le -4$	$-2l+16$	*if* $t \le -3$
	20	*if* $-6 \le q \le 14$	21	*if* $-3 \le t \le 6$	21	*if* $-2 \le t \le 7$
	$q+6$	*if* $q \ge 15$	$2t+8$	*if* $t \ge 7$	$2l+7$	*if* $t \ge 8$
8_5 	$-q+22$	*if* $q \le 1$	$-2t+23$	*if* $t \le 0$	$-2l+24$	*if* $t \le 1$
	20	*if* $2 \le q \le 22$	21	*if* $1 \le t \le 10$	21	*if* $2 \le t \le 11$
	$q-2$	*if* $q \ge 23$	$2t$	*if* $t \ge 11$	$2l-1$	*if* $t \ge 12$
8_6^* 	$-q+18$	*if* $q \le -3$	$-2t+19$	*if* $t \le -2$	$-2l+20$	*if* $t \le -1$
	20	*if* $-2 \le q \le 18$	21	*if* $-1 \le t \le 8$	21	*if* $0 \le t \le 9$
	$q+2$	*if* $q \ge 19$	$2t+4$	*if* $t \ge 9$	$2l+3$	*if* $t \ge 10$

K	$\alpha(K^{(2,q)})$		$\alpha(K^{(+,t)})$		$\alpha(K^{(-,t)})$	
8_7 	$-q+16$	*if* $q \leq -5$	$-2t+17$	*if* $t \leq -3$	$-2t+18$	*if* $t \leq -2$
	20	*if* $-4 \leq q \leq 16$	21	*if* $-2 \leq t \leq 7$	21	*if* $-1 \leq t \leq 8$
	$q+4$	*if* $q \geq 17$	$2t+6$	*if* $t \geq 8$	$2t+5$	*if* $t \geq 9$
8_8^* 	$-q+8$	*if* $q \leq -13$	$-2t+9$	*if* $t \leq -7$	$-2t+10$	*if* $t \leq -6$
	20	*if* $-12 \leq q \leq 8$	21	*if* $-6 \leq t \leq 3$	21	*if* $-5 \leq t \leq 4$
	$q+12$	*if* $q \geq 9$	$2t+14$	*if* $t \geq 4$	$2t+13$	*if* $t \geq 5$
8_9 	$-q+10$	*if* $q \leq -11$	$-2t+11$	*if* $t \leq -6$	$-2t+12$	*if* $t \leq -5$
	20	*if* $-10 \leq q \leq 10$	21	*if* $-5 \leq t \leq 4$	21	*if* $-4 \leq t \leq 5$
	$q+10$	*if* $q \geq 11$	$2t+12$	*if* $t \geq 5$	$2t+11$	*if* $t \geq 6$
8_{10} 	$-q+16$	*if* $q \leq -5$	$-2t+17$	*if* $t \leq -3$	$-2t+18$	*if* $t \leq -2$
	20	*if* $-4 \leq q \leq 16$	21	*if* $-2 \leq t \leq 7$	21	*if* $-1 \leq t \leq 8$
	$q+4$	*if* $q \geq 17$	$2t+6$	*if* $t \geq 8$	$2t+5$	*if* $t \geq 9$
8_{11} 	$-q+2$	*if* $q \leq -19$	$-2t+3$	*if* $t \leq -10$	$-2t+4$	*if* $t \leq -9$
	20	*if* $-18 \leq q \leq 2$	21	*if* $-9 \leq t \leq 0$	21	*if* $-8 \leq t \leq 1$
	$q+18$	*if* $q \geq 3$	$2t+20$	*if* $t \geq 1$	$2t+19$	*if* $t \geq 2$

K	$\alpha(K^{(2,q)})$		$\alpha(K^{(+,t)})$		$\alpha(K^{(-,t)})$	
8_{12} 	$-q+10$	*if* $q \leq -11$	$-2t+11$	*if* $t \leq -6$	$-2t+12$	*if* $t \leq -5$
	20	*if* $-10 \leq q \leq 10$	21	*if* $-5 \leq t \leq 4$	21	*if* $-4 \leq t \leq 5$
	$q+10$	*if* $q \geq 11$	$2t+12$	*if* $t \geq 5$	$2t+11$	*if* $t \geq 6$
8^*_{13} 	$-q+8$	*if* $q \leq -13$	$-2t+9$	*if* $t \leq -7$	$-2t+10$	*if* $t \leq -6$
	20	*if* $-12 \leq q \leq 8$	21	*if* $-6 \leq t \leq 3$	21	*if* $-5 \leq t \leq 4$
	$q+12$	*if* $q \geq 9$	$2t+14$	*if* $t \geq 4$	$2t+13$	*if* $t \geq 5$
8^*_{14} 	$-q+18$	*if* $q \leq -3$	$-2t+19$	*if* $t \leq -2$	$-2t+20$	*if* $t \leq -1$
	20	*if* $-2 \leq q \leq 18$	21	*if* $-1 \leq t \leq 8$	21	*if* $0 \leq t \leq 9$
	$q+2$	*if* $q \geq 19$	$2t+4$	*if* $t \geq 9$	$2t+3$	*if* $t \geq 10$
8^*_{15} 	$-q+26$	*if* $q \leq 5$	$-2t+27$	*if* $t \leq 2$	$-2t+28$	*if* $t \leq 3$
	20	*if* $6 \leq q \leq 26$	21	*if* $3 \leq t \leq 12$	21	*if* $4 \leq t \leq 13$
	$q-6$	*if* $q \geq 27$	$2t-4$	*if* $t \geq 13$	$2t-5$	*if* $t \geq 14$
8^*_{16} 	$-q+16$	*if* $q \leq -5$	$-2t+17$	*if* $t \leq -3$	$-2t+18$	*if* $t \leq -2$
	20	*if* $-4 \leq q \leq 16$	21	*if* $-2 \leq t \leq 7$	21	*if* $-1 \leq t \leq 8$
	$q+4$	*if* $q \geq 17$	$2t+6$	*if* $t \geq 8$	$2t+5$	*if* $t \geq 9$

