Surface-links and marked graph diagrams

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Outline

- Surface-links
- Marked graph diagrams of surface-links
- Polynomials for marked graph diagrams via classical link invariants
- Ideal coset invariants for surface-links
Surface-links

- A surface-link is a closed surface smoothly embedded in $\mathbb{R}^4$ (or in $S^4$).
- A surface-knot is a one component surface-link.
  - A 2-sphere-link is sometimes called a 2-link.
  - A 2-link of 1-component is called a 2-knot.
- Two surface-links $\mathcal{L}$ and $\mathcal{L}'$ in $\mathbb{R}^4$ are equivalent if they are ambient isotopic, i.e.,
  $$\exists \text{ orient. pres. homeo. } h : \mathbb{R}^4 \to \mathbb{R}^4 \text{ s.t. } h(\mathcal{L}) = \mathcal{L}'$$
  $$\iff \exists \text{ a smooth family of diffeomorphisms } f_s : \mathbb{R}^4 \to \mathbb{R}^4$$
  $$(s \in [0, 1]) \text{ s.t. } f_0 = \text{id}_{\mathbb{R}^4} \text{ and } f_1(\mathcal{L}) = \mathcal{L}'$$.
- If each component $\mathcal{K}_i$ of a surface-link $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_\mu$ is oriented, $\mathcal{L}$ is called an oriented surface-link. Two oriented surface-links $\mathcal{L}$ and $\mathcal{L}'$ are equivalent if the restriction $h|_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}'$ is also orientation preserving.
Examples of surface-knots

- Artin’s spinning construction:

\[ \pi_1(\mathbb{R}^3 - K) \cong \pi_1(\mathbb{R}^4 - \mathcal{K}). \]
Methods of describing surface-links

- Motion pictures (Movies)
- Normal forms
- Broken surface diagrams/Roseman moves
- Charts/Chart moves
- Two dimensional braids/Markov equivalence
- Braid charts/Braid chart moves
- Marked graph diagrams/Yoshikawa moves
Some known invariants of surface-links

- The complement $X = \mathbb{R}^4 - \mathcal{L}$
  $\implies$ Homotopy type of $X$: $\pi_1(X)$, $\pi_2(X)$, etc.
  Homology of infinite cyclic covering $X_\infty$ of $X$:
  Alexander module $H_*(X_\infty; \mathbb{Z}[t, t^{-1}])$

- Normal Euler number, ....

- Broken surface diagram $\implies$
  Triple point number, Quandle cocycle invariants, Fundamental biquandles, ....

- Braid presentation of orientable surface-link $\mathcal{L}$
  $\implies$ Braid index $b(\mathcal{L})$, ....

- Marked vertex diagrams $\implies$ ch-index, Quandle cocycle invariants, Fundamental biquandles, ....
Marked graphs in $\mathbb{R}^3$

- A marked graph is a spatial graph $G$ in $\mathbb{R}^3$ which satisfies the following:
  - $G$ is a finite regular graph possibly with 4-valent vertices, say $v_1, v_2, \ldots, v_n$.
  - Each $v_i$ is a rigid vertex, i.e., we fix a sufficiently small rectangular neighborhood
    \[ N_i \cong \{(x, y) \in \mathbb{R}^2 | -1 \leq x, y \leq 1\}, \]
    where $v_i$ corresponds to the origin and the edges incident to $v_i$ are represented by $x^2 = y^2$.
  - Each $v_i$ has a marker, which is the interval on $N_i$ given by \[ \{(x, 0) \in \mathbb{R}^2 | -\frac{1}{2} \leq x \leq \frac{1}{2}\}. \]
Orientations of marked graphs

- An orientation of a marked graph $G$ is a choice of an orientation for each edge of $G$ in such a way that every vertex in $G$ looks like $\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow
\end{array}$ or $\begin{array}{c}
\downarrow \uparrow \\
\uparrow \downarrow
\end{array}$.

- A marked graph is said to be orientable if it admits an orientation. Otherwise, it is said to be non-orientable.

- By an oriented marked graph we mean an orientable marked graph with a fixed orientation.

