Introduction to Heegaard Floer homology

In the viewpoint of Low-dimensional Topology

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§1 What is Heegaard Floer homology?

Theoretical physician says.

4-dimensional gauge theory "=" 2-dimensional non-linear sigma models

Floer's idea(Floer's will)

This equivalence means mathematically an isomorphism of homology groups!!

Consider the "Morse theory" on Chern-Simons functional.

Topologist considers in this situation.

This isomorphism can apply to make interesting invariant.

Atiyah-Floer conjecture (Yang-Mills theorey)

(gauge theory side)

Y: homology 3-sphere

$P_Y \to Y$: $SU(2), SO(3)$-bundle with $w_2(P_Y) \neq 0$.

Generators: $R(Y)$ flat connections

Differential: $\mathcal{M}_{Y\times\mathbb{R}}$: ASD connections on $Y \times \mathbb{R}$

$$\rightsquigarrow H\mathbb{F}^{\text{inst}}(Y, P_Y)$$

(symplectic geometry side)

$Y = U_0 \cup_\Sigma U_1$: Heegaard decomposition

$R(\Sigma)$: flat connections on $\Sigma \times SO(3)$

$\mathcal{L}_0, \mathcal{L}_1 \subset R(\Sigma)$: subset of flat connections extending to U_0, U_1.

Generators: $\mathcal{L}_0 \cap \mathcal{L}_1$

Differential: holomorphic disk connecting two intersection points

$$\rightsquigarrow H\mathbb{F}^{\text{symp}}(R(\Sigma), \mathcal{L}_0, \mathcal{L}_1)$$

$$H\mathbb{F}^{\text{inst}}(Y, P_Y) \cong H\mathbb{F}^{\text{symp}}(R(\Sigma), \mathcal{L}_0, \mathcal{L}_1)$$

$$\begin{array}{c}\text{Homology theory}\\ \text{on Gauge theory}\end{array} \cong \begin{array}{c}\text{Homology theory}\\ \text{on Symplectic theory}\end{array}$$

Theorem 1 (Fukaya '15)

This isomorphism of $SO(3)$-version is true by using bounding cochain b.

Theorem 2 (Taubes)

$\chi(H\mathbb{F}^{inst}(Y, P_Y)) = \lambda(Y)$ (*Casson invariant*)

Topological invariant appears!

Ozsváth-Szabó's Dream

Parallel thinking

Seiberg-Witten invariant is "Tractable".

$+$

Atiyah-Floer isomorphism is mysterious relationship

Seiberg-Witten's sympletic side should definitely construct a TRUELY interesting topological invariant!!!!

They swore to make the invariants in their mind (about 1998??).

Ozsváth and Szabó said in 2001

Look!
We did it.
And put the following gently.

arXiv:math/0101206

As expected...

The invariants are interesting.

This excitement is beyond the expectation.

In their mind.

(gauge theory side)
$(Y, g, \mathfrak{s})$: a Spin^c Riemannian 3-manifold.

- Generators: a critical point x of CSD functional=the solution of Seiberg-Witten equation on $(Y^3, g, \mathfrak{s})$:

$$CSD : \mathcal{M}^{SW}(Y^3, \mathfrak{s}, g) \to \mathbb{R}.$$

- Differential: $s(t)$: a solution of the Seiberg-Witten equation on $Y \times \mathbb{R}$ with

$$\lim_{x(t)\to-\infty} s(t) = x, \lim_{x(t)\to\infty} s(t) = y$$

$$\partial x = y + \cdots$$

(symplectic geometry side)
$Y = U_0 \cup_\Sigma U_1$: Heegaard splitting
$\mathcal{S}_\Sigma$: moduli space of the solutions of SW-equation over Σ
Generaotrs:??
Differential:??

$$\rightsquigarrow SWF^{\mathrm{symp}}(Y, \mathfrak{s})$$

$$SWF(Y, \mathfrak{s}) \cong \text{"} SWF^{\mathrm{symp}}(Y, \mathfrak{s}) \text{"} ??$$

Their final picture.

