A polynomial invariant and the forbidden move of virtual knots

Migiwa Sakurai

migiwa@ge.ibaraki-ct.ac.jp

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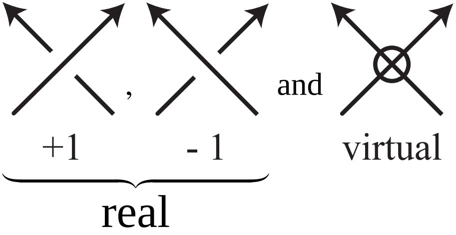
Ibaraki National College of Technology
Natural Sciences

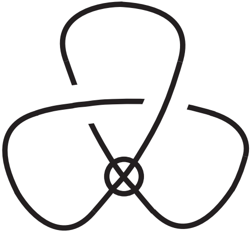
Contents

§1. Preliminaries

D : a virtual knot diagram

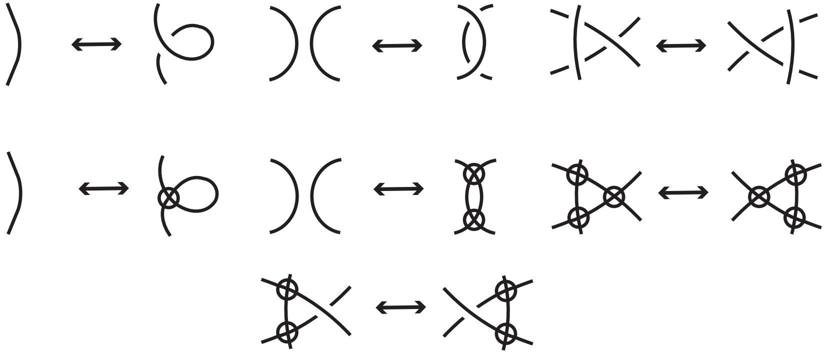
$\overset{\text{def}}{\Leftrightarrow}$ D ; a knot diagram with



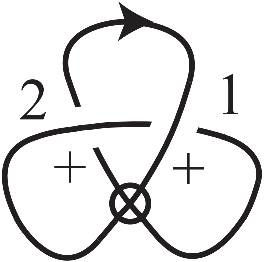


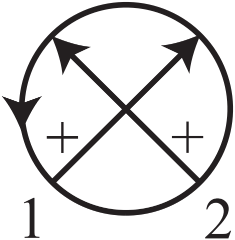
K : a virtual knot

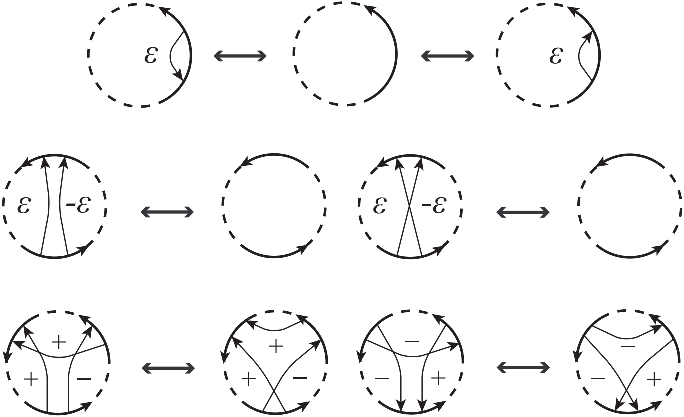
$\overset{\text{def}}{\Leftrightarrow}$ K ; an eq. class of virtual knot diagrams under GRM



A Gauss daigram : a preimage of a virtual knot diagram with info. of real crossings

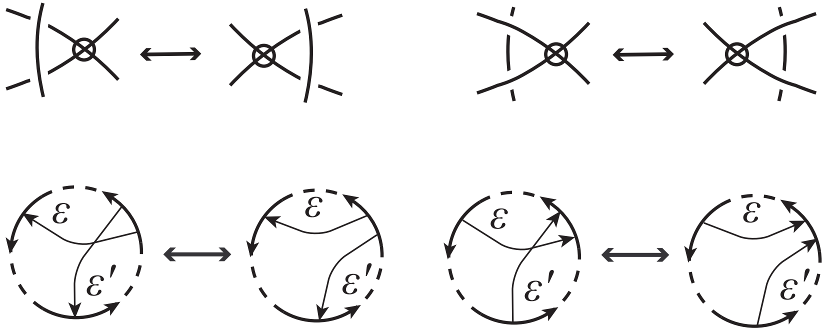






GRM of Gauss diagrams

{All virtual knots} $\overset{\text{one-to-one}}{\longleftrightarrow}$ $\left\{ \begin{array}{l} \text{All eq. classes of Gauss} \\ \text{diagrams by GRM} \end{array} \right\}$