K	$\alpha(K^{(2,q)})$		$\alpha(K^{(+,t)})$		$\alpha(K^{(-,t)})$	
8_{17} 	$-q+10$	*if* $q \leq -11$	$-2t+11$	*if* $t \leq -6$	$-2t+12$	*if* $t \leq -5$
	20	*if* $-10 \leq q \leq 10$	21	*if* $-5 \leq t \leq 4$	21	*if* $-4 \leq t \leq 5$
	$q+10$	*if* $q \geq 11$	$2t+12$	*if* $t \geq 5$	$2t+11$	*if* $t \geq 6$
8_{18} 	$-q+10$	*if* $q \leq -11$	$-2t+11$	*if* $t \leq -6$	$-2t+12$	*if* $t \leq -5$
	20	*if* $-10 \leq q \leq 10$	21	*if* $-5 \leq t \leq 4$	21	*if* $-4 \leq t \leq 5$
	$q+10$	*if* $q \geq 11$	$2t+12$	*if* $t \geq 5$	$2t+11$	*if* $t \geq 6$
8_{19} 	$-q+24$	*if* $q \leq 9$	$-2t+25$	*if* $t \leq 4$	$-2t+26$	*if* $t \leq 5$
	14	*if* $10 \leq q \leq 24$	15	*if* $5 \leq t \leq 11$	15	*if* $6 \leq t \leq 12$
	$q-10$	*if* $q \geq 25$	$2t-8$	*if* $t \geq 12$	$2t-9$	*if* $t \geq 13$
8_{20} 	$-q+4$	*if* $q \leq -13$	$-2t+5$	*if* $t \leq -7$	$-2t+6$	*if* $t \leq -6$
	16	*if* $-12 \leq q \leq 4$	17	*if* $-6 \leq t \leq 1$	17	*if* $-5 \leq t \leq 2$
	$q+12$	*if* $q \geq 5$	$2t+14$	*if* $t \geq 2$	$2t+13$	*if* $t \geq 3$
8^*_{21} 	$-q+18$	*if* $q \leq 1$	$-2t+19$	*if* $t \leq 0$	$-2t+20$	*if* $t \leq 1$
	16	*if* $2 \leq q \leq 18$	17	*if* $1 \leq t \leq 8$	17	*if* $2 \leq t \leq 9$
	$q-2$	*if* $q \geq 19$	$2t$	*if* $t \geq 9$	$2t-1$	*if* $t \geq 10$

Questions

Question 1

$$\alpha(K^{(p,q)}) = \alpha_c(K^{(p,q)})?$$

Question 2

For two minimal grid diagrams G, G' of a knot K, we have

$$\alpha(G^{(p,q)}) = \alpha(G'^{(p,q)})?$$

Question 3

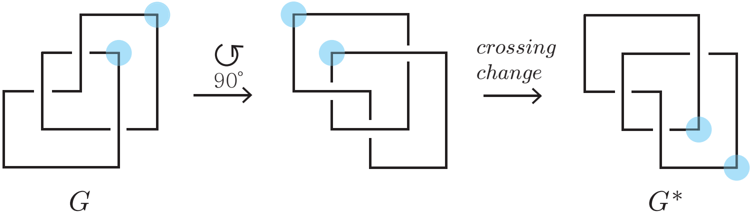
If G is a minimal grid diagram of a knot K, then we have

$$\alpha_c(K^{(p,q)}) = \alpha(G^{(p,q)})?$$

Terminology

A corner of G is called *se* if it is a SouthEast corner, and similarly *sw, ne* or *nw* when SouthWest, NorthEast or NorthWest, respectively.

- ★ $\mathrm{ne}(G)$: the number of *ne* corners of G
- ★ $\mathrm{se}(G)$: the number of *se* corners of G
- ★ $\mathrm{tb}(G) = w(G) - \mathrm{se}(G)$.



Note : $\mathrm{ne}(G) = \mathrm{se}(G^*)$ and $\mathrm{se}(G) = \mathrm{ne}(G^*)$.

Terminology

A corner of G is called *se* if it is a SouthEast corner, and similarly *sw, ne* or *nw* when SouthWest, NorthEast or NorthWest, respectively.

- ★ $\mathrm{ne}(G)$: the number of *ne* corners of G
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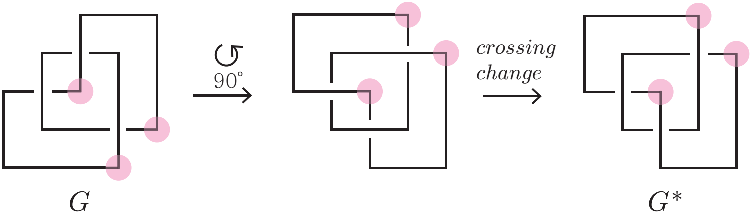
- ★ $\mathrm{tb}(G) = w(G) - \mathrm{se}(G)$.

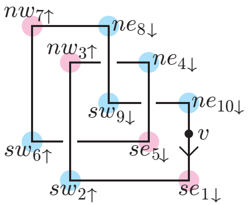


Note : $\mathrm{ne}(G) = \mathrm{se}(G^*)$ and $\mathrm{se}(G) = \mathrm{ne}(G^*)$.

