Surface-links and marked graph diagrams

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Outline

- Surface-links
- Marked graph diagrams of surface-links
- Polynomials for marked graph diagrams via classical link invariants [ ]
- Ideal coset invariants for surface-links

Surface-links

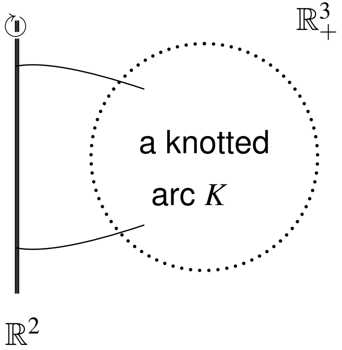
- A surface-link is a closed surface smoothly embedded in $\mathbb{R}^4$ (or in S^4).
- A surface-knot is a one component surface-link.
  · A 2-sphere-link is sometimes called a 2-link.
  · A 2-link of 1-component is called a 2-knot.
- Two surface-links $\mathscr{L}$ and $\mathscr{L}'$ in $\mathbb{R}^4$ are equivalent if they are ambient isotopic, i.e.,

  $\exists$ orient. pres. homeo. $h : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $h(\mathscr{L}) = \mathscr{L}'$

  $\iff$ $\exists$ a smooth family of diffeomorphisms $f_s : \mathbb{R}^4 \to \mathbb{R}^4$ $(s \in [0,1])$ s.t. $f_0 = \mathrm{id}_{\mathbb{R}^4}$ and $f_1(\mathscr{L}) = \mathscr{L}'$.
- If each component $\mathscr{K}_i$ of a surface-link $\mathscr{L} = \mathscr{K}_1 \cup \cdots \cup \mathscr{K}_\mu$ is oriented, $\mathscr{L}$ is called an oriented surface-link. Two oriented surface-links $\mathscr{L}$ and $\mathscr{L}'$ are equivalent if the restriction $h|_{\mathscr{L}} : \mathscr{L} \to \mathscr{L}'$ is also orientation preserving.

Examples of surface-knots

- Artin's spinning construction:



$\rightsquigarrow$ a spun 2-knot $\mathscr{K}$

$$\pi_1(\mathbb{R}^3 - K) \cong \pi_1(\mathbb{R}^4 - \mathscr{K}).$$

Methods of describing surface-links

- Motion pictures (Movies)
- Normal forms
- Broken surface diagrams/Roseman moves
- Charts/Chart moves
- Two dimensional braids/Markov equivalence
- Braid charts/Braid chart moves
- **Marked graph diagrams/Yoshikawa moves**

Some known invariants of surface-links

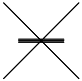
- The complement $X = \mathbb{R}^4 - \mathscr{L}$
  $\Longrightarrow$ Homotopy type of X: $\pi_1(X)$, $\pi_2(X)$, etc.
  Homology of infinite cyclic covering X_∞ of X:
  Alexander module $H_*(X_\infty; \mathbb{Z}[t, t^{-1}])$
- Normal Euler number, ...
- Broken surface diagram $\Longrightarrow$
  Triple point number, Quandle cocycle invariants,
  Fundamental biquandles, ...
- Braid presentation of orientable surface-link $\mathscr{L}$
  $\Longrightarrow$ Braid index $b(\mathscr{L})$, ...
- Marked vertex diagrams $\Longrightarrow$ ch-index, Quandle cocycle invariants, Fundamental biquandles, ...

Marked graphs in $\mathbb{R}^3$

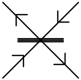
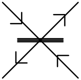
- A marked graph is a spatial graph G in $\mathbb{R}^3$ which satisfies the following
  - G is a finite regular graph possibly with 4-valent vertices, say $v_1, v_2, \ldots, v_n$.
  - Each v_i is a rigid vertex, i.e., we fix a sufficiently small rectangular neighborhood

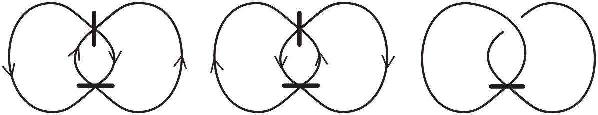
    $$N_i \cong \{(x,y) \in \mathbb{R}^2 | -1 \leq x, y \leq 1\},$$

    where v_i corresponds to the origin and the edges incident to v_i are represented by $x^2 = y^2$.
  - Each v_i has a marker, which is the interval on N_i given by $\{(x,0) \in \mathbb{R}^2 | -\frac{1}{2} \leq x \leq \frac{1}{2}\}$.