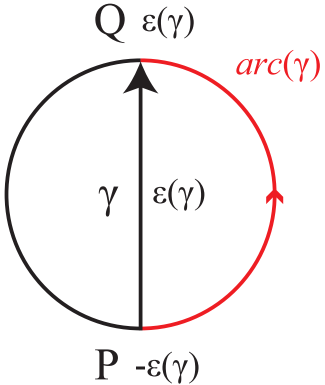


Forbidden moves (F)

$$u_F(K) = min\left\{\text{the number of } F \text{ s.t } K \overset{F \text{ or GRM}}{\rightleftharpoons} \cdots \overset{F \text{ or GRM}}{\rightleftharpoons} \bigcirc \right\}$$

Invariants for virtual knots

K : a virtual knot
G : a Gauss diagram of K
P, Q : points on the circle $\mathbb{S}^1$ of G
$\gamma = \overrightarrow{PQ}$: a chord oriented from P into Q
$\varepsilon(\gamma)$: the sign of γ
$\varepsilon(P) = -\varepsilon(\gamma)$
$\varepsilon(Q) = \varepsilon(\gamma)$
$arc(\gamma)$: the arc form P to Q along the ori. in $\mathbb{S}^1$



$$i(\gamma) = \sum_{R\ :\ \text{points on } arc(\gamma)} \varepsilon(R) \quad : \text{the index of } \gamma$$

Definition 1

1. ([Satoh-Taniguchi]) $J_n(K) = \sum_{i(\gamma)=n} \varepsilon(\gamma)$: n-writhe
2. ([Henrich]) $\mathbf{p}_t(K) = \sum_{\gamma} \varepsilon(\gamma)(t^{|i(\gamma)|} - 1)$: index polynomial
3. ([Cheng]) $f_K(t) = \sum_{i(\gamma):odd} \varepsilon(\gamma)t^{1-i(\gamma)}$: odd writhe poly.

Proposition 2 (Satoh-Taniguchi)

1. *$\mathbf{p}_t(K)$ is induced from $J_n(K)$.*

$$\mathbf{p}_t(K) = \sum_{n>0} \{J_n(K) + J_{-n}(K)\}(t^n - 1).$$

2. *$f_K(t)$ is induced from $J_n(K)$.*

$$f_K(t) = \sum_{n\in\mathbb{Z}} J_{1-2n}(K)t^{2n}.$$

Definition 3

$$\mathbf{q}_t(K) = \sum_{n\in\mathbb{Z}} J_n(K)(t^n - 1)$$
$$= \sum_{\gamma} \varepsilon(\gamma)(t^{i(\gamma)} - 1).$$

1. $\mathbf{p}_t(K)$ is induced from $\mathbf{q}_t(K)$.

$$\mathbf{q}_t(K) = \sum_{n\geq 0} \alpha_n t^n + \sum_{n<0} \alpha_n t^n \Rightarrow \mathbf{p}_t(K) = \sum_{n\geq 0} \alpha_n t^n + \sum_{n<0} \alpha_n t^{-n} \quad (1)$$

2. $f_K(t)$ is induced from $\mathbf{q}_t(K)$.

$$\mathbf{q}_{t^{-1}}(K) = \sum_{n\in\mathbb{Z}} \alpha_{2n} t^{2n} + t^{-1} f_K(t).$$

§2. forbidden moves and $\mathbf{q}_t(K)$

Theorem 4

K, K' s.t. $K \overset{F}{\rightleftharpoons} K'$

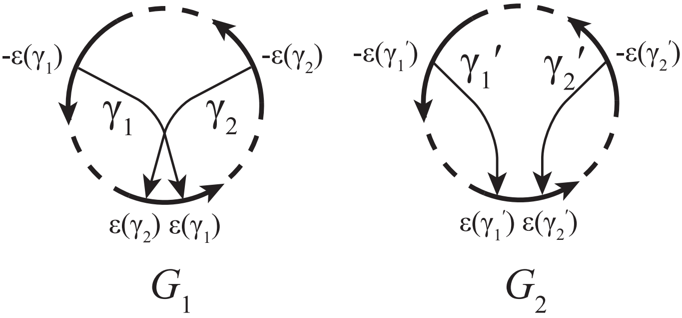
$$\mathbf{q}_t(K) - \mathbf{q}_t(K') = (t-1)(\pm t^{\ell} \pm t^{m}) \qquad (\ell,\ m \in \mathbb{Z}).$$