Definition 3 (Heegaard Floer homology)

$$HF(Y, \mathfrak{s}) \rightsquigarrow SWF^{symp}(Y, \mathfrak{s})$$

$$\chi(HF(Y, \mathfrak{s})) = |H_1(Y)|$$

This invariant is weak??
No, nevertheless,....

§2 Floer theory

$(\mathcal{X}, \omega)$: Symplectic manifold ((almost) Complex manifold)
$\mathcal{L}_0, \mathcal{L}_1 \subset \mathcal{X}$: Lagrangian submanifolds (Totally real submanifolds)

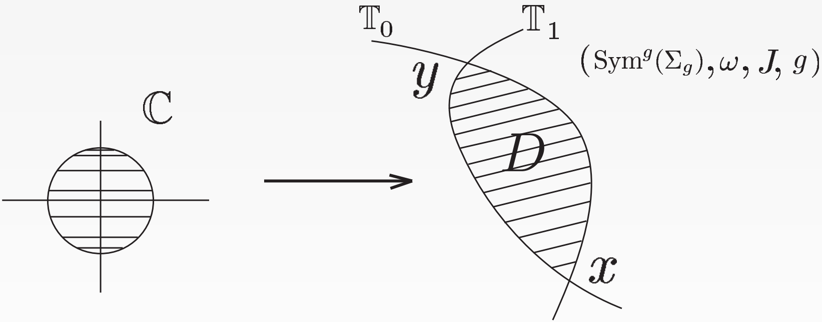
Lagrangian intersection Floer homology

- Generator: $\mathcal{L}_0 \cap \mathcal{L}_1$
- Differential: Counting of holomorphic disk connecting $\mathbf{x}, \mathbf{y} \in \mathcal{L}_0 \cap \mathcal{L}_1$.

$$\partial x = \sum_{y \in \mathcal{L}_0 \cap \mathcal{L}_1} \sum_{\phi \in \pi_2(x,y)} \#\left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) y$$

$CF(\mathcal{X}, \mathcal{L}_0, \mathcal{L}_1)$: Lagrangian intersection Floer homology

Differential operator



Holomorphic disk $D \in \mathcal{M}(\phi)$, $[D] = \phi$.

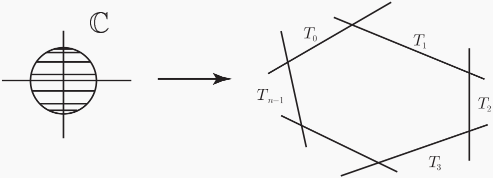
A_∞-structure

The graded map m_n

There exists the following:

$$m_k : CF(T_0, T_1) \otimes CF(T_1, T_2) \otimes CF(T_{n-2}, T_{n-1}) \to CF(T_0, T_{n-1})$$

by counting the holomorphic polygon satisfying an A_∞ relation.



This map is applied to (Dehn) surgery exact sequence.

Low-dimensional applicaction of Heegaard Floer homology

Geometrization $\pi_1(M)$ $g(M)$

$\text{vol}(M)$
$CS(M)$

Knot theory Foliation

$\hat{Z}^{LMO}(M)$ $\tau^{\mathfrak{g}}(M)$ $\Delta(M)$ $H_\star(M)$

$\overline{\mu}(M)$

Contact cobordism $h(M)$

$SW(M)$

$\tau(M)$ $\lambda(M)$

Figure: 2007

Geometrization $\pi_1(M)$ $g(M)$

$\mathrm{vol}(M)$
$CS(M)$

Instanton Floer

Knot theory Foliation

$\hat{Z}^{LMO}(M)$ $\tau^{\mathfrak{g}}(M)$ $\Delta(M)$ homology cobordism $\mu(M)$ $H_\star(M)$