Orientations of marked graphs

- An orientation of a marked graph G is a choice of an orientation for each edge of G in such a way that every vertex in G looks like  or  .
- A marked graph is said to be orientable if it admits an orientation. Otherwise, it is said to be non-orientable.
- By an oriented marked graph we mean an orientable marked graph with a fixed orientation.



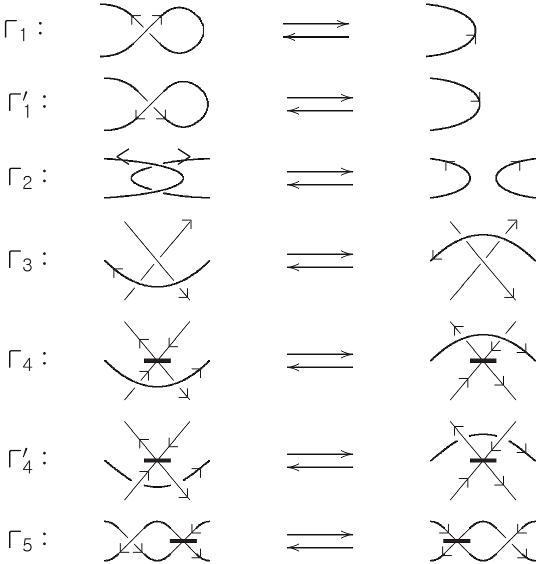
- Two (oriented) marked graphs are said to be equivalent if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods, (orientation) and markers.

Oriented marked graph diagrams

- An oriented marked graph G in $\mathbb{R}^3$ can be described as usual by a diagram D in $\mathbb{R}^2$, which is an oriented link diagram in $\mathbb{R}^2$ possibly with some marked 4-valent vertices whose incident edges are oriented illustrated as above, and is called an oriented marked graph diagram (simply, oriented MG diagram) of G .

- Two oriented MG diagrams in $\mathbb{R}^2$ represent equivalent oriented marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of the oriented rigid vertex 4-regular spatial graph moves (simply RV4 moves) $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_4'$ and Γ_5 shown in Figure below.

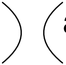
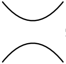
RV4 moves

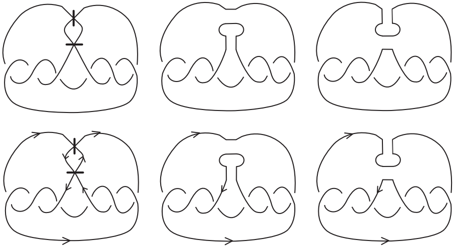


Unoriented marked graph diagrams

- An unoriented marked graph diagram or simply a marked graph diagram (MG diagram) means a nonorientable or an orientable but not oriented marked graph diagram in $\mathbb{R}^2$, and so it represents marked graphs in $\mathbb{R}^3$ without orientations.
- Two MG diagrams in $\mathbb{R}^2$ represent equivalent marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of the moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_4'$ and Ω_5, where Ω_i stands for the move Γ_i without orientation.

Admissible MG diagrams

For an (oriented) MG diagram D, let $L_-(D)$ and $L_+(D)$ be the (oriented) link diagrams obtained from D by replacing each marked vertex  with  and , respectively.



D L_(D) L+(D)

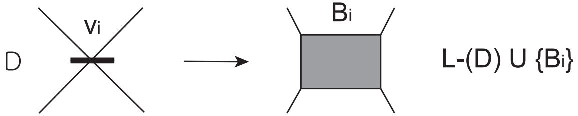
- We call $L_-(D)$ and $L_+(D)$ the negative resolution and the positive resolution of D, respectively.
- An (oriented) MG diagram D is admissible if both resolutions $L_-(D)$ and $L_+(D)$ are trivial link diagrams.