Corollary 5

$\mathbf{q}_t(K) := (t-1)\sum_{n\in\mathbb{Z}} a_n t^n$

$$u_F(K) \geq \frac{\sum_{n\in\mathbb{Z}} |a_n|}{2}.$$

Sketch Proof



$$\begin{cases} i(\gamma_1') = i(\gamma_1) - \varepsilon(\gamma_2) \\ i(\gamma_2') = i(\gamma_2) + \varepsilon(\gamma_1). \end{cases}$$

$$\varepsilon(\gamma_i) = \varepsilon(\gamma_i') \quad (i = 1, 2).$$

forbidden moves and $J_n(K)$

$$
\begin{aligned}
\mathbf{q}_t(K_1)-&\mathbf{q}_t(K_2)\\
&=\varepsilon(\gamma_1)(t^{i(\gamma_1)}-1)+\varepsilon(\gamma_2)(t^{i(\gamma_2)}-1)\\
&\quad-\varepsilon(\gamma_1')(t^{i(\gamma_1')}-1)-\varepsilon(\gamma_2')(t^{i(\gamma_2')}-1)\\
&=\begin{cases}
(t-1)(t^{i(\gamma_1)-1}-t^{i(\gamma_2)}) & (\varepsilon(\gamma_i)=1)\\
(t-1)(-t^{i(\gamma_1)}+t^{i(\gamma_2)}) & (\varepsilon(\gamma_1)=1,\ \varepsilon(\gamma_2)=-1)\\
(t-1)(-t^{i(\gamma_1)-1}+t^{i(\gamma_2)-1}) & (\varepsilon(\gamma_1)=-1,\ \varepsilon(\gamma_2)=1)\\
(t-1)(t^{i(\gamma_1)}-t^{i(\gamma_2)-1}) & (\varepsilon(\gamma_i)=-1)
\end{cases}
\end{aligned}
\tag{2}
$$

□

From Thm. 4,

$$\begin{aligned}&\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)\\ &= \varepsilon(\gamma_1)t^{i(\gamma_1)} + \varepsilon(\gamma_2)t^{i(\gamma_2)} - \varepsilon(\gamma_1)t^{i(\gamma_1)-\varepsilon(\gamma_2)} - \varepsilon(\gamma_2)t^{i(\gamma_2)+\varepsilon(\gamma_1)}\end{aligned} \quad (3)$$

If 2 terms clear in $\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)$,

- $\varepsilon(\gamma_1)t^{i(\gamma_1)} + \varepsilon(\gamma_2)t^{i(\gamma_2)} = 0$
  $\Rightarrow \varepsilon(\gamma_1) = -\varepsilon(\gamma_2),\ i(\gamma_1) = i(\gamma_2)$
  $\Rightarrow \varepsilon(\gamma_1) = -\varepsilon(\gamma_2),\ i(\gamma_1) - \varepsilon(\gamma_2) = i(\gamma_2) + \varepsilon(\gamma_1)$
  $\Rightarrow -\varepsilon(\gamma_1)t^{i(\gamma_1)-\varepsilon(\gamma_2)} - \varepsilon(\gamma_2)t^{i(\gamma_2)+\varepsilon(\gamma_1)} = 0$
  $\therefore \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = 0$
- Otherwise, $\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = 0$.

Since " the coefficient of t^n " $= J_n$ $(n \neq 0)$,

$$\sum_{n\neq 0} |J_n(K_1) - J_n(K_2)|$$
$$= \begin{cases} 4 & (\text{Any term isn't constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 3 & (\text{Only 1 term is constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 2 & (\text{Only 2 terms are constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 0 & (\text{All terms are constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) \text{ or} \\ & \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = 0) \end{cases} \quad (4)$$

Since " the coefficient of t^n " $= J_n$ $(n \neq 0)$,

$$
\begin{aligned}
&\sum_{n\neq 0} |J_n(K_1) - J_n(K_2)| \\
&= \begin{cases} 4 & (\text{Any term isn't constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 3 & (\text{Only 1 term is constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 2 & (\text{Only 2 terms are constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 0 & (\text{All terms are constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) \text{ or} \\ & \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = 0) \end{cases}
\end{aligned} \tag{4}
$$

$$
\therefore u_F(K) \geq \frac{\sum_{n\neq 0} |J_n(K)|}{4}.
$$

Theorem 6

1. ([S])
$\mathbf{p}_t(K) := (t-1)\sum_{n\geq 0} b_n t^n$
$$u_F(K) \geq \frac{\sum_{n\geq 0}|b_n|}{2}.$$