Canonical (p, q)-cabling algorithm

We start a grid diagram G of a knot K.

- ▷ Let v be a point on the rightmost vertical line segment which is not any corner.



Step I : Identify each corner of G

- ▷ Let C_G denote the sequence of oriented indexed corners of G where the index i indicates the ith meeting corner when we travel along G starting from v away from the rightmost ne-corner.

$$C_G : (se_{1\downarrow}, sw_{2\uparrow}, nw_{3\uparrow}, ne_{4\downarrow}, se_{5\downarrow}, sw_{6\uparrow}, nw_{7\uparrow}, ne_{8\downarrow}, sw_{9\downarrow}, ne_{10\downarrow})$$

- ▷ Let C_G^+ and C_G^- be the subsequences of C_G which is made up of all sw, ne-corners and all se, nw-corners of C_G, respectively.

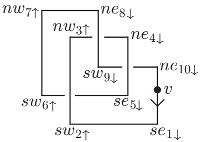
$$C_G^+ : (sw_{2\uparrow}, ne_{4\downarrow}, sw_{6\uparrow}, ne_{8\downarrow}, sw_{9\downarrow}, ne_{10\downarrow}),$$

$$C_G^- : (se_{1\downarrow}, nw_{3\uparrow}, se_{5\downarrow}, nw_{7\uparrow}).$$

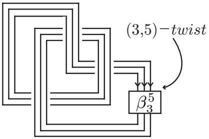
Canonical (p, q)-cabling algorithm

Step II : Obtain the standard diagram $S_G^{(p,q)}$

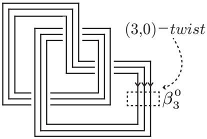
Given two integers $p > 0$ and q, we have $S_G^{(p,q)}$ of $K^{(p,q)}$ from G.



G



$S_G^{(3,14)}$



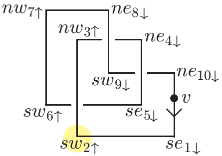
$S_G^{(3,9)} = G^{(3,9)}$

$w(G) = 3$ $\quad$ $\beta_p^{q-pw(G)} = \beta_3^5$ $\quad$ $\beta_p^{q-pw(G)} = \beta_3^0$

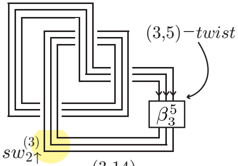
If $q = pw(G)$ then $S_G^{(p,pw(G))}$ is the canonical grid diagram of $K^{(p,pw(G))}$.

Suppose that $q - pw(G) \neq 0$.

Canonical (p, q)-cabling algorithm



G



$S_G^{(3,14)}$

We denote

- the corners corresponding to se, sw, nw and ne by $se^{(p)}, sw^{(p)}, nw^{(p)}$ and $ne^{(p)}$, respectively.
- the sequences corresponding to C_G, C_G^+ and C_G^- by $C_{S_G^{(p,q)}}$, $C^+_{S_G^{(p,q)}}$ and $C^-_{S_G^{(p,q)}}$ in $S_G^{(p,q)}$, respectively,

$$C_{S_G^{(3,14)}}: (se^{(3)}_{1\downarrow}, sw^{(3)}_{2\uparrow}, nw^{(3)}_{3\uparrow}, ne^{(3)}_{4\downarrow}, se^{(3)}_{5\downarrow}, sw^{(3)}_{6\uparrow}, nw^{(3)}_{7\uparrow}, ne^{(3)}_{8\downarrow}, sw^{(3)}_{9\downarrow}, ne^{(3)}_{10\downarrow}),$$

$$C^+_{S_G^{(3,14)}}: (sw^{(3)}_{2\uparrow}, ne^{(3)}_{4\downarrow}, sw^{(3)}_{6\uparrow}, ne^{(3)}_{8\downarrow}, sw^{(3)}_{9\downarrow}, ne^{(3)}_{10\downarrow}),$$

$$C^-_{S_G^{(3,14)}}: (se^{(3)}_{1\downarrow}, nw^{(3)}_{3\uparrow}, se^{(3)}_{5\downarrow}, nw^{(3)}_{7\uparrow}).$$

Canonical (p, q)-cabling algorithm

Step III (1) : Define the canonical grid form of $(p, q - pw(G))$-twist

A *grid form* of a p-strand braid is a diagram such that each strand of the braid is expressed as a union of vertical and horizontal line segments and at each crossing the vertical line segment crosses over the horizontal line segment.

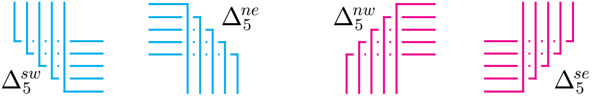
Let $\beta_p = \sigma_1\sigma_2 \cdots \sigma_{p-1}$. Applying the rule of the figure below to the $\beta_p^{\pm 1}$

we can obtain a grid form of β_p^s for an integer s which is called the *canonical grid form* of β_p^s.

Canonical (p, q)-cabling algorithm

Step III (2) : Understand half-twisted corners

For a positive integer r, let $\Delta_r^{sw}, \Delta_r^{ne}, \Delta_r^{nw}$ and Δ_r^{se} be a half-twisted r-strand sw, ne, nw and se composed of r vertical line segments and the same number of horizontal line segments in a grid form, respectively.