- Two (oriented) marked graphs are said to be equivalent if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods, (orientation) and markers.
Oriented marked graph diagrams

An oriented marked graph $G$ in $\mathbb{R}^3$ can be described as usual by a diagram $D$ in $\mathbb{R}^2$, which is an oriented link diagram in $\mathbb{R}^2$ possibly with some marked 4-valent vertices whose incident edges are oriented illustrated as above, and is called an oriented marked graph diagram (simply, oriented MG diagram) of $G$.

Two oriented MG diagrams in $\mathbb{R}^2$ represent equivalent oriented marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of the oriented rigid vertex 4-regular spatial graph moves (simply RV4 moves) $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma'_4$ and $\Gamma_5$ shown in Figure below.
RV4 moves

$\Gamma_1 :$ 

$\Gamma'_1 :$ 

$\Gamma_2 :$ 

$\Gamma_3 :$ 

$\Gamma_4 :$ 

$\Gamma'_4 :$ 

$\Gamma_5 :$
Unoriented marked graph diagrams

- An unoriented marked graph diagram (MG diagram) means a nonorientable or an orientable but not oriented marked graph diagram in $\mathbb{R}^2$, and so it represents marked graphs in $\mathbb{R}^3$ without orientations.

- Two MG diagrams in $\mathbb{R}^2$ represent equivalent marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of the moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_4'$ and $\Omega_5$, where $\Omega_i$ stands for the move $\Gamma_i$ without orientation.
Admissible MG diagrams

For an (oriented) MG diagram \( D \), let \( L^{-}(D) \) and \( L^{+}(D) \) be the (oriented) link diagrams obtained from \( D \) by replacing each marked vertex with \( \times \) and \( \bigcirc \), respectively.

We call \( L^{-}(D) \) and \( L^{+}(D) \) the negative resolution and the positive resolution of \( D \), respectively.

An (oriented) MG diagram \( D \) is admissible if both resolutions \( L^{-}(D) \) and \( L^{+}(D) \) are trivial link diagrams.
Surface-links from adm. MG diagrams

Let $D$ be a given admissible MG diagram with marked vertices $v_1, \ldots, v_n$. Define a surface $F(D) \subset \mathbb{R}^3 \times [-1, 1]$ by

$$(\mathbb{R}_t^3, F(D) \cap \mathbb{R}_t^3) = \begin{cases} (\mathbb{R}^3, L_+(D)) & \text{for } 0 < t \leq 1, \\ (\mathbb{R}^3, L_-(D) \cup \left( \bigcup_{i=1}^n B_i \right)) & \text{for } t = 0, \\ (\mathbb{R}^3, L_-(D)) & \text{for } -1 \leq t < 0, \end{cases}$$

where $\mathbb{R}_t^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$ and $B_i (1 \leq i \leq n)$ is a band attached to $L_-(D)$ at each marked vertex $v_i$ as

We call $F(D)$ the proper surface associated with $D$. 
Surface-links from adm. MG diagrams

- When $D$ is oriented, $L_-(D)$ and $L_+(D)$ have the orientations induced from the orientation of $D$. We assume that the proper surface $F(D)$ is oriented so that the induced orientation on $L_+(D) = \partial F(D) \cap \mathbb{R}^3_1$ matches the orientation of $L_+(D)$.

- Since $D$ is admissible, we can obtain a surface-link from $F(D)$ by attaching trivial disks in $\mathbb{R}^3 \times [1, \infty)$ and another trivial disks in $\mathbb{R}^3 \times (-\infty, 1]$. We denote the resulting (oriented) surface-link by $\mathcal{L}(D)$, and call it the (oriented) surface-link associated with $D$.

- It is well known that the isotopy type of $\mathcal{L}(D)$ does not depend on the choices of trivial disks (Horibe-Yanakawa Lemma).
Adm. MG diagram $D \rightarrow$ Surface-link $\mathcal{L}(D)$
Surface-links presented by MG diagrams

Definition
Let $\mathcal{L}$ be an (oriented) surface-link in $\mathbb{R}^4$. We say that $\mathcal{L}$ is presented by an (oriented) MG diagram $D$ if $\mathcal{L}$ is ambient isotopic to the (oriented) surface-link $\mathcal{L}(D)$ in $\mathbb{R}^4$.

