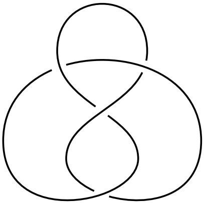
Quandle cocycle invariants of cabled surface knots

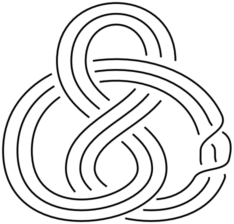
Katsumi Ishikawa

RIMS, Kyoto univ., D1 / JSPS research fellow DC



K

cabling $\longmapsto$



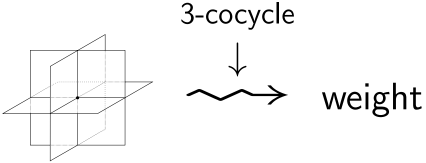
$K^{(m,n)}$

$$\text{inv. of } K^{(m,n)} \quad = \quad \text{inv. of } K$$

↑ cabling formula

- (paral.) Jones polyn. (J. Murakami; 1989)
- Kontsevich inv.
  (D. Bar-Natan, T.T.Q. Le, D.P. Thurston; 2003)

Quandle cocycle invariants (CJKLS; 2003):



Aim QCI of $F^{(m,\nu)}$ $=$ inv. of F

T. Naruse (2015) - I. (2016):

QCI of cabling = kink cocycle inv. ↕ eqv. RCI

$\left(= \begin{cases} \text{3-CI} \\ \text{2-CI (CSS; 2006)} \end{cases} \right.$ (Alex. cases) $\Big)$

kink cocycle inv.

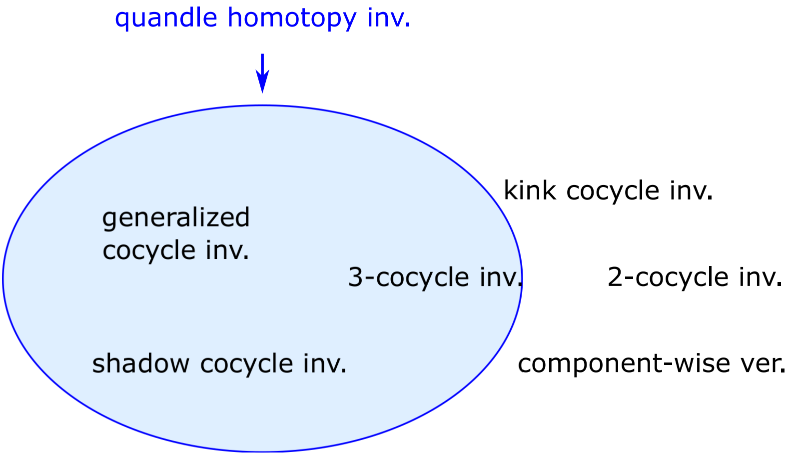
generalized
cocycle inv.

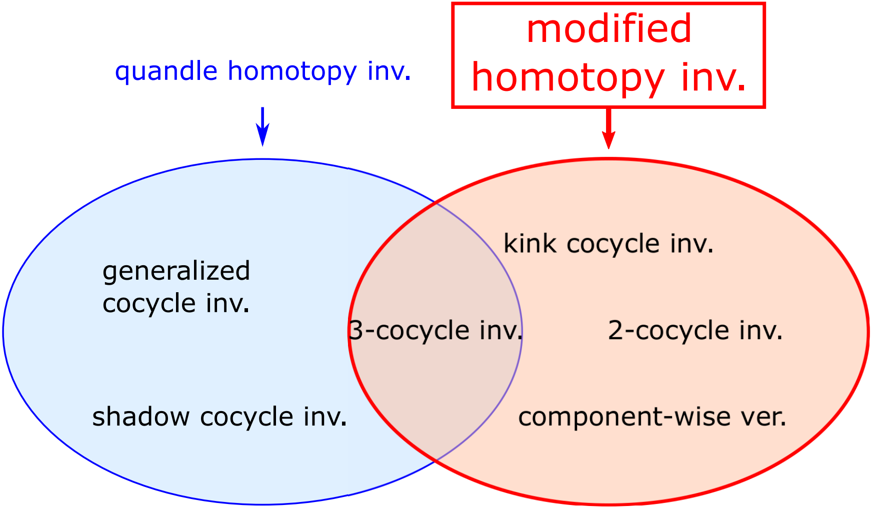
3-cocycle inv.