Proposition C1

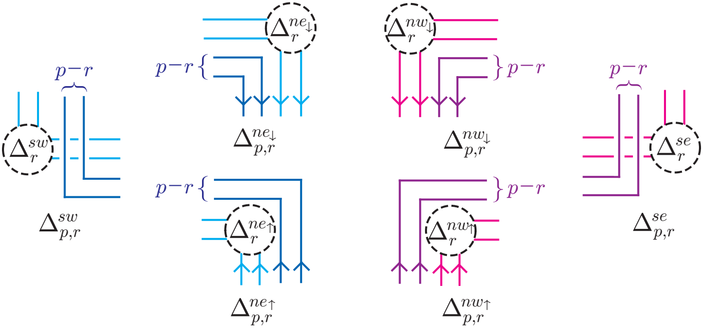
(1) The products $\Delta_r^{sw}\Delta_r^{ne}$ and $\Delta_r^{ne}\Delta_r^{sw}$ of two half twists $\Delta_r^{sw}, \Delta_r^{ne}$ is presented by a positive full twist word β_r^r.

(2) The products $\Delta_r^{nw}\Delta_r^{se}$ and $\Delta_r^{se}\Delta_r^{nw}$ of two half twists $\Delta_r^{nw}, \Delta_r^{se}$ is presented by a negative full twist word $(\beta_r^{-1})^r$.

Canonical (p, q)-cabling algorithm

Step III (3) : Understand partially half-twisted corners

For $p > r > 0$, let $\Delta^{sw}_{p,r}$, $\Delta^{ne}_{p,r}$, $\Delta^{nw}_{p,r}$ and $\Delta^{se}_{p,r}$ be a partially half-twisted *sw*, *ne*, *nw* and *se* corner in the grid form whose diagram is given in the figure below, respectively.



Canonical (p, q)-cabling algorithm

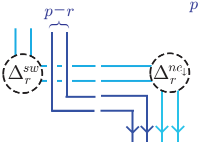
For $p > r > 0$, we consider two subsequences of C_G for a grid diagram G,

$$C_G^+ = (\cdots, sw_i, ne_{j\downarrow} \text{or } ne_{j\uparrow}, \cdots),\ C_G^- = (\cdots, se_{i'}, nw_{j'\downarrow} \text{or } nw_{j'\uparrow}, \cdots).$$

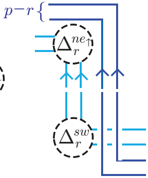
Proposition C2

(1) The products $\Delta_{p,r}^{sw_i}\Delta_{p,r}^{ne_{j\downarrow}}$ and $\Delta_{p,r}^{sw_i}\Delta_{p,r}^{ne_{j\uparrow}}$ of partially half twisted corners $\Delta_{p,r}^{sw_i}$, $\Delta_{p,r}^{ne_{j\downarrow}}$ and $\Delta_{p,r}^{ne_{j\uparrow}}$ is presented by a positive (p, r)-twist word β_p^r.

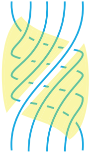
(2) The products $\Delta_{p,r}^{se_{i'}}\Delta_{p,r}^{nw_{j'\downarrow}}$ and $\Delta_{p,r}^{se_{i'}}\Delta_{p,r}^{nw_{j'\uparrow}}$ of partially half twisted corners $\Delta_{p,r}^{se_{i'}}$, $\Delta_{p,r}^{ne_{j'\downarrow}}$ and $\Delta_{p,r}^{ne_{j'\uparrow}}$ is presented by a negative (p, r) twist word $(\beta_p^{-1})^r$.



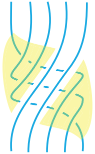
$\Delta_{p,r}^{sw}\ \Delta_{p,r}^{ne_\downarrow}$



$\Delta_{p,r}^{sw}\ \Delta_{p,r}^{ne_\uparrow}$



β_5^4



β_5^3

Canonical (p, q)-cabling algorithm

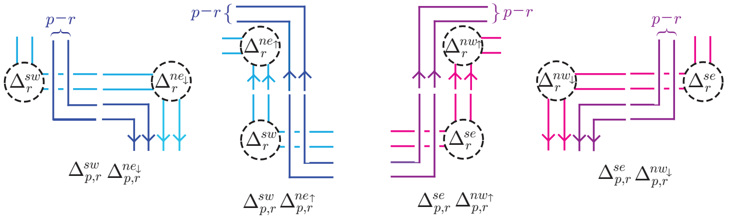
For $p > r > 0$, we consider two subsequences of C_G for a grid diagram G,

$$C_G^+ = (\cdots, sw_i, ne_{j\downarrow} \text{or } ne_{j\uparrow}, \cdots), \;\; C_G^- = (\cdots, se_{i'}, nw_{j'\downarrow} \text{or } nw_{j'\uparrow}, \cdots).$$

Proposition C2

(1) The products $\Delta_{p,r}^{sw_i}\Delta_{p,r}^{ne_{j\downarrow}}$ and $\Delta_{p,r}^{sw_i}\Delta_{p,r}^{ne_{j\uparrow}}$ of partially half twisted corners $\Delta_{p,r}^{sw_i}$, $\Delta_{p,r}^{ne_{j\downarrow}}$ and $\Delta_{p,r}^{ne_{j\uparrow}}$ is presented by a positive (p, r)-twist word β_p^r.

(2) The products $\Delta_{p,r}^{se_{i'}}\Delta_{p,r}^{nw_{j'\downarrow}}$ and $\Delta_{p,r}^{se_{i'}}\Delta_{p,r}^{nw_{j'\uparrow}}$ of partially half twisted corners $\Delta_{p,r}^{se_{i'}}$, $\Delta_{p,r}^{ne_{j'\downarrow}}$ and $\Delta_{p,r}^{ne_{j'\uparrow}}$ is presented by a negative (p, r) twist word $(\beta_p^{-1})^r$.