$\mathfrak{ds}$ $\overline{\mu}(M)$

Contact cobordism $h(M)$

$\alpha\beta\gamma$ $SW(M)$

$\tau(M)$ $\lambda(M)$

Figure: 2016

Knot theory

Curvature

Energy

minimal crossing

E_K

Mutation

u

bridge number

Quandle

volume

hyperbolic

det

fiberdness

$\pi_1(E_K)$

A-polynomial

CG

$g(K)$

Khovanov homology

branched cover

Δ_K

$\mathcal{C}$

Homfly

$g_4(K)$

$Z(K)$

lens space

σ

Gauss code

alternating

$\mathcal{C}_{top}$

Kauffmann

$J(K)$

Figure: Red:Strongly related. Cyan: Less known. Magenta: Gently related.

Related topics

- 3-manifold theory (contact structure, foliation, homology cobordism, Dehn surgery, Seiberg-Witten Floer, Casson invariant)
- 4-manifold theory (cobordism, Seiberg-Witten invariant, exotic structure)
- Knot theory (Seifert genus, 4-ball genus, concordance, fiberness,)
- Categorical action of mapping class group

However, it is less known whether the following is related:

- Fundamental group (expected)
- Geometric structure, $\pi_1(Y)$,
- Branched cover or covering
- Chern-Simons invariant,
- Quantum invariants (Jones polynomial, $Z(K)$ LMO, Ohtsuki etc.)
- Purely topological invariant (Heegaard genus, minimal crossing number bridge number, Morse-Novikov number,)

The role of low-dimensional topology

$$\text{Floer theory} \Rightarrow \text{Low-dimensional topology}$$

In the feture,

$$\text{Floer theory} \Leftarrow \text{Low-dimensional topology}$$

§3 Overview of Heegaard Floer homology

$$SWF(Y) = HF(Y)$$

Atiyah-Floer view point.

4-dim

Gauge theory ↓

SWF ═══ HF (3-dim)

↑ Symplectic theory

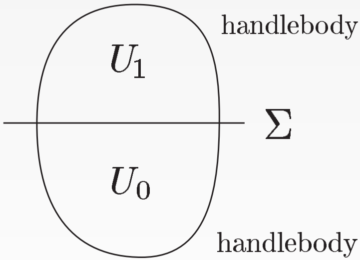
2-dim

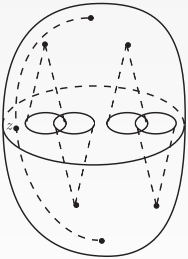
Isomorphism:Kutluhan(Tr)Lee(Ch)T(US)
&Colin(Fr.)Ghiggini(It.)Honda(Jp)

Heegaard decomposition

$^{\forall}$3-manifold

$^{\exists}$ Heegaard decomposition





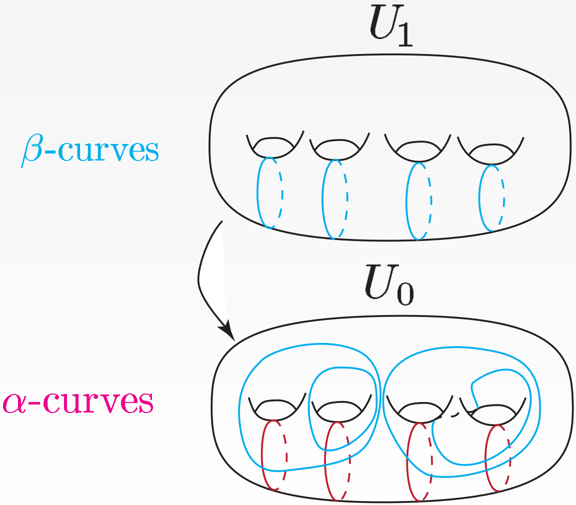
Seiberg-Witten solutions on Σ.