Surface-links from adm. MG diagrams

Let D be a given admissible MG diagram with marked vertices $v_1, \dots, v_n$. Define a surface $F(D) \subset \mathbb{R}^3 \times [-1,1]$ by

$$(\mathbb{R}^3_t, F(D) \cap \mathbb{R}^3_t) = \begin{cases} (\mathbb{R}^3, L_+(D)) & \text{for } 0 < t \leq 1, \\ \left(\mathbb{R}^3, L_-(D) \cup \left(\bigcup_{i=1}^{n} B_i\right)\right) & \text{for } t = 0, \\ (\mathbb{R}^3, L_-(D)) & \text{for } -1 \leq t < 0, \end{cases}$$

where $\mathbb{R}^3_t := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$ and $B_i (1 \leq i \leq n)$ is a band attached to $L_-(D)$ at each marked vertex v_i as

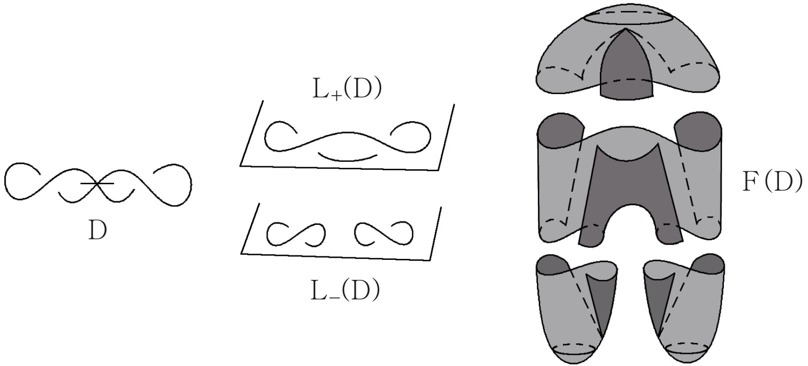


We call $F(D)$ the proper surface associated with D.

Surface-links from adm. MG diagrams

- When D is oriented, $L_-(D)$ and $L_+(D)$ have the orientations induced from the orientation of D. We assume that the proper surface $F(D)$ is oriented so that the induced orientation on $L_+(D) = \partial F(D) \cap \mathbb{R}^3_1$ matches the orientation of $L_+(D)$.
- Since D is admissible, we can obtain a surface-link from $F(D)$ by attaching trivial disks in $\mathbb{R}^3 \times [1, \infty)$ and another trivial disks in $\mathbb{R}^3 \times (-\infty, 1]$. We denote the resulting (oriented) surface-link by $\mathscr{L}(D)$, and call it the (oriented) surface-link associated with D.
- It is well known that the isotopy type of $\mathscr{L}(D)$ does not depend on the choices of trivial disks (Horibe-Yanakawa Lemma).

Adm. MG diagram $D \longrightarrow$ Surface-link $\mathscr{L}(D)$



Surface-links presented by MG diagrams

Definition

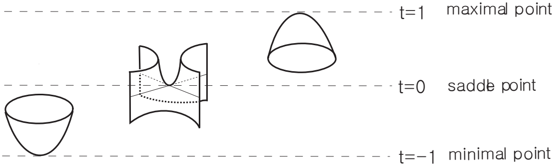
Let $\mathscr{L}$ be an (oriented) surface-link in $\mathbb{R}^4$. We say that $\mathscr{L}$ is presented by an (oriented) MG diagram D if $\mathscr{L}$ is ambient isotopic to the (oriented) surface-link $\mathscr{L}(D)$ in $\mathbb{R}^4$.

- Let D be an admissible (oriented) MG diagram. By definition, $\mathscr{L}(D)$ is presented by D.

From now on, we explain that any (oriented) surface-link is presented by an admissible (oriented) MG diagram.

MG diagrams from surface-links

- It is well known that any surface link $\mathscr{L}$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be deformed into a surface link $\mathscr{L}'$, called a hyperbolic splitting of $\mathscr{L}$, by an ambient isotopy of $\mathbb{R}^4$ in such a way that the projection $p : \mathscr{L}' \to \mathbb{R}$ satisfies the followings:
  - all critical points are non-degenerate,
  - all the index 0 critical points (minimal points) are in $\mathbb{R}^3_{-1}$,
  - all the index 1 critical points (saddle points) are in $\mathbb{R}^3_0$,
  - all the index 2 critical points (maximal points) are in $\mathbb{R}^3_1$.