2-cocycle inv.

shadow cocycle inv.

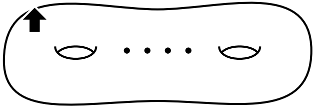
component-wise ver.


quandle homotopy inv.
generalized
cocycle inv.
3-cocycle inv.
shadow cocycle inv.
kink cocycle inv.
2-cocycle inv.
component-wise ver.


quandle homotopy inv.
modified
homotopy inv.
generalized
cocycle inv.
kink cocycle inv.
3-cocycle inv.
2-cocycle inv.
shadow cocycle inv.
component-wise ver.

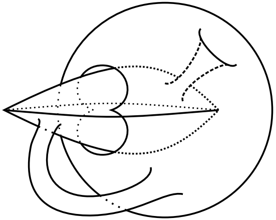
Contents

A **surface knot** F is the image of



$\xhookrightarrow[\text{smooth}]{}$ $\mathbb{R}^4$

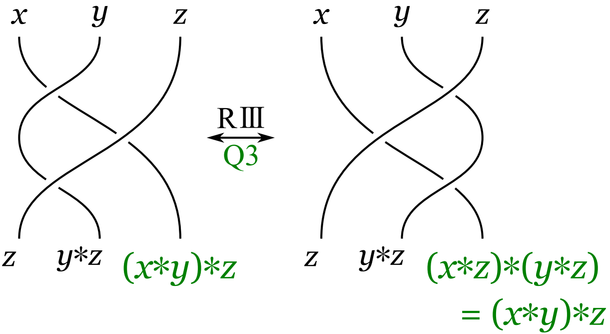
$$
\begin{array}{ccc}
F & \subset & \mathbb{R}^4 \\
\downarrow & & \downarrow
\end{array}
$$



$\subset$ $\mathbb{R}^3$: a **diagram** of F

$X = (X, *)$ is a **quandle**

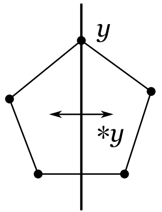
$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{(Q1) } x * x = x, \\ \text{(Q2) the map } X \ni a \mapsto a * x \in X \text{ is a bijection,} \\ \text{(Q3) } (x * y) * z = (x * z) * (y * z), \\ \qquad\qquad\qquad\qquad \text{for } {}^\forall x, y, z \in X. \end{cases}$$



Ex.

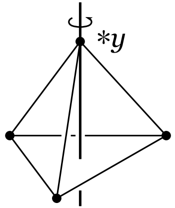
- **dihedral quandle** $R_k = \mathbb{Z}/k\mathbb{Z}$.

$$x * y = 2y - x.$$



- **tetrahedral quandle**
  $Q_4 = \mathbb{Z}[T]/(2, T^2 + T + 1)$.

$$x * y = Tx + (1 - T)y.$$



- **Alexander quandle**
  X : a $\mathbb{Z}[T^{\pm}]$-module

$$x * y = Tx + (1 - T)y.$$

$\phi : X \times X \to \underset{\text{ab.grp.}}{\underline{A}}$ is a **quandle 2-cocycle**

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{l} \phi(x,x) = 0. \\ \phi(x,y) - \phi(x,z) - \phi(x*y,z) + \phi(x*z, y*z) = 0. \end{array}$$

$\psi : X \times X \times X \to A$ is a **quandle 3-cocycle**

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{l} \psi(x,x,y) = \psi(x,y,y) = 0. \\ \psi(w,y,z) - \psi(w,x,z) + \psi(w,x,y) \\ -\psi(w*x,y,z) + \psi(w*y,x*y,z) - \psi(w*z,x*z,y*z) = 0. \end{array}$$

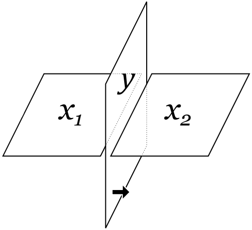
<u>Ex.</u> $\cdot H^2_Q(R_p;\mathbb{Z}/p\mathbb{Z}) = 0$

$H^3_Q(R_p;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ (Mochizuki 3-cocycle: a gen.)