2. ([Crans, Ganzell, Mellor])
$f_K(t) := \sum_{n\neq 0} c_n t^n$
$$u_F(K) \geq \frac{\sum_{n\neq 0}|c_n|}{2}.$$

Proposition 7

$$u_F(K) \geq \frac{\sum_{n\neq 0}|a_n|}{2} \geq \frac{\sum_{n\geq 0}|b_n|}{2}, \frac{\sum_{n\neq 0}|c_n|}{2}, \frac{\sum_{n\neq 0}|J_n(K)|}{4}.$$

$\mathbf{q}_t$ VS $\mathbf{p}_t$

In $\mathbf{q}_t$,

$$t^n - 1 = (t-1)(1 + t + \cdots + t^{n-1}),$$
$$t^{-n} - 1 = (t-1)(-t^{-1} - t^{-2} - \cdots - t^{-n}).$$

From (1),

$$b_n = a_n - a_{-n-1}.$$
$$|b_n| = |a_n - a_{-n-1}| \leq |a_n| + |a_{-n-1}|.$$

$$\sum_{n\geq 0}|b_n| \leq \sum_{n\geq 0}\left\{|a_n| + |a_{-n-1}|\right\} = \sum_{n\in\mathbb{Z}}|a_n|$$

The equation (1),

$$\mathbf{q}_t(K) = \sum_{n\geq 0} \alpha_n t^n + \sum_{n<0} \alpha_n t^n \Rightarrow \mathbf{p}_t(K) = \sum_{n\geq 0} \alpha_n t^n + \sum_{n<0} \alpha_n t^{-n}$$

$\boxed{\mathbf{q}_t \text{ VS } J_n}$

$K_1 \overset{\mathrm{F}}{\rightleftharpoons} K_2$

Assuming $\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = (t-1)(\pm t^{\ell} \pm t^m)$ $(\pm t^{\ell} \pm t^m \neq 0)$

$$\frac{\sum_{n\neq 0} |a_n|}{2} = 1 \text{ but } \frac{\sum_{n\neq 0} |J_n(K)|}{4} = 1,\ \frac{3}{4},\ \frac{1}{2},\ 0 \text{ from (4).}$$

$$\therefore\ \frac{\sum_{n\neq 0} |a_n|}{2} \geq \frac{\sum_{n\neq 0} |J_n(K)|}{4}$$

The equation (4),

$$\sum_{n\neq 0} |J_n(K_1) - J_n(K_2)|$$

$$= \begin{cases} 4 & (\text{Any term isn't constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 3 & (\text{Only 1 term is constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 2 & (\text{Only 2 terms are constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2)) \\ 0 & (\text{All terms are constant in } \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) \text{ or} \\ & \mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = 0) \end{cases}$$

$\mathbf{q}_t$ VS $f_K(t)$

In (3),

$i(\gamma_1)$ is odd $\Rightarrow$ $i(\gamma_1) - \varepsilon(\gamma_2)$ is even.

$i(\gamma_2)$ is odd $\Rightarrow$ $i(\gamma_2) + \varepsilon(\gamma_1)$ is even.

$$f_{K_1}(t) - f_{K_2}(t)$$
$$= \begin{cases} \varepsilon(\gamma_1)t^{1-i(\gamma_1)} - \varepsilon(\gamma_2)t^{1-i(\gamma_2)} & (i(\gamma_i) : \text{odd}) \\ \varepsilon(\gamma_1)t^{1-i(\gamma_1)} - \varepsilon(\gamma_2)t^{1-i(\gamma_2)-\varepsilon(\gamma_1)} & (i(\gamma_1) : \text{odd}, i(\gamma_2) : \text{even}) \\ \varepsilon(\gamma_2)t^{1-i(\gamma_2)} - \varepsilon(\gamma_1)t^{1-i(\gamma_1)+\varepsilon(\gamma_2)} & (i(\gamma_1) : \text{even}, i(\gamma_2) : \text{odd}) \\ -\varepsilon(\gamma_1)t^{1-i(\gamma_1)+\varepsilon(\gamma_2)} - \varepsilon(\gamma_2)t^{1-i(\gamma_2)-\varepsilon(\gamma_1)} & (i(\gamma_i) : \text{even}) \end{cases}$$