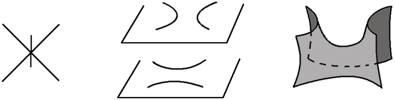


MG diagrams from surface-links

- Then the cross-section

$$\mathscr{L}'_0 = \mathscr{L}' \cap \mathbb{R}^3_0 \text{ at } t = 0$$

  is a spatial 4-valent regular graph in $\mathbb{R}^3_0$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Figure:



- When $\mathscr{L}$ is an oriented surface-link, we choose an orientation for each edge of $\mathscr{L}'_0$ so that it coincides with the induced orientation on the boundary of $\mathscr{L}' \cap \mathbb{R}^3 \times (-\infty, 0]$ by the orientation of $\mathscr{L}'$ inherited from the orientation of $\mathscr{L}$.

MG diagrams from surface-links

- The resulting (oriented) marked graph $G := \mathscr{L}'_0$ is called an (oriented) marked graph presenting $\mathscr{L}$.
- A diagram D of an (oriented) marked graph $G := \mathscr{L}'_0$ is clearly admissible, and is called an (oriented) MG diagram or (oriented) ch-diagram presenting $\mathscr{L}$.

In conclusion,

Theorem (Kawauchi-Shibuya-Suzuki)

(i) *Let D be an admissible (oriented) MG diagram. Then there is an (oriented) surface-link $\mathscr{L}$ presented by D.*

(ii) *Let $\mathscr{L}$ be an (oriented) surface-link. Then there is an admissible (oriented) MG diagram D presenting L.*

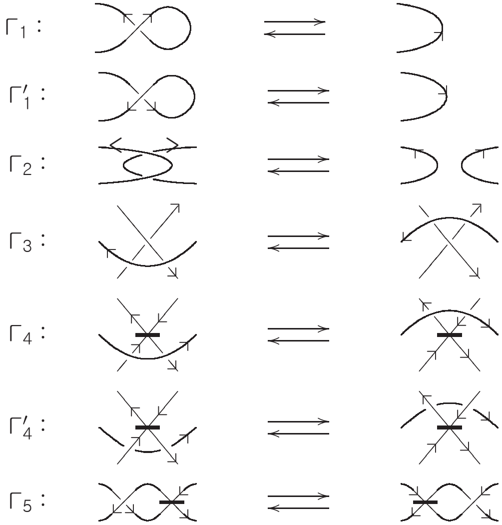
Surface-links & MG diagrams

$$
\begin{tikzcd}
\{\text{adm. (ori) MG diag. } D\} \arrow[r, "\text{(i)}"] \arrow[d, leftrightarrow, "?"'] & \{\text{(ori) surface-link } \mathscr{L}(D)\} \arrow[d, "\text{Morse modification}"] \\
\{\text{adm. (ori) MG diag. } D'\} & \{\text{hyperbolic split. } \mathscr{L}'(D)\} \arrow[l, "(ii)"]
\end{tikzcd}
$$

Theorem (Kearton-Kurlin, Swenton)

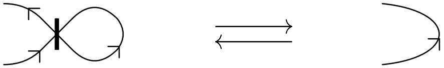
Two (oriented) marked graph diagrams present the same (oriented) surface-link if and only if they are transformed into each other by a finite sequence of RV4 moves (called (oriented) Yoshikawa moves of type I) and (oriented) Yoshikawa moves of type II in Figure below.

Oriented Yoshikawa moves of type I (=RV4 moves)

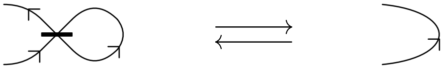


Oriented Yoshikawa moves of type II

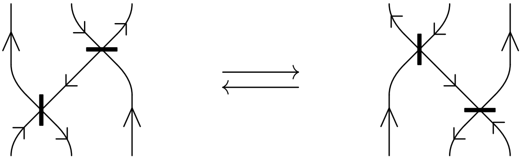
Γ_6 :



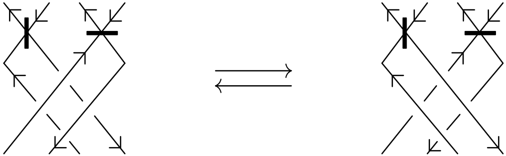
Γ_6' :



Γ_7 :



Γ_8 :

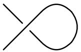
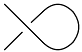


Classical link invariants

Let R be a commutative ring with the additive identity 0 and the multiplicative identity 1 and let

$$[\quad]: \{\text{classical knots and links in } \mathbb{R}^3\} \longrightarrow R$$

be a regular or an ambient isotopy invariant such that for a unit $\alpha \in R$ and $\delta \in R$,