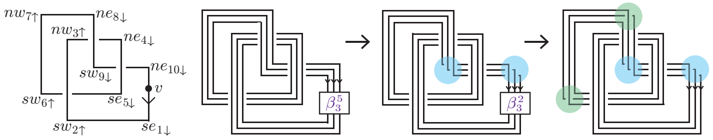
Canonical arc index of $K^{(p,q)}$

Our main concerns :

- ★ How many β_p^p and β_p^{-p} can be expresses as products of two p-strands half-twisted corners?
- ★ How many β_p^r and β_p^{-r} for $p > r > 0$ can be expresses as products of two p-strands partially half-twisted corners?

Canonical (p, q)-cabling algorithm

Step IV : A canonical grid diagram of $K^{(3,14)}$



G $\qquad$ $S_G^{(3,14)}$ $\qquad$ $G^{(3,14)}$

Since $p = 3, q = 14$ and $w(G) = 3$, $(p, q - pw(G))$-twisted braid is β_3^5.

$$C_G^+ : (sw_{2\uparrow}, ne_{4\downarrow}, sw_{6\uparrow}, ne_{8\downarrow}, sw_{9\downarrow}, ne_{10\downarrow}),$$

$$[C^+_{S_G^{(3,14)}}, \beta_3^5] : [(sw_{2\uparrow}^{(3)}, ne_{4\downarrow}^{(3)}, sw_{6\uparrow}^{(3)}, ne_{8\downarrow}^{(3)}, sw_{9\downarrow}^{(3)}, ne_{10\downarrow}^{(3)}), \beta_3^5]$$

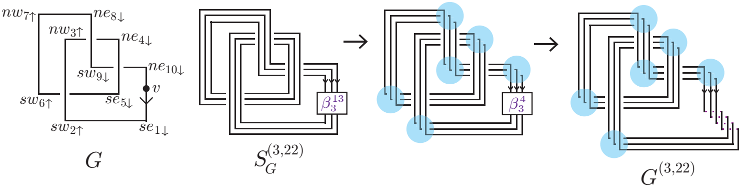
$$\to [(sw_{2\uparrow}^{(3)}, ne_{4\downarrow}^{(3)}, sw_{6\uparrow}^{(3)}, ne_{8\downarrow}^{(3)}, \Delta_3^{sw_9}, \Delta_3^{ne_{10}}), \beta_3^2]$$

$$\to [(sw_{2\uparrow}^{(3)}, ne_{4\downarrow}^{(3)}, \Delta_{3,2}^{sw_6}, \Delta_{3,2}^{ne_{8\downarrow}}, \Delta_3^{sw_9}, \Delta_3^{ne_{10}}), \beta_3^0] : [C^+_{G^{(3,14)}}, \beta_3^0]$$

Therefore we have $\alpha(G^{(3,14)}) = 3 \times \alpha(G) = 15$.

Canonical (p, q)-cabling algorithm

Step IV : A canonical grid diagram of $K^{(3,22)}$



Since $p = 3, q = 22$ and $w(G) = 3$, $(p, q - pw(G))$-twisted braid is β_3^{13}.

$$C_G^+ : (sw_{2\uparrow}, ne_{4\downarrow}, sw_{6\uparrow}, ne_{8\downarrow}, sw_{9\downarrow}, ne_{10\downarrow}),$$

$$[C^+_{S_G^{(3,22)}}, \beta_3^{13}] : [(sw^{(3)}_{2\uparrow}, ne^{(3)}_{4\downarrow}, sw^{(3)}_{6\uparrow}, ne^{(3)}_{8\downarrow}, sw^{(3)}_{9\downarrow}, ne^{(3)}_{10\downarrow}), \beta_3^{13}]$$

$$\rightarrow [(sw^{(3)}_{2\uparrow}, ne^{(3)}_{4\downarrow}, sw^{(3)}_{6\uparrow}, ne^{(3)}_{8\downarrow}, \Delta_3^{sw9}, \Delta_3^{ne10}), \beta_3^{10}]$$

$$\rightarrow [(sw^{(3)}_{2\uparrow}, ne^{(3)}_{4\downarrow}, \Delta_3^{sw6}, \Delta_3^{ne8}, \Delta_3^{sw9}, \Delta_3^{ne10}), \beta_3^{7}]$$

$$\rightarrow [(\Delta_3^{sw2}, \Delta_3^{ne4}, \Delta_3^{sw6}, \Delta_3^{ne8}, \Delta_3^{sw9}, \Delta_3^{ne10}), \beta_3^{4}] : [C^+_{G^{(3,22)}}, \beta_3^{4}]$$

Therefore we have $\alpha(G^{(3,22)}) = 3 \times \alpha(G) + 4 = 19$.

Canonical arc index of $K^{(p,q)}$

Let G be a grid diagram of a knot K and p, q be integers with $p > 0$.

Theorem C1

Suppose that $n(G) = q - pw(G)$.

(1) If $n(G) \geq 0$, $\exists!\ m(G) \geq 0$ s.t. $pm(G) \leq n(G) < p(m(G) + 1)$. Then,

$$\alpha_c(K^{(p,q)}) \leq \begin{cases} p\alpha(G) & \text{if } ne(G) > m(G) \\ p(\alpha(G) + tb(G^*)) + q & \text{if } ne(G) \leq m(G) \end{cases}$$

(2) If $n(G) < 0$, $\exists!\ m'(G) < 0$ s.t. $p(m'(G) - 1) < n(G) \leq pm'(G)$. Then,

$$\alpha_c(K^{(p,q)}) \leq \begin{cases} p\alpha(G) & \text{if } se(G) > -m'(G) \\ p(\alpha(G) + tb(G)) - q & \text{if } se(G) \leq -m'(G) \end{cases}$$

Arc index of $K^{(p,q)}$

Let G be a grid diagram of a knot K and p, q be integers with $p > 0$.