The equation (3),

$$
\begin{aligned}
&\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) \\
&= \varepsilon(\gamma_1)t^{i(\gamma_1)} + \varepsilon(\gamma_2)t^{i(\gamma_2)} - \varepsilon(\gamma_1)t^{i(\gamma_1)-\varepsilon(\gamma_2)} - \varepsilon(\gamma_2)t^{i(\gamma_2)+\varepsilon(\gamma_1)}
\end{aligned}
$$

$\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) = (t-1)(\pm t^\ell \pm t^m)$
From (2),
$\pm t^\ell \pm t^m = 0 \Leftrightarrow$

1. If $\varepsilon(\gamma_i) = 1$, $i(\gamma_1) - 1 = i(\gamma_2)$.
2. If $\varepsilon(\gamma_1) = 1,\ \varepsilon(\gamma_2) = -1$, $i(\gamma_1) = i(\gamma_2)$.
3. If $\varepsilon(\gamma_1) = -1,\ \varepsilon(\gamma_2) = 1$, $i(\gamma_1) - 1 = i(\gamma_2) - 1$.
4. If $\varepsilon(\gamma_i) = -1$, $i(\gamma_1) = i(\gamma_2) - 1$.

$$f_{K_1}(t) - f_{K_2}(t) = 0 \text{ in } 1, 2, 3, 4$$

$$\therefore \frac{\sum_{n\neq 0} |a_n|}{2} \geq \frac{\sum_{n\neq 0} |c_n|}{2}$$

The equation (2),

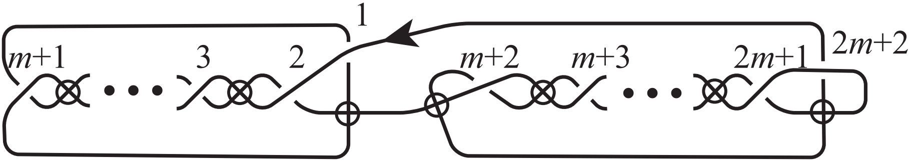
$$
\begin{aligned}
\mathbf{q}_t(K_1) - \mathbf{q}_t(K_2) \\
&= \varepsilon(\gamma_1)(t^{i(\gamma_1)} - 1) + \varepsilon(\gamma_2)(t^{i(\gamma_2)} - 1) \\
&\quad - \varepsilon(\gamma'_1)(t^{i(\gamma'_1)} - 1) - \varepsilon(\gamma'_2)(t^{i(\gamma'_2)} - 1) \\
&= \begin{cases}
(t-1)(t^{i(\gamma_1)-1} - t^{i(\gamma_2)}) & (\varepsilon(\gamma_i) = 1) \\
(t-1)(-t^{i(\gamma_1)} + t^{i(\gamma_2)}) & (\varepsilon(\gamma_1) = 1,\ \varepsilon(\gamma_2) = -1) \\
(t-1)(-t^{i(\gamma_1)-1} + t^{i(\gamma_2)-1}) & (\varepsilon(\gamma_1) = -1,\ \varepsilon(\gamma_2) = 1) \\
(t-1)(t^{i(\gamma_1)} - t^{i(\gamma_2)-1}) & (\varepsilon(\gamma_i) = -1)
\end{cases}
\end{aligned}
$$

§3. Examples

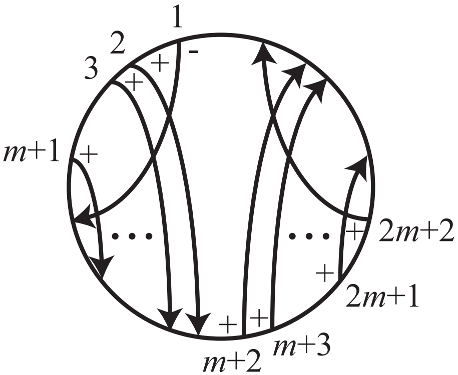
Example 8

$^\forall k \in \mathbb{N}$, $^\exists K_k$ s.t. "$u_F(K_k)$ can not be determined by $\mathbf{p}_t(K_k)$, $f_{K_k}(t)$ $J_n(K_k)$", "it can be determined by $\mathbf{q}_t(K_k)$".

m : *even* (> 0)