$\cdot H^2_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

$H^3_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$

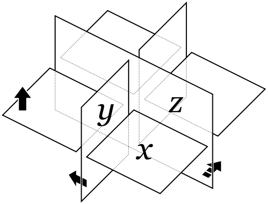
X-**coloring** of a diagram D: $\{\text{sheets}\} \to X$ such that



$$x_1 * y = x_2$$

quandle cocycle invariant:

ψ : a quandle 3-cocycle of $\underset{\text{fin.qdle.}}{\underline{X}}$



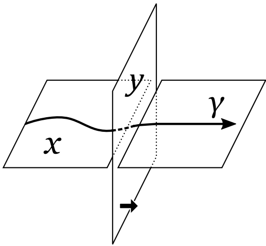
$$\Psi_\psi(F) = \sum_{\text{col.}} \prod_{\text{tri. pt.}} \psi(x, y, z)^{\pm} \in \mathbb{Z}[A].$$

$$\left(\Psi_\psi(\mathcal{C}) = \prod \psi(x, y, z)^{\pm} \quad \text{for } \mathcal{C} \in \underbrace{\mathrm{Col}_X(D)}_{= \{X\text{-cols.}\}}\right)$$

X : a finite quandle $\quad \phi$: a quandle 2-cocycle on X
F : a surface knot $\quad D$: a diagram of F

$\mathcal{C}$: an X-coloring on D

$$\begin{array}{ccccc} & F & & A \\ & \cup & & \Cup \\ \text{a loop} & \gamma & \underset{\Phi_\phi(\mathcal{C})}{\longmapsto} & \prod \phi(x,y)^{\pm} \end{array}$$

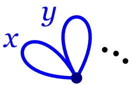


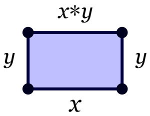
$$\rightsquigarrow \Phi_\phi(\mathcal{C}) \in \mathrm{Hom}(H_1(F), A) \cong H^1(F; A)$$

The **quandle 2-cocycle invariant**:

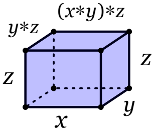
$$\Phi_\phi(F) := \sum_{\mathcal{C}:\mathrm{col.}} \Phi_\phi(\mathcal{C}) \in \mathbb{Z}[H^1(F;A)]$$

The **quandle space** $B^Q X$ (FRS '95, Nosaka '13)

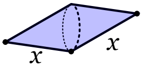
1-skeleton: 

2-cells: 

(x,y)

3-cells: 

(x,y,z)



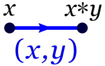
(x,x)

4-cells: (x,y,z,w) (x,y,y) (x,x,y)

The **modified quandle space** $B'X$

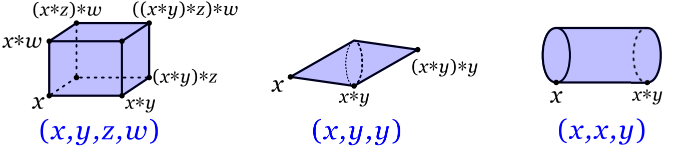
0-cells:

1-cells:

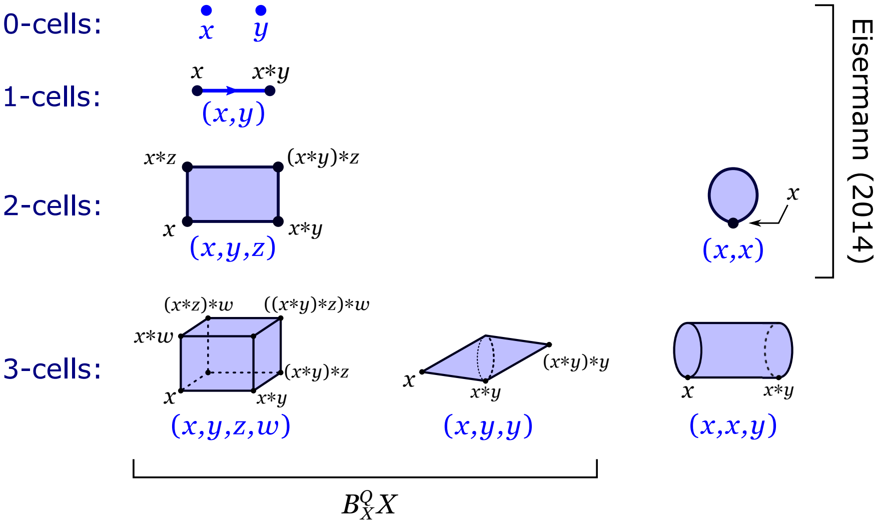


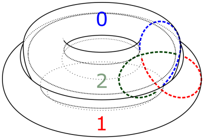
2-cells:

3-cells:



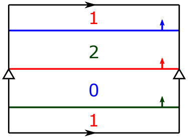
The **modified quandle space** $B'X$





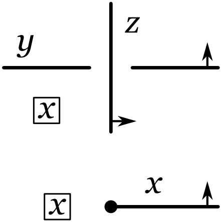
colored diagram

$\rightsquigarrow$

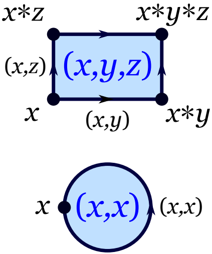


lower graph

$F \supset$



$\xrightarrow[\Xi'_X(\mathcal{C})]{}$

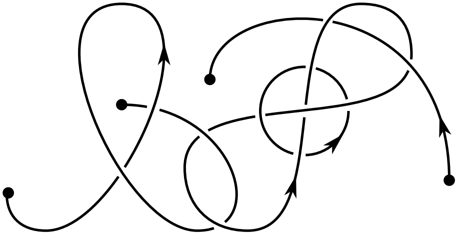


$\subset B'X$

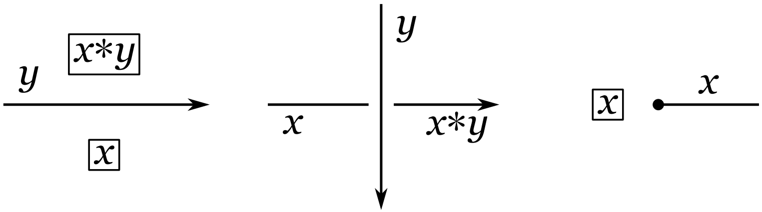
The **modified homotopy invariant**:

$$\Xi'_X : \mathrm{Col}_X(D) \to [F; B'X]$$

generalized link diagram (CKS; 2001):



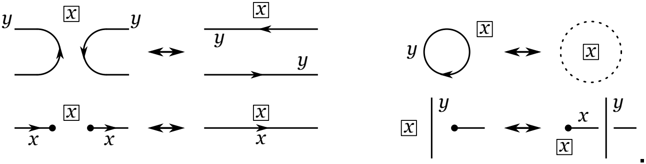
shadow coloring on gen. link diag.



Another definition:

$$\Pi'_F(X) \stackrel{\text{def}}{=} \left\{ \begin{array}{c} \text{gen. diag.} \\ \text{on } F \\ \text{w/color} \end{array} \right\} \Big/$$

Reidemeister moves I II III



$$\Xi'_X : \mathrm{Col}_X(D) \to \Pi'_F(X) \stackrel{\text{bij.}}{\cong} [F; B'X]$$

- n-cells of $B^Q X \leftrightarrow (n-1)$-cells of $B'X$

$$\rightsquigarrow \begin{array}{l} H_n^Q(X, A) = H_n(B^Q X, A) \cong H_{n-1}(B'X, A), \\ H_Q^n(X; A) = H^n(B^Q X; A) \cong H^{n-1}(B'X; A). \end{array}$$

If X is connected ($\Leftrightarrow B'X$ is connected),

- $\pi_1(B'X) \cong$ the "fundamental group" of X.
- $\pi_2(B'X) \cong \pi_2(B^Q X)$.