- Let $D$ be an admissible (oriented) MG diagram. By definition, $\mathcal{L}(D)$ is presented by $D$.

From now on, we explain that any (oriented) surface-link is presented by an admissible (oriented) MG diagram.
It is well known that any surface link $\mathcal{L}$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be deformed into a surface link $\mathcal{L}'$, called a hyperbolic splitting of $\mathcal{L}$, by an ambient isotopy of $\mathbb{R}^4$ in such a way that the projection $p : \mathcal{L}' \to \mathbb{R}$ satisfies the followings:

- all critical points are non-degenerate,
- all the index 0 critical points (minimal points) are in $\mathbb{R}^3_{-1}$,
- all the index 1 critical points (saddle points) are in $\mathbb{R}^3_0$,
- all the index 2 critical points (maximal points) are in $\mathbb{R}^3_1$. 

$t=0$ saddle point

$t=-1$ minimal point

$t=1$ maximal point
MG diagrams from surface-links

- Then the cross-section

\[ \mathcal{L}_0' = \mathcal{L}' \cap \mathbb{R}_0^3 \text{ at } t = 0 \]

is a spatial 4-valent regular graph in \( \mathbb{R}_0^3 \). We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Figure:

- When \( \mathcal{L} \) is an oriented surface-link, we choose an orientation for each edge of \( \mathcal{L}_0' \) so that it coincides with the induced orientation on the boundary of \( \mathcal{L}' \cap \mathbb{R}^3 \times (-\infty, 0] \) by the orientation of \( \mathcal{L}' \) inherited from the orientation of \( \mathcal{L} \).
MG diagrams from surface-links

- The resulting (oriented) marked graph $G := L_0'$ is called an (oriented) marked graph presenting $L$.
- A diagram $D$ of an (oriented) marked graph $G := L_0'$ is clearly admissible, and is called an (oriented) MG diagram or (oriented) ch-diagram presenting $L$.

In conclusion,

**Theorem (Kawauchi-Shibuya-Suzuki)**

(i) Let $D$ be an admissible (oriented) MG diagram. Then there is an (oriented) surface-link $L$ presented by $D$.

(ii) Let $L$ be an (oriented) surface-link. Then there is an admissible (oriented) MG diagram $D$ presenting $L$. 
Surface-links & MG diagrams

\[
\begin{align*}
\{\text{adm. (ori) MG diag. } D\} & \xrightarrow{(i)} \{\text{(ori) surface-link } \mathcal{L}(D)\} \\
\{\text{adm. (ori) MG diag. } D'\} & \xleftarrow{(ii)} \{\text{hyperbolic split. } \mathcal{L}'(D)\}
\end{align*}
\]

Theorem (Kearton-Kurlin, Swenton)

Two (oriented) marked graph diagrams present the same (oriented) surface-link if and only if they are transformed into each other by a finite sequence of RV4 moves (called (oriented) Yoshikawa moves of type I) and (oriented) Yoshikawa moves of type II in Figure below.
Oriented Yoshikawa moves of type I
(=RV4 moves)
Oriented Yoshikawa moves of type II

\( \Gamma_6 : \)

\( \Gamma'_6 : \)

\( \Gamma_7 : \)

\( \Gamma_8 : \)
Classical link invariants

Let $R$ be a commutative ring with the additive identity $0$ and the multiplicative identity $1$ and let

$$[\ ] : \{\text{classical knots and links in } \mathbb{R}^3\} \rightarrow R$$

be a regular or an ambient isotopy invariant such that for a unit $\alpha \in R$ and $\delta \in R$,

$$[\otimes] = \alpha [\ )], \quad [\infty] = \alpha^{-1} [\ )]. \quad (1)$$

$$[K \circ] = \delta [K], \quad (2)$$

where $K \circ$ denotes any addition of a disjoint circle $\circ$ to a classical knot or link diagram $K$. 
Polynomial \([D]\) for MG diagrams via \([D]\)