K_m


1
2
3
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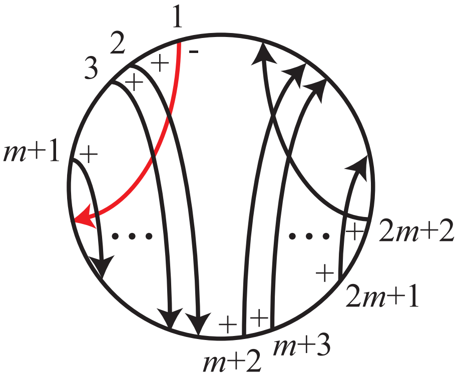
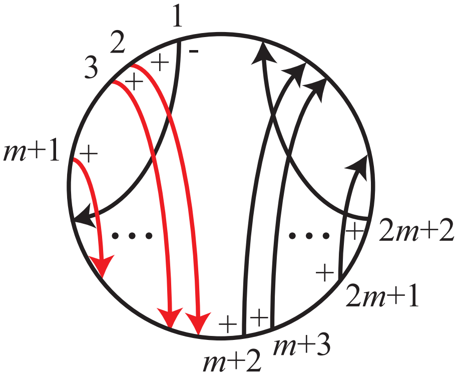
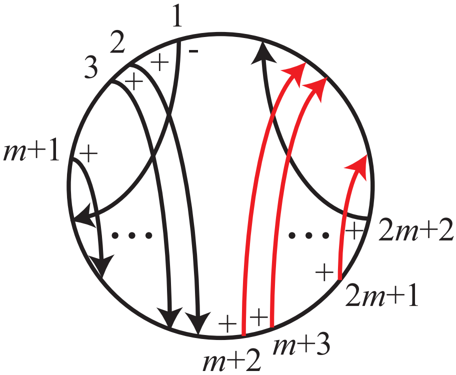


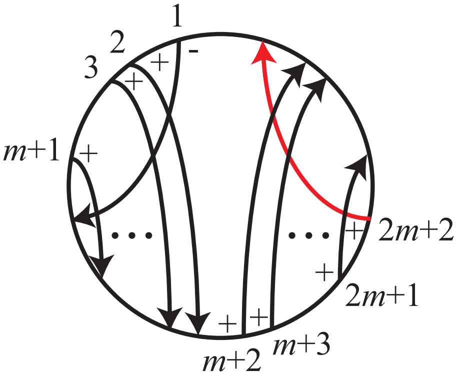
$i(1) = -m$



$$i(1) = -m$$
$$i(2) = -1$$
$$\vdots$$
$$i(m+1) = -1$$



$$i(1) = -m$$
$$i(2) = -1$$
$$\vdots$$
$$i(m+1) = -1$$
$$i(m+2) = -1$$
$$\vdots$$
$$i(2m+1) = -1$$



$$i(1) = -m$$
$$i(2) = -1$$
$$\vdots$$
$$i(m+1) = -1$$
$$i(m+2) = -1$$
$$\vdots$$
$$i(2m+1) = -1$$
$$i(2m+2) = m$$

$\mathbf{p}_t(K_m)$

$$\mathbf{p}_t(K_m) = 2m(t-1).$$
$$\to \sum_{n>0} |b_n| = 2m.$$

$f_{K_m}(t)$

$$f_{K_m}(t) = 2mt^2$$
$$\to \sum_{n\in\mathbb{Z}} |c_n| = 2m.$$

$J_n(K_m)$

$$J_{-m}(K_m) = -1,\ J_{-1}(K_m) = 2m,\ J_m(K_m) = 1$$

$\mathbf{p}_t(K_m)$

$$\mathbf{p}_t(K_m) = 2m(t-1).$$

$$\rightarrow \sum_{n>0} |b_n| = 2m. \quad \therefore \; u_F(K_m) \geq m.$$

$f_{K_m}(t)$

$$f_{K_m}(t) = 2mt^2$$

$$\rightarrow \sum_{n\in\mathbb{Z}} |c_n| = 2m.$$

$J_n(K_m)$

$$J_{-m}(K_m) = -1, \; J_{-1}(K_m) = 2m, \; J_m(K_m) = 1$$

$\mathbf{p}_t(K_m)$

$$\mathbf{p}_t(K_m) = 2m(t-1).$$
$$\to \sum_{n>0} |b_n| = 2m. \quad \therefore\ u_F(K_m) \geq m.$$

$f_{K_m}(t)$

$$f_{K_m}(t) = 2mt^2$$
$$\to \sum_{n\in\mathbb{Z}} |c_n| = 2m. \quad \therefore\ u_F(K_m) \geq m.$$