Moduli space of Seiberg-Witten solution over Σ

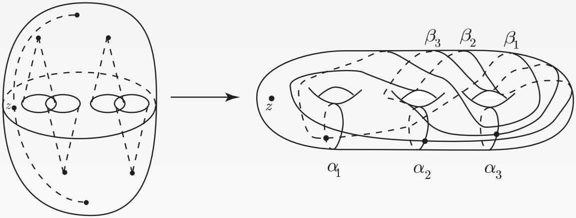
" = "$\mathrm{Sym}^g(\Sigma)$ (the g-fold Symmetric product of Σ)

g: the genus of Σ.

α-curves, β-curves



pointed Heegaard diagram.



pointed Heegaard decomposition $\rightarrow$ pointed Heegaard diagram

$$(\mathrm{Sym}^g(\Sigma_g), \omega, J, g)$$

Ozsváth-Szabó's Lagrangian submanifolds in $\mathrm{Sym}^g(\Sigma_g)$.

$$T^g \cong \mathbb{T}_\alpha = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_g \subset \mathrm{Sym}^g(\Sigma_g)$$

$$T^g \cong \mathbb{T}_\beta = \beta_1 \times \beta_2 \times \cdots \times \beta_g \subset \mathrm{Sym}^g(\Sigma_g)$$

Definition of $\widehat{HF}$

$$n_z(\phi) = \#[(\text{disk } D) \cap (z \times \text{Sym}^g(\Sigma_g))]$$

$$[D] = \phi$$

$$\widehat{CF}(\Sigma, \alpha, \beta, \mathfrak{s}) = CF(Sym^g(\Sigma_g), \mathbb{T}_\alpha, \mathbb{T}_\beta)$$

- Generators: $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$: a g-tuple of points over Σ_g
  $\coprod_{\sigma \in S_g} \{\alpha_1 \cap \beta_{\sigma(1)}\} \times \{\alpha_2 \cap \beta_{\sigma(2)}\} \times \cdots \times \{\alpha_g \cap \beta_{\sigma(g)}\}$
- Differential: counting of holomorphic disks with $n_z(\phi) = 0$

$$\hat{\partial}\mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\phi) = 0 \\ \mu(\phi) = 1}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \mathbf{y}$$

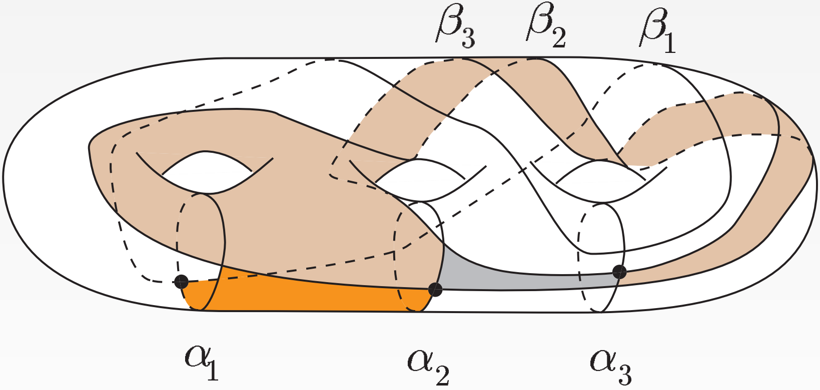
$\pi_2(\mathbf{x}, \mathbf{y})$: homotopy classes of disks from $\mathbf{x}$ to $\mathbf{y}$
$\mathcal{M}(\phi)$: the Moduli space of holomorphic curve representing ϕ

- $\widehat{HF}(Y) = H_*(\widehat{CF}(\Sigma, \alpha, \beta, \mathfrak{s}))$ Topological invariant
- $\widehat{HF}(Y)$: $H_1(Y, \mathbb{Z})$-module
- $\widehat{HF}(Y) = \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s})$
- Euler number: $\chi(\widehat{HF}(Y)) = |H_1(Y)|$.
- $\widehat{SWF}(Y) \cong \widehat{HF}(Y)$ (Atiyah-Floer conjecture)