Let \(D\) be an (oriented) MG diagram. Let \([[D]](x,y)\) (\([[D]]\) for short) be the polynomial in \(R[x,y]\) defined by the following two rules:

(L1) \([[D]] = [D] \) if \(D\) is an (oriented) link diagram,

(L2) \([\[\begin{array}{c} \hline \hline \end{array}\]\]] = \([[\begin{array}{c} \hline \hline \end{array}\}\]]x + [[\begin{array}{c} \hline \hline \end{array}\}\]]y,

\([\begin{array}{c} \hline \hline \end{array}\][\begin{array}{c} \hline \hline \end{array}\}] = \([[\begin{array}{c} \hline \hline \end{array}\}\]]x + [[\begin{array}{c} \hline \hline \end{array}\}\]]y,

\([[\begin{array}{c} \hline \hline \end{array}\}\]] = \([[\begin{array}{c} \hline \hline \end{array}\}\]]x + [[\begin{array}{c} \hline \hline \end{array}\}\]]y.\)
Self-writhe for MG diagrams

Let $D = D_1 \cup \cdots \cup D_m$ be an oriented link diagram and let $w(D_i)$ be the usual writhe of the component $D_i$. The **self-writhe** $sw(D)$ of $D$ is defined to be the sum

$$sw(D) = \sum_{i=1}^{m} w(D_i).$$

Let $D$ be a MG diagram. We choose an arbitrary orientation for each component of $L_+(D)$ and $L_-(D)$. Define the **self-writhe** $sw(D)$ of $D$ by

$$sw(D) = \frac{sw(L_+(D)) + sw(L_-(D))}{2},$$

where $sw(L_+(D))$ and $sw(L_-(D))$ are independent of the choice of orientations because the writhe $w(D_i)$ is independent of the choice of orientation for $D_i$. 
Normalization of \[ \] 

Let $D$ be a MG diagram. Then $sw(D)$ is invariant under the Yoshikawa moves except the move $\Gamma_1$. For $\Gamma_1$ and its mirror move,

\[
sw \begin{pmatrix} \infty \end{pmatrix} = sw \begin{pmatrix} \end{pmatrix} + 1,
\]

\[
sw \begin{pmatrix} \end{pmatrix} = sw \begin{pmatrix} \end{pmatrix} - 1.
\]

Definition

Let $D$ be an (oriented) MG diagram. We define $\ll D \gg (x, y)$ ($\ll D \gg$ for short) to be the polynomial in variables $x$ and $y$ with coefficients in $R$ given by

\[
\ll D \gg = \alpha^{-sw(D)}[[D]](x, y).
\]
Let $D$ be an (oriented) MG diagram. A state of $D$ is an assignment of $T_\infty$ or $T_0$ to each marked vertex in $D$. Let $\mathcal{S}(D)$ be the set of all states of $D$. For $\sigma \in \mathcal{S}(D)$, let $D_\sigma$ denote the link diagram obtained from $D$ by

$$
\begin{align*}
T_\infty & \rightarrow \text{ ,}\quad T_0 \rightarrow , \\
T_\infty & \rightarrow \text{ ,}\quad T_0 \rightarrow ,
\end{align*}
$$

Then

$$\ll D \gg = \alpha^{-\text{sw}(D)} \sum_{\sigma \in \mathcal{S}(D)} [D_\sigma] x^{\sigma(\infty)} y^{\sigma(0)},$$

where $\sigma(\infty)$ and $\sigma(0)$ denote the numbers of the assignment $T_\infty$ and $T_0$ of the state $\sigma$, respectively.

\[\ll D \gg = \alpha^{-\text{sw}(D)} \sum_{\sigma \in \mathcal{S}(D)} [D_\sigma] x^{\sigma(\infty)} y^{\sigma(0)},\]
Polynomial invariants for marked graphs in $\mathbb{R}^3$

Theorem (L)