$J_n(K_m)$

$$J_{-m}(K_m) = -1,\ J_{-1}(K_m) = 2m,\ J_m(K_m) = 1$$

$\mathbf{p}_t(K_m)$

$$\mathbf{p}_t(K_m) = 2m(t-1).$$

$$\to \sum_{n>0} |b_n| = 2m. \quad \therefore\ u_F(K_m) \geq m.$$

$f_{K_m}(t)$

$$f_{K_m}(t) = 2mt^2$$

$$\to \sum_{n\in\mathbb{Z}} |c_n| = 2m. \quad \therefore\ u_F(K_m) \geq m.$$

$J_n(K_m)$

$$J_{-m}(K_m) = -1,\ J_{-1}(K_m) = 2m,\ J_m(K_m) = 1$$

$$\therefore\ u_F(K_m) \geq \frac{m+1}{2}.$$

$\mathbf{q}_t(K_m)$

$$\mathbf{q}_t(K_m) = (t-1)\{t^{m-1} + \cdots + 1 + (-2m+1)t^{-1} + \cdots + t^{-m}\}$$

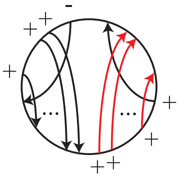
$$\rightarrow \sum_{n \in \mathbb{Z}} |a_k| = 4m - 2.$$

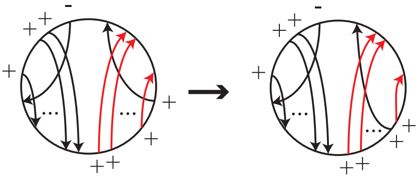
$\mathbf{q}_t(K_m)$

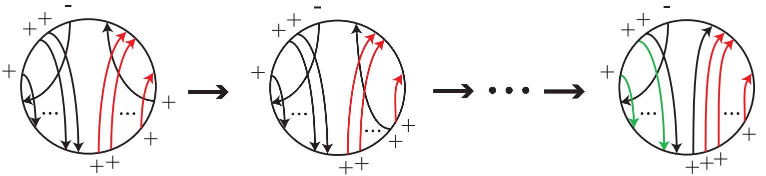
$$\mathbf{q}_t(K_m) = (t-1)\{t^{m-1} + \cdots + 1 + (-2m+1)t^{-1} + \cdots + t^{-m}\}$$

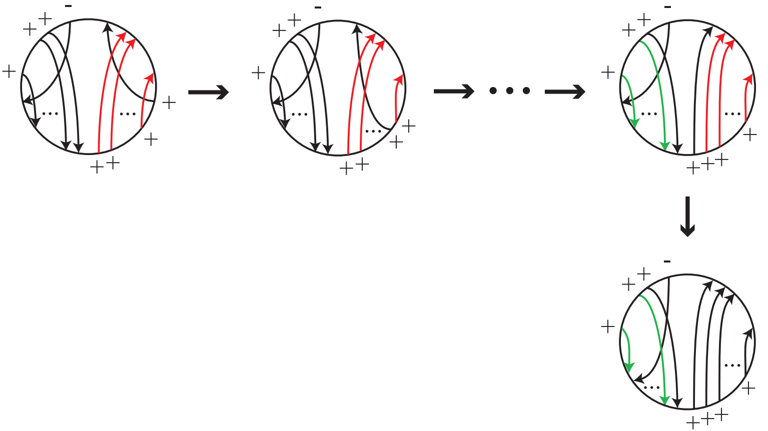
$$\rightarrow \sum_{n\in\mathbb{Z}} |a_k| = 4m - 2.$$