$$\rightsquigarrow \#X < \infty \Rightarrow \#[F; B'X] < \infty$$

Universality among 2- and 3-cocycle invariants

ϕ : a quandle 2-cocycle $\leftarrow$ a 1-cocycle on $B'X$
ψ : a quandle 3-cocycle $\leftarrow$ a 2-cocycle on $B'X$
$\mathcal{C}$: an X-coloring of a surface link diagram D

rem $\Xi'_X(\mathcal{C}) : F \to B'X$

- universality among 2-cocycle invariants:

$$\Phi_\phi(\mathcal{C}) = (\Xi'_X(\mathcal{C}))^*\phi \qquad \in H^1(F; A)$$

- universality among 3-cocycle invariants:

$$\Psi_\psi(\mathcal{C}) = \langle(\Xi'_X(\mathcal{C}))^*\psi, [F]\rangle \qquad \in A$$

Ex.

- dihedral quandle R_k (k: odd)

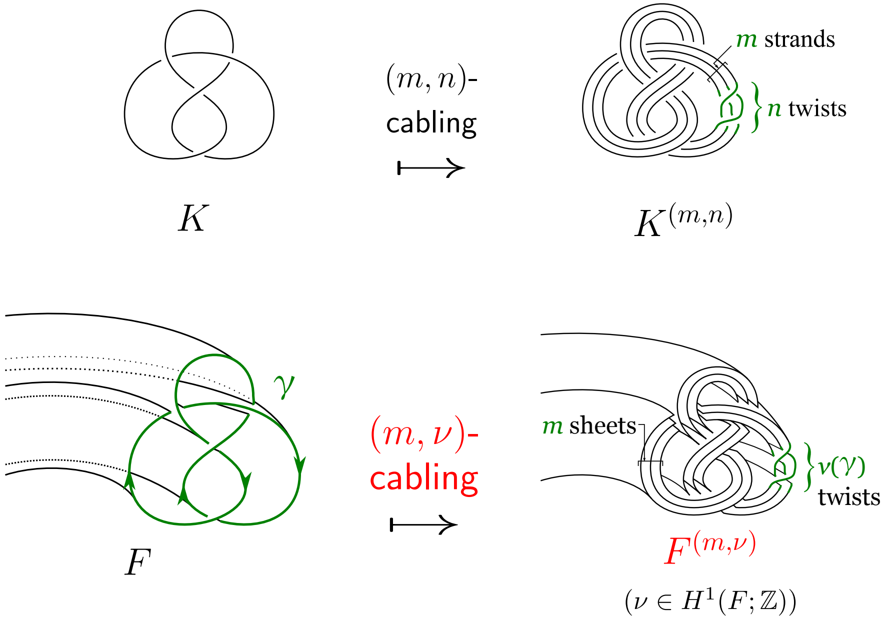
  $\pi_1(B'R_k) = 0, \pi_2(B'R_k) \cong \mathbb{Z}/k\mathbb{Z}$.

  $\rightsquigarrow \Xi_{R_k} \overset{\text{eqv.}}{\longleftrightarrow}$ a 3-cocycle inv.

- tetrahedral quandle Q_4

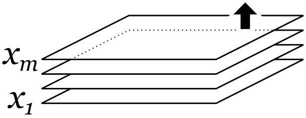
  $\pi_1(B'Q_4) \cong \mathbb{Z}/2\mathbb{Z}, \pi_2(B'Q_4) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.
  $H_2(B'Q_4) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

  $\rightsquigarrow \Xi_{Q_4}$ is stronger than 2- and 3 cocycle invs.



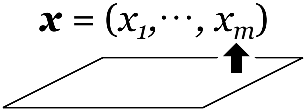
K
(m, n)-
cabling
m strands
n twists
K^(m,n)
F
γ
(m, ν)-
cabling
m sheets
ν(γ)
twists
F^(m,ν)
(ν ∈ H^1(F; ℤ))

QCI of cabled surface knots — idea

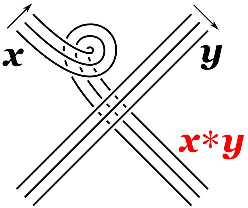


X-coloring on $D^{(m,\nu)}$

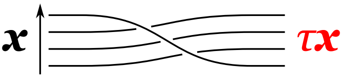
$\longleftrightarrow$



$\underline{X^m}$-coloring on D



$\rightsquigarrow$ X^m is a quandle.