Let $G$ be an (oriented) marked graph in $\mathbb{R}^3$ and let $D$ be an (oriented) marked graph diagram representing $G$. For any given regular or ambient isotopy invariant

$$
\bigl[ \quad \bigr] : \{ \text{classical (oriented) links in } \mathbb{R}^3 \} \longrightarrow \mathbb{R}
$$

with the properties (1) and (2), the associated polynomial

$$
\langle \langle D \rangle \rangle = \alpha^{-\text{sw}(D)} \sum_{\sigma \in \mathcal{S}(D)} [D_{\sigma}] x^{\sigma(\infty)} y^{\sigma(0)} \in \mathbb{R}[x, y],
$$

is an invariant for (oriented) Yoshikawa moves of type I, and therefore it is an invariant of the (oriented) marked graph $G$. 
**n-tangle diagrams**

An oriented $n$-tangle diagram $(n \geq 1)$ we mean an oriented link diagram $\mathcal{T}$ in the rectangle $I^2 = [0, 1] \times [0, 1]$ in $\mathbb{R}^2$ such that $\mathcal{T}$ transversely intersect with $(0, 1) \times \{0\}$ and $(0, 1) \times \{1\}$ in $n$ distinct points, respectively, called the endpoints of $\mathcal{T}$.

Let $\mathcal{T}_{3}^{ori}$ and $\mathcal{T}_{4}^{ori}$ denote the set of all oriented 3- and 4-tangle diagrams such that the orientations of the arcs of the tangles intersecting the boundary of $I^2$ coincide with the orientations as shown in Figure (a) and (b) below, respectively.

![Diagram](attachment:diagram.png)

(a) \hspace{2cm} (b)
Closing operations of 3- and 4-tangles

For $U \in \mathcal{T}_3^{\text{ori}}$ and $V \in \mathcal{T}_4^{\text{ori}}$, let $R(U), R^*(U), S(V), S^*(V)$ denote the oriented link diagrams obtained from the tangles $U$ and $V$ by closing as shown in Figures below:

Let $\mathcal{T}_3$ and $\mathcal{T}_4$ denote the set of all 3- and 4-tangle diagrams without orientations, respectively. For $U \in \mathcal{T}_3$ and $V \in \mathcal{T}_4$, let $R(U), R^*(U), S(V), S^*(V)$ be the link diagrams defined as above forgetting orientations.
Ideals of $R[x,y]$ ass. w/ classical link invariants

Definition
For any given regular or ambient isotopy invariant

$[\ ] : \{\text{classical oriented links in } \mathbb{R}^3\} \to \mathbb{R}$

with the properties (1) and (2). The $[\ ]$-obstruction ideal (or simply $[\ ]$ ideal), denoted by $I$, is defined to be the ideal of $R[x,y]$ generated by the polynomials:

$P_1 = \delta x + y - 1,$
$P_2 = x + \delta y - 1,$
$P_U = ([R(U)] - [R^*(U)])xy, U \in \mathcal{T}_3^{\text{ori}}$
$P_V = ([S(V)] - [S^*(V)])xy, V \in \mathcal{T}_4^{\text{ori}}.$
Ideal coset invariants for surface-links

Theorem (Joung-Kim-L)

The map

\[
\overline{[\_]} : \{(\text{oriented}) \text{ MG diagrams}\} \longrightarrow R[x,y]/I
\]

defined by

\[
\overline{[\_]}(D) = \overline{[D]} := \langle \langle D \rangle \rangle + I
\]

is an invariant for (oriented) surface-links.
Ideal coset invariants for surface-links

Remark. Let $F$ be an extension field of $R$. By Hilbert Basis Theorem, the ideal $I$ is completely determined by a finite number of polynomials in $F[x, y]$, say $p_1, p_2, \ldots, p_r$, i.e.,

$$I = \langle p_1, p_2, \ldots, p_r \rangle.$$ 

Example (Kauffman bracket ideal)

Let $K$ be a virtual knot or link diagram. The Kauffman bracket polynomial of $K$ is a Laurent polynomial $\langle K \rangle = \langle K \rangle(A) \in R = \mathbb{Z}[A, A^{-1}]$ defined by the following rules:

(B1) $\langle \bigcirc \rangle = 1$,

(B2) $\langle \bigcirc K' \rangle = \delta \langle K' \rangle$, where $\delta = -A^2 - A^{-2}$,

(B3) $\langle \bigotimes \rangle = A \langle \bigotimes \rangle + A^{-1} \langle \bigcirlce \bigotimes \rangle$,

where $\bigcirc K'$ denotes any addition of a disjoint circle $\bigcirc$ to a knot or link diagram $K'$. 
Example (Kauffman bracket ideal)

- The Kauffman bracket ideal $I$ is the ideal of $\mathbb{Z}[A, A^{-1}][x, y]$ generated by

  \[
  (-A^2 - A^{-2})x + y - 1,
  x + (-A^2 - A^{-2})y - 1,
  (A^8 + A^4 + 1)xy.
  \]

- The map

  \[\langle \rangle : \{\text{marked graph diagrams}\} \longrightarrow \mathbb{Z}[A, A^{-1}][x, y]/I\]

  defined by $\langle D \rangle = \ll D \gg + I$ is an invariant for unoriented surface-links.

Example (Quantum $A_2$ bracket ideal)

Theorem (Kuperberg, 1994)

There is a regular isotopy invariant $\langle \cdot \rangle_{A_2} \in \mathbb{Z}[a, a^{-1}]$ for links and TTG diagrams, called the quantum $A_2$ bracket, which is defined by the following recursive rules:

(K1) $\langle \emptyset \rangle_{A_2} = 1$.

(K2) $\langle D \sqcup O \rangle_{A_2} = (a^{-6} + 1 + a^6)\langle D \rangle_{A_2}$ for any diagram $D$.

(K3) $\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \end{tikzpicture} \rangle_{A_2} = (a^{-3} + a^3)\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0); \end{tikzpicture} \rangle_{A_2}$,

(K4) $\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \draw [->] (1,0) -- (2,0); \draw [->] (1,0) -- (1,1); \end{tikzpicture} \rangle_{A_2} = \langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \end{tikzpicture} \rangle_{A_2} + \langle \begin{tikzpicture} [baseline] \draw [->] (1,0) -- (2,0); \end{tikzpicture} \rangle_{A_2}$,

(K5) $\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \draw [->] (1,0) -- (1,1); \end{tikzpicture} \rangle_{A_2} = -a\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \draw [->] (1,0) -- (1,1); \draw [->] (1,0) -- (2,0); \end{tikzpicture} \rangle_{A_2} + a^{-2}\langle \begin{tikzpicture} [baseline] \draw [->] (1,0) -- (2,0); \end{tikzpicture} \rangle_{A_2}$,

(K6) $\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \draw [->] (1,0) -- (1,1); \draw [->] (1,0) -- (2,0); \end{tikzpicture} \rangle_{A_2} = -a^{-1}\langle \begin{tikzpicture} [baseline] \draw [->] (0,0) -- (1,0) (0,0) -- (0,1); \draw [->] (1,0) -- (1,1); \draw [->] (1,0) -- (2,0); \end{tikzpicture} \rangle_{A_2} + a^2\langle \begin{tikzpicture} [baseline] \draw [->] (1,0) -- (2,0); \end{tikzpicture} \rangle_{A_2}$. 
Example (Quantum $A_2$ bracket ideal)

- The quantum $A_2$ bracket ideal $I$ is the ideal of $\mathbb{Z}[a, a^{-1}][x, y]$ generated by
  
  $$(a^{-6} + 1 + a^6)x + y - 1,$$
  
  $$x + (a^{-6} + 1 + a^6)y - 1,$$
  
  $$(a^{12} + 1)(a^6 + 1)^2xy.$$

- The map
  
  $$\langle \overline{D} \rangle_{A_2} : \{\text{oriented MG diagrams}\} \rightarrow \mathbb{Z}[a, a^{-1}][x, y]/I$$
  
  defined by $\overline{\langle D \rangle}_{A_2} = \langle \overline{D} \rangle + I$ is an invariant for oriented surface-links.

Question:

Is there a classical link invariant \([\ ]\) such that the \([\ ]\) ideal is trivial?
Thank you!