Corollary

Suppose that $n(G) = q - pw(G)$.

(1) If $n(G) \geq 0$, $\exists!\, m(G)$ s.t. $pm(G) \leq n(G) < p(m(G) + 1)$. Then,

$$\alpha(K^{(p,q)}) \leq \begin{cases} p\alpha(G) & \text{if } ne(G) > m(G) \\ p(\alpha(G) + tb(G^*)) + q & \text{if } ne(G) \leq m(G) \end{cases}$$

(2) If $n(G) < 0$, $\exists!\, m'(G)$ s.t. $p(m'(G) - 1) < n(G) \leq pm'(G)$. Then,

$$\alpha(K^{(p,q)}) \leq \begin{cases} p\alpha(G) & \text{if } se(G) > -m'(G) \\ p(\alpha(G) + tb(G)) - q & \text{if } se(G) \leq -m'(G) \end{cases}$$

Canonical arc index of $K^{(p,q)}$

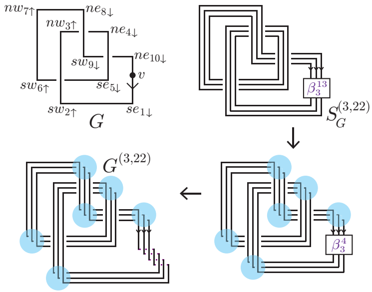
Let G be a grid diagram of a knot K and p, q be integers with $p > 0$.

Theorem C1

Suppose that $n(G) = q - pw(G)$.

(1) If $n(G) \geq 0$, $\exists!\ m(G)$ s.t. $pm(G) \leq n(G) < p(m(G)+1)$. Then,

$$\alpha_c(K^{(p,q)}) \leq \begin{cases} p\alpha(G), & \text{if } ne(G) > m(G) \\ p(\alpha(G) + tb(G^*)) + q. & \text{if } ne(G) \leq m(G) \end{cases}$$



$$\begin{aligned} \alpha(G^{(p,q)}) &= p\alpha(K) + n(G) - p\text{ne}(G) \\ &= p\alpha(G) + (q - pw(G)) - p\text{ne}(G) \\ &= p\alpha(G) + q + p(-w(G) - \text{ne}(G)) \\ &= p\alpha(G) + q + p(w(G^*) - \text{se}(G^*)) \\ &= p\alpha(G) + q + p\text{tb}(G^*) \\ &= p(\alpha(G) + \text{tb}G^*) + q. \end{aligned}$$

Arc index of Kanenobu knots

Main results

Theorem K1

Let $1 \le p \le q$ and $pq \ge 3$. Then

$$\alpha(K(p,q)) = p + q + 6.$$

Theorem K2

Let $p = 0$ and $q \ge 3$. Then

$$q + 6 \le \alpha(K(0,q)) \le q + 7.$$

Theorem K3

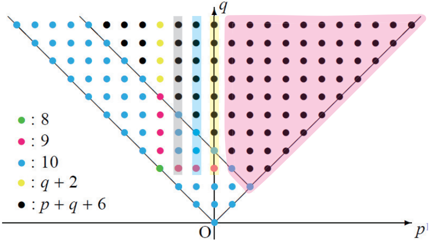
Let $p = -1$ and $q \ge 3$. Then

$$q + 5 \le \alpha(K(-1,q)) \le q + 7.$$

Theorem K4

Let $p = -2$ and $q \ge 3$. Then

$$q + 4 \le \alpha(K(-2,q)) \le q + 7.$$



How to determine $\alpha(K(p,q))$?

Our strategy is ...

- For the lower bound of $\alpha(L)$, we compute the $\text{spread}_a(F_L(a,z))$.

Morton-Beltrami, 1998

For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a,z)) + 2$.

Note : $\text{spread}_a(F_L(a,z)) = \text{spread}_a(\Lambda_D(a,z))$.

- For the upper bound of $\alpha(L)$, we find an arc presentation of L with the minimum number of arcs.

Lemma K1

$$\Lambda_{K(p,q)}(a,z) = \sigma_{p-1}\sigma_{q-1}(\Lambda_{K(0,0)} - 1) + (\sigma_p\sigma_{q+1} - \sigma_{p-1}\sigma_q)(\Lambda_{K(0,1)} - a^{-1}) - \sigma_p\sigma_q(\Lambda_{K(-1,1)} - 1) + a^{-(p+q)},$$