[] $= \alpha \left[\,)\,\right]$, [] $= \alpha^{-1} \left[\,)\,\right]$. (1)

$$\left[K \bigcirc\right] = \delta \left[K\right], \tag{2}$$

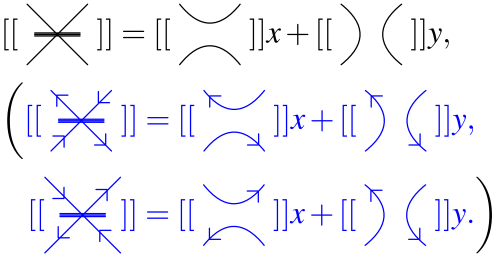
where $K \bigcirc$ denotes any addition of a disjoint circle $\bigcirc$ to a classical knot or link diagram K.

Polynomial [[ ]] for MG diagrams via [ ]

Let D be an (oriented) MG diagram.
Let $[[D]](x,y)$ ([[D]] for short) be the polynomial in $R[x,y]$ defined by the following two rules:

(L1) $[[D]] = [D]$ if D is an (oriented) link diagram,

(L2)



Self-writhe for MG diagrams

- Let $D = D_1 \cup \cdots \cup D_m$ be an oriented link diagram and let $w(D_i)$ be the usual writhe of the component D_i. The self-writhe $sw(D)$ of D is defined to be the sum

$$sw(D) = \sum_{i=1}^{m} w(D_i).$$

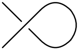
- Let D be a MG diagram. We choose an arbitrary orientation for each component of $L_+(D)$ and $L_-(D)$. Define the self-writhe $sw(D)$ of D by

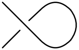
$$sw(D) = \frac{sw(L_+(D)) + sw(L_-(D))}{2},$$

where $sw(L_+(D))$ and $sw(L_-(D))$ are independent of the choice of orientations because the writhe $w(D_i)$ is independent of the choice of orientation for D_i.

Normalization of $[[\ \ ]]$

Let D be a MG diagram. Then $sw(D)$ is invariant under the Yoshikawa moves except the move $\overline{\Gamma}_1$. For $\overline{\Gamma}_1$ and its mirror move,

$sw\Bigg($  $\Bigg) = sw\Bigg($  $\Bigg) + 1,$

$sw\Bigg($  $\Bigg) = sw\Bigg($  $\Bigg) - 1.$

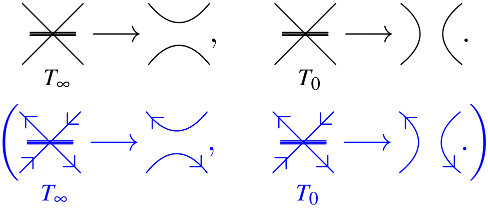
Definition

Let D be an (oriented) MG diagram. We define $\ll D \gg (x,y)$ ($\ll D \gg$ for short) to be the polynomial in variables x and y with coefficients in R given by

$$\ll D \gg = \alpha^{-sw(D)}[[D]](x,y).$$

State-sum formula for $\ll \quad \gg$

Let D be an (oriented) MG diagram. A state of D is an assignment of T_∞ or T_0 to each marked vertex in D. Let $\mathscr{S}(D)$ be the set of all states of D. For $\sigma \in \mathscr{S}(D)$, let D_σ denote the link diagram obtained from D by



Then

$$\ll D \gg = \alpha^{-sw(D)} \sum_{\sigma \in \mathscr{S}(D)} [D_\sigma]\, x^{\sigma(\infty)} y^{\sigma(0)},$$

where $\sigma(\infty)$ and $\sigma(0)$ denote the numbers of the assignment T_∞ and T_0 of the state σ, respectively.

Polynomial invariants for marked graphs in $\mathbb{R}^3$

Theorem (L)

Let G be an (oriented) marked graph in $\mathbb{R}^3$ and let D be an (oriented) marked graph diagram representing G. For any given regular or ambient isotopy invariant

$$[\quad] : \{\textit{classical (oriented) links in } \mathbb{R}^3\} \longrightarrow R$$

with the properties (1) and (2), the associated polynomial

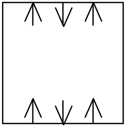
$$\ll D \gg = \alpha^{-sw(D)} \sum_{\sigma \in \mathscr{S}(D)} [D_\sigma]\, x^{\sigma(\infty)} y^{\sigma(0)} \in R[x,y],$$

is an invariant for (oriented) Yoshikawa moves of type I, and therefore it is an invariant of the (oriented) marked graph G.