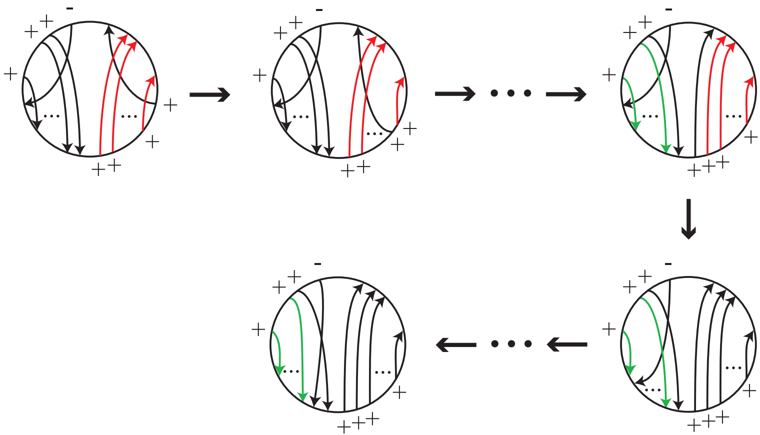
$$\therefore\ u_F(K_m) \geq 2m - 1.$$

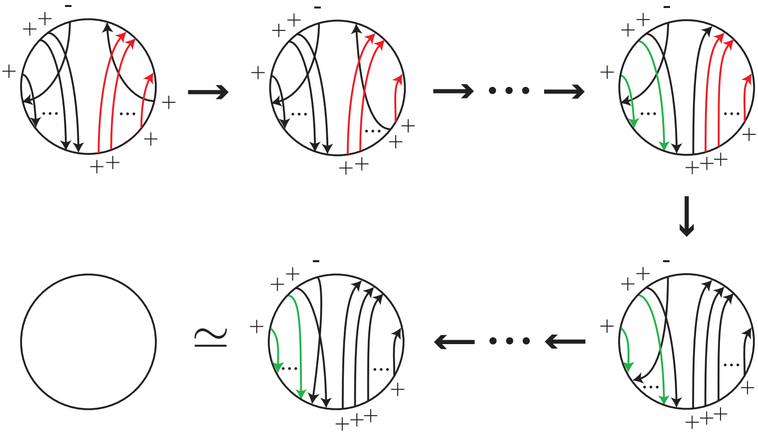




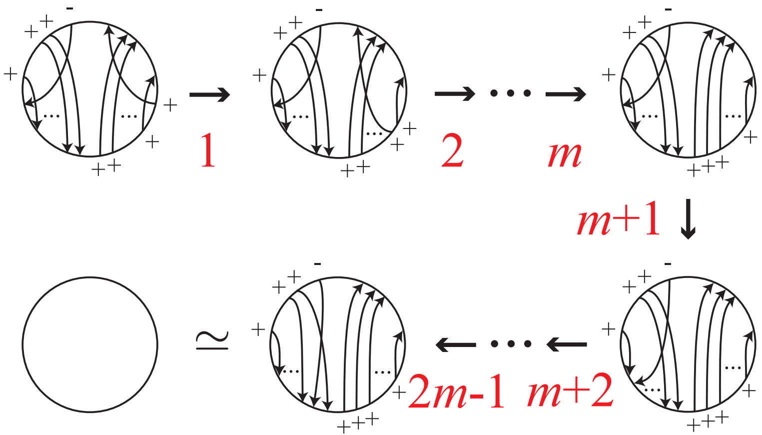








$\simeq$



$$u_F(K_m) = 2m - 1.$$

Unknotting numbers for virtual knots with with up to 4 real crossing points

K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$
0.1	0	4.6	1-2	4.20	1	4.34	1	4.48	3
2.1	1	4.7	2	4.21	2	4.35	1	4.49	1
3.1	1	4.8	1-2	4.22	1	4.36	2	4.50	1
3.2	1	4.9	1-2	4.23	1	4.37	3	4.51	1
3.3	2	4.10	1-2	4.24	2-3	4.38	1	4.52	1
3.4	1	4.11	2	4.25	2	4.39	1	4.53	2
3.5	2-3	4.12	1-2	4.26	2	4.40	1	4.54	1
3.6	1-4	4.13	1-2	4.27	1-2	4.41	1	4.55	1
3.7	2-3	4.14	1-2	4.28	2	4.42	1	4.56	1
4.1	2	4.15	2	4.29	2	4.43	2	4.57	1
4.2	1-2	4.16	1	4.30	1-2	4.44	1-2	4.58	1
4.3	2	4.17	1	4.31	1	4.45	2	4.59	1
4.4	1	4.18	1	4.32	1	4.46	1-2	4.60	1
4.5	1	4.19	1-2	4.33	1	4.47	2	4.61	1-4

K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$	K	$u_F(K)$
4.62	3-4	4.76	1	4.90	1-2	4.104	3-5
4.63	2	4.77	1	4.91	4-5	4.105	1-5
4.64	1-2	4.78	3-4	4.92	3-5	4.106	2-3
4.65	2-4	4.79	1	4.93	2	4.107	1-4
4.66	2-3	4.80	3	4.94	1-5	4.108	1-4
4.67	1-2	4.81	2	4.95	3-5		
4.68	1-3	4.82	3	4.96	2-3		
4.69	1-2	4.83	2	4.97	1-2		
4.70	1-2	4.84	1-2	4.98	1-3		
4.71	1-2	4.85	2	4.99	1-3		
4.72	1-2	4.86	2-3	4.100	2-7		
4.73	2	4.87	4-5	4.101	3-4		
4.74	1	4.88	1	4.102	2-4		
4.75	1-2	4.89	4	4.103	3-6		

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