$\rightsquigarrow$ τ is a “kink map”.

QCI of cabled surface knots

X : a finite quandle $\quad\quad \psi$: a 3-cocycle on X

$X_\tau^m \stackrel{\text{def}}{=} X^m / x \sim \tau^k x$: the quotient quandle

Theorem

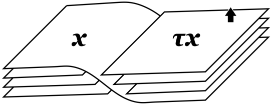
There exists a map $f_\psi^{(m,\nu)} : [F; B' X_\tau^m] \to \mathbb{Z}[A]$ such that

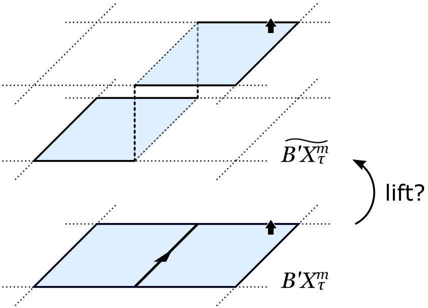
$$\Psi_\psi(F^{(m,\nu)}) = \sum_{\mathcal{C}} f_\psi^{(m,\nu)}(\Xi'_{X_\tau^m}(\mathcal{C})) \in \mathbb{Z}[A],$$

where $\mathcal{C}$ is taken over the X_τ^m-colorings on F.

Proof

$$X^m : \text{quandle} \; + \; \tau : \text{kink map} \quad \longleftrightarrow \quad \widetilde{B'X^m_\tau} \curvearrowleft \tau \;\; \fallingdotseq B'X^m \;\; \downarrow \;\; B'X^m_\tau \quad : \text{cyclic cov.}$$





$$X^m_\tau\text{–col on } F$$
$$\underbrace{[\pi_1(F) \to \mathbb{Z}/N\mathbb{Z}]}_{\text{monodromy rep.}} = -\nu \quad \overset{\text{lift}}{\rightsquigarrow} \quad X\text{-col. on } F^{(m,\nu)}$$

Proof

$\nu \rightsquigarrow$ lift $\tilde{\Xi}_\nu(\mathcal{C})$:

$$\begin{array}{ccc} & & Y \quad (\supset \widetilde{B'X^m_\tau}) : \text{pr. } U(1)\text{-b'dle} \\ & \overset{\tilde{\Xi}_\nu(\mathcal{C})}{\nearrow} & \downarrow \\ F & \underset{\Xi'_{X^m_\tau}(\mathcal{C})}{\longrightarrow} & B'X^m_\tau \end{array}$$

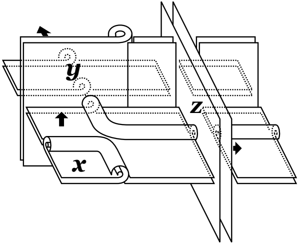
· $\tilde{\Xi}_\nu(\mathcal{C}) \in [F; Y]$ is determined by $\Xi'_{X^m_\tau}(\mathcal{C}) \in [F, B'X^m_\tau]$

<u>claim</u> $^\exists \tilde{\psi} \in H^2(Y; A)$ s.t. $\Psi_\psi(\tilde{\mathcal{C}}) = \langle \tilde{\Xi}_\nu(\mathcal{C})^* \tilde{\psi}, [F] \rangle$.

· 2-cells of Y : $\begin{cases} 2-\text{cells of } \widetilde{B'X^m_\tau} \\ (1-\text{cells of } \widetilde{B'X^m_\tau}) \times [0,1] \quad (\text{fiber}) \end{cases}$

Proof

$$\tilde{\psi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \quad \text{“}=\text{”} \quad \Psi_\psi \left(\vphantom{\int} \right)$$



$$\tilde{\psi}(\boldsymbol{x}, \boldsymbol{y}) \quad \text{“}=\text{”} \quad \Psi_\psi \left(\vphantom{\int} \right)$$