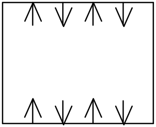
n-tangle diagrams

An oriented n-tangle diagram ($n \geq 1$) we mean an oriented link diagram $\mathscr{T}$ in the rectangle $I^2 = [0,1] \times [0,1]$ in $\mathbb{R}^2$ such that $\mathscr{T}$ transversely intersect with $(0,1) \times \{0\}$ and $(0,1) \times \{1\}$ in n distinct points, respectively, called the endpoints of $\mathscr{T}$.

Let $\mathscr{T}_3^{\mathrm{ori}}$ and $\mathscr{T}_4^{\mathrm{ori}}$ denote the set of all oriented 3- and 4-tangle diagrams such that the orientations of the arcs of the tangles intersecting the boundary of I^2 coincide with the orientations as shown in Figure (a) and (b) below, respectively.



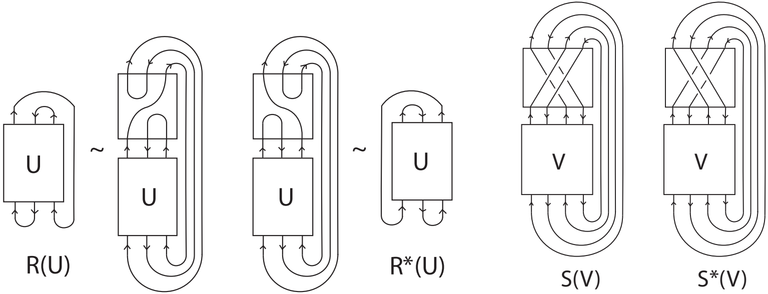
(a)



(b)

Closing operations of 3- and 4-tangles

For $U \in \mathscr{T}_3^{\mathrm{ori}}$ and $V \in \mathscr{T}_4^{\mathrm{ori}}$, let $R(U), R^*(U), S(V), S^*(V)$ denote the oriented link diagrams obtained from the tangles U and V by closing as shown in Figures below:



Let $\mathscr{T}_3$ and $\mathscr{T}_4$ denote the set of all 3- and 4-tangle diagrams without orientations, respectively. For $U \in \mathscr{T}_3$ and $V \in \mathscr{T}_4$, let $R(U), R^*(U), S(V), S^*(V)$ be the link diagrams defined as above forgetting orientations.

Ideals of $R[x,y]$ ass. w/ classical link invariants

Definition

For any given regular or ambient isotopy invariant

$$[\ \ ]: \{\text{classical oriented links in } \mathbb{R}^3\} \longrightarrow R$$

with the properties (1) and (2). The $[\ \ ]$-obstruction ideal (or simply $[\ \ ]$ ideal), denoted by I, is defined to be the ideal of $R[x,y]$ generated by the polynomials:

$$
\begin{aligned}
P_1 &= \delta x + y - 1, \\
P_2 &= x + \delta y - 1, \\
P_U &= ([R(U)] - [R^*(U)])xy, U \in \mathscr{T}_3^{\text{ori}} \\
P_V &= ([S(V)] - [S^*(V)])xy, V \in \mathscr{T}_4^{\text{ori}}.
\end{aligned}
$$

Ideal coset invariants for surface-links

Theorem (Joung-Kim-L)

The map

$$\overline{[\ \ ]} : \{\textit{(oriented) MG diagrams}\} \longrightarrow R[x,y]/I$$

defined by

$$\overline{[\ \ ]}(D) = \overline{[D]} := \ll D \gg + I$$

is an invariant for (oriented) surface-links.

Ideal coset invariants for surface-links

Remark. Let F be an extension field of R. By Hilbert Basis Theorem, the $[\quad]$ ideal I is completely determined by a finite number of polynomials in $F[x,y]$, say $p_1, p_2, \ldots, p_r$, i.e.,

$$I = < p_1, p_2, \ldots, p_r > .$$

Y. Joung, J. Kim and S. Y. Lee, Ideal coset invariants for surface-links in $\mathbb{R}^4$, *J. Knot Theory Ramifications* **22** (2013), no. 9, 1350052 (25 pages).