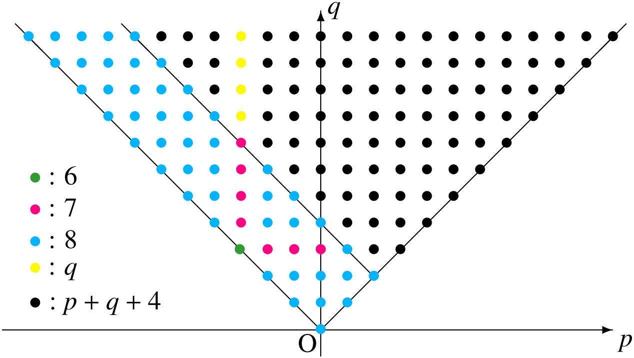
where

$$\sigma_n(\alpha,\beta) = \begin{cases} \frac{\alpha^n-\beta^n}{\alpha-\beta} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\frac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta} & \text{if } n < 0 \end{cases}$$

for

$$\alpha + \beta = z, \quad \alpha\beta = 1.$$

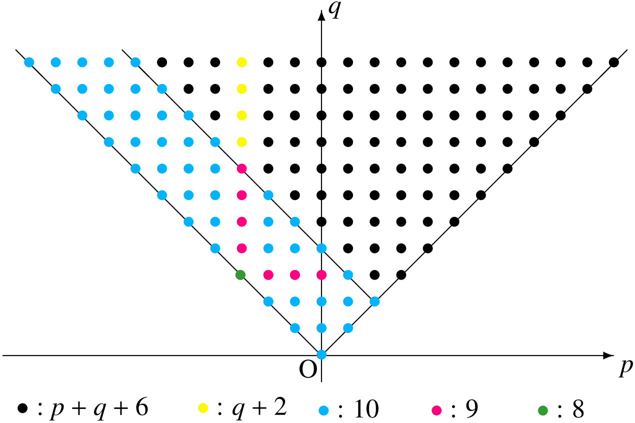
Proposition K0 : spread$_a(F_{K(p,q)}(a,z))$.



Morton-Beltrami, 1998

For any link K, $\alpha(K) \geq \text{spread}_a(F_K(a,z)) + 2$.

Corollary K1 : A lower bound of $\alpha(K(p,q))$



An arc presentation on knot diagrams

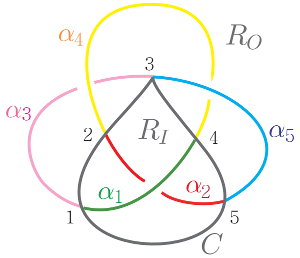
Let D be a diagram of a knot or a link L. Suppose that there is a simple closed curve C meeting D in k distinct points which divide D into k arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following properties:

(1) Each α_i has no self-crossings.

(2) If α_i crosses over α_j at a crossing in R_I(resp. R_O), then $i < j$(resp. $i > j$) and it crosses over α_j at any other crossings with α_j, respectively

(3) For each i, there exists an embedded disk d_i such that $\partial d_i = C$ and $\alpha_i \subset d_i$.

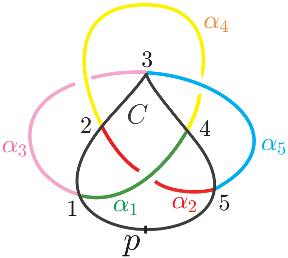
(4) $d_i \cap d_j = C$, for distinct i and j.

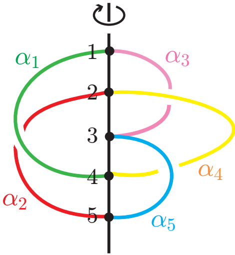


Then the pair (D, C) is called an *arc presentation* of L with k arcs. C is called the *binding circle* of the arc presentation.

An arc presentation on knot diagrams (Cont.)

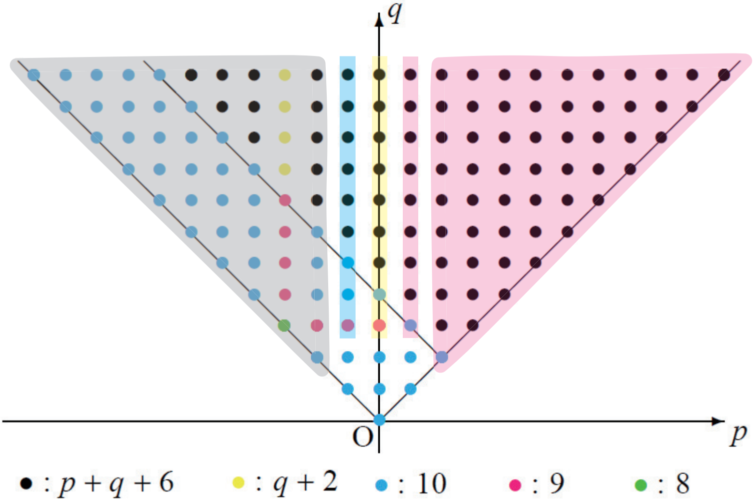
By removing a point P from C away from D, we may identify $C \setminus P$ with the z-axis and each $d_i \setminus P$ with a vertical half plane along the z-axis.





This shows that an arc presentation onto knot diagrams is equivalent to an arc presentation.