Example (Kauffman bracket ideal)

Let K be a virtual knot or link diagram. The Kauffman bracket polynomial of K is a Laurent polynomial $\langle K\rangle = \langle K\rangle(A) \in R = \mathbb{Z}[A, A^{-1}]$ defined by the following rules:

$(\mathbf{B1})$ $\langle \bigcirc \rangle = 1$,

$(\mathbf{B2})$ $\langle \bigcirc\ K' \rangle = \delta \langle K' \rangle$, where $\delta = -A^2 - A^{-2}$,

$(\mathbf{B3})$ $\left\langle \diagup\!\!\!\!\diagdown \right\rangle = A \left\langle\ )\ (\ \right\rangle + A^{-1} \left\langle \asymp \right\rangle$,

where $\bigcirc\ K'$ denotes any addition of a disjoint circle $\bigcirc$ to a knot or link diagram K'.

Example (Kauffman bracket ideal)

- The Kauffman bracket ideal I is the ideal of $\mathbb{Z}[A,A^{-1}][x,y]$ generated by

$$
\begin{aligned}
&(-A^2-A^{-2})x+y-1,\\
&x+(-A^2-A^{-2})y-1,\\
&(A^8+A^4+1)xy.
\end{aligned}
$$

- The map

$$\overline{\langle\ \ \rangle} : \{\text{marked graph diagrams}\} \longrightarrow \mathbb{Z}[A,A^{-1}][x,y]/I$$

defined by $\overline{\langle D\rangle} = \ll D \gg + I$ is an invariant for unoriented surface-links.

S. Y. Lee, Towards invariants of surfaces in 4-space via classical link invariants, *Trans. Amer. Math. Soc.* **361** (2009), no. 1, 237–265.

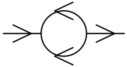
Example (Quantum A_2 bracket ideal)

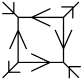
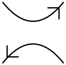
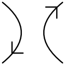
Theorem (Kuperberg, 1994)

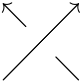
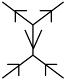
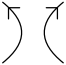
There is a regular isotopy invariant $\langle \cdot \rangle_{A_2} \in \mathbb{Z}[a, a^{-1}]$ for links and TTG diagrams, called the quantum A_2 bracket, which is defined by the following recursive rules:

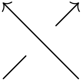
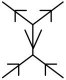
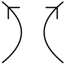
(K1) $\langle \emptyset \rangle_{A_2} = 1.$

(K2) $\langle D \sqcup O \rangle_{A_2} = (a^{-6} + 1 + a^{6}) \langle D \rangle_{A_2}$ for any diagram D.

(K3) $\langle$  $\rangle_{A_2} = (a^{-3} + a^{3}) \langle$  $\rangle_{A_2}$,

(K4) $\langle$  $\rangle_{A_2} = \langle$  $\rangle_{A_2} + \langle$  $\rangle_{A_2}$,

(K5) $\langle$  $\rangle_{A_2} = -a \langle$  $\rangle_{A_2} + a^{-2} \langle$  $\rangle_{A_2}$,

(K6) $\langle$  $\rangle_{A_2} = -a^{-1} \langle$  $\rangle_{A_2} + a^{2} \langle$  $\rangle_{A_2}$.

Example (Quantum A_2 bracket ideal)

- The quantum A_2 bracket ideal I is the ideal of $\mathbb{Z}[a,a^{-1}][x,y]$ generated by

$$(a^{-6}+1+a^{6})x+y-1,$$
$$x+(a^{-6}+1+a^{6})y-1,$$
$$(a^{12}+1)(a^{6}+1)^{2}xy.$$

- The map

$$\overline{\langle\ \ \rangle}_{A_2} : \{\text{oriented MG diagrams}\} \longrightarrow \mathbb{Z}[a,a^{-1}][x,y]/I$$

defined by $\overline{\langle D\rangle}_{A_2} = \ll D \gg + I$ is an invariant for oriented surface-links.

Y. Joung, S. Kamada, A. Kawauchi and S. Y. Lee, Polynomial of an oriented surface-link diagram via quantum A_2 invariant, arXiv:1602.01558.

Question:

Is there a classical link invariant [ ] such that the [ ] ideal is trivial?

Thank you!