Distinguish into five cases



An upper bound of $\alpha(K(p,q))$

Propositon K1

Let $1 \leq p \leq q$ and $pq \geq 3$. Then

$$\alpha(K(p,q)) \leq p + q + 6.$$

Propositon K3

Let $p = -1$ and $q \geq 3$. Then

$$\alpha(K(-1,q)) \leq q + 7.$$

Proposition K2

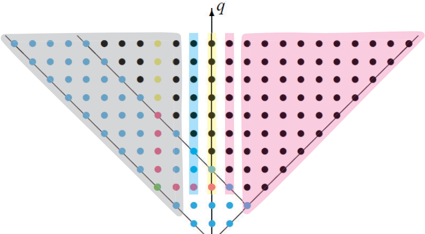
Let $p = 0$ and $q \geq 3$. Then

$$\alpha(K(0,q)) \leq q + 7.$$

Proposition K4

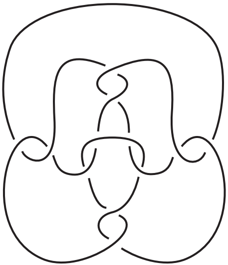
Let $p = -2$ and $q \geq 3$. Then

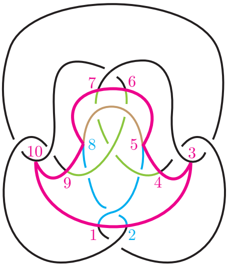
$$\alpha(K(-2,q)) \leq q + 7.$$

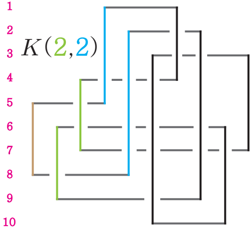


Let $2 \le p \le q$**.**

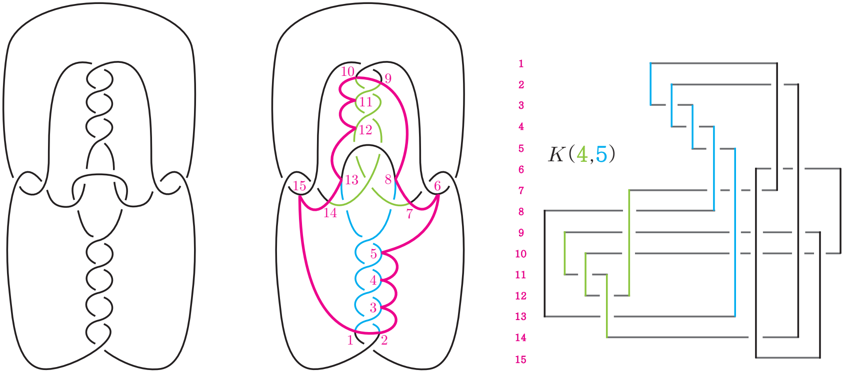
Then $\alpha(K(p,q)) \le p + q + 6$**.**





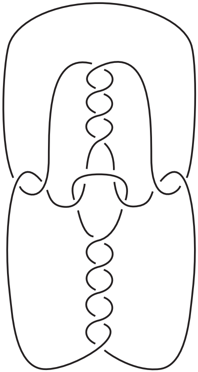


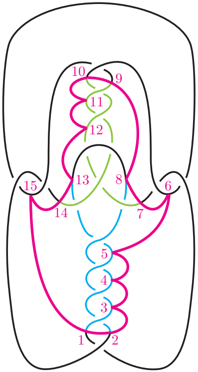
Let $2 \leq p \leq q$. Then $\alpha(K(p,q)) \leq p + q + 6$.

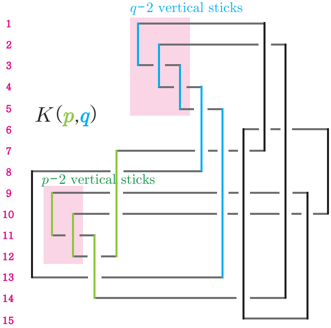


Let $2 \leq p \leq q$.

Then $\alpha(K(p,q)) \leq p+q+6$.







An sharpper upper bound than Proposition K4.

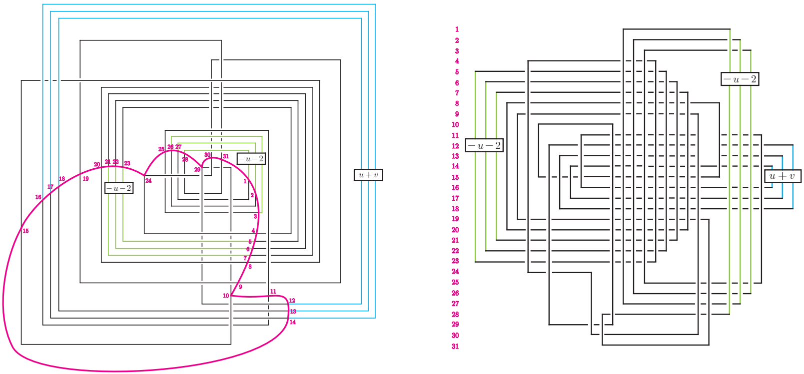
Proposition K4

Let $2 \leq -p \leq q$. Then $\alpha(K(p,q)) \leq -p+q+6$.

Let $p = 2u+\varepsilon$ and $q = 2v+\delta$ with $2 \leq -p \leq q$, where $\varepsilon, \delta = 0, 1$. Then

$$\alpha(K(p,q)) \leq \begin{cases} -p+q+5 & \text{if } \; p=-2,\; q \geq 3 \\ -p+q+4 & \text{if } \; (\varepsilon,\delta)=(0,0),\; q=-p \geq 4 \\ -p+q+3 & \text{if } \; (\varepsilon,\delta)=(0,0),\; q>-p \geq 4 \\ -p+q+4 & \text{if } \; (\varepsilon,\delta)=(1,0),\; -p \geq 3 \\ -p+q+4 & \text{if } \; (\varepsilon,\delta)=(0,1),\; -p \geq 4 \\ -p+q+5 & \text{if } \; (\varepsilon,\delta)=(1,1),\; -p \geq 3 \end{cases} \qquad \text{[Prop.K5]}$$

Prop.5. Let $v + 1 \geq -u \geq 2$. Then $\alpha(K(2u+1, 2v+1)) \leq -(2u+1)+(2v+1)+5$.



An arc presentation of $K(2u + 1, 2v + 1)$ with $v \geq -u \geq 2$

$\beta(K(2u + 1, 2v + 1)) \leq -u + v + 2$ for $v \geq -u \geq 2$

(H.Takioka, *On the braid index of Kanenobu knots II* (preprint))

Thank you very much.