Ozsváth-Szabó's idea

$$\begin{array}{ccccc} \widehat{\mathbb{D}} & \longrightarrow & \{(x,v) \in \Sigma \times \mathrm{Sym}^g(\Sigma_g) | x \in v\} & \longrightarrow & \Sigma \\ \Big\downarrow \text{g-fold branched cover} & & \Big\downarrow & & \\ \mathbb{D} & \longrightarrow & \mathrm{Sym}^g(\Sigma_g) & & \end{array}$$

$\widehat{\mathbb{D}}$: holomorphic curve with boundary
Disks in $\mathrm{Sym}^g(\Sigma_g)$ =Holomorphic regions on Σ
The region is the lift of a disk via the cover.



Examples of holomorphic regions

Definition of HF^∞

$z \in \Sigma_g - \alpha_1 \cdots - \beta_g$
$z \times \mathrm{Sym}^{g-1}(\Sigma_g) \subset \mathrm{Sym}^g(\Sigma_g)$: hypersurface

$$(CF^\infty(\Sigma, \alpha, \beta, z), \partial^\infty)$$

$\mathbb{Z}[U, U^{-1}]$-module chain complex

- Generators: $U^n\mathbf{x}, \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$
- Differential:

$$\partial^\infty(U^i\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x},\mathbf{y}), gr(\phi)=1} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) U^{i-n_z(\phi)}\mathbf{y}$$

- $HF^\infty(Y) = H_*(CF^\infty(\Sigma, \alpha, \beta, z))$ topological invariant

Definition of HF^-, HF^+

- $CF^-(\Sigma, \alpha, \beta)$ (subchain complex of CF^∞)
  - **Generators**: $U^n \cdot \mathbf{x}$ ($n > 0$ and $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$)
  - **Differential**: (restricted) ∂^∞
- Filtered chain complex:

$$\{0\} \subset CF^-(\Sigma, \alpha, \beta, z) \overset{i}{\subset} CF^\infty(\Sigma, \alpha, \beta, z)$$

Exact sequence

$$0 \to CF^-(\Sigma, \alpha, \beta, z) \xrightarrow{i} CF^\infty(\Sigma, \alpha, \beta, z) \xrightarrow{\pi} CF^+(\Sigma, \alpha, \beta, z) \to 0$$

(CF^-, ∂^-): sub-chain complex
(CF^+, ∂^+): quotient complex

Module structure

$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$
$HF^\infty(Y, \mathfrak{s})$: $\mathbb{F}[U, U^{-1}]$-module
$HF^+(Y, \mathfrak{s})$: $\mathbb{F}[U]$-module
$HF^-(Y, \mathfrak{s})$: $\mathbb{F}[U]$-module

$$\pi_* : HF^\infty(Y, \mathfrak{s}) \to HF^+(Y, \mathfrak{s})$$

$\mathrm{Im}(\pi_*)$ is a finite direct sum of T^+:

$$T^+ = \mathbb{F} \cdot 1 \oplus \mathbb{F} \cdot U^{-1} \oplus \mathbb{F} \cdot U^{-2} \oplus \cdots$$
$$\cong \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$$

(Infinite dimensional $\mathbb{F}$-module with U-action)

$HF_{\mathbf{red}}(Y, \mathfrak{s})$

$HF_{\text{red}}(Y, \mathfrak{s}) := HF^{+}(Y, \mathfrak{s})/\pi_{*}(HF^{\infty}(Y, \mathfrak{s}))$

$$T^{+}[n] = \mathbb{F} \cdot 1 \oplus \mathbb{F} \cdot U^{-1} \oplus \mathbb{F} \cdot U^{-2} \oplus \cdots \mathbb{F} \cdot U^{-n+1}$$

$HF_{\text{red}}(Y, \mathfrak{s})$ is a finite sum of $T^{+}[n]$.

Definition 4 (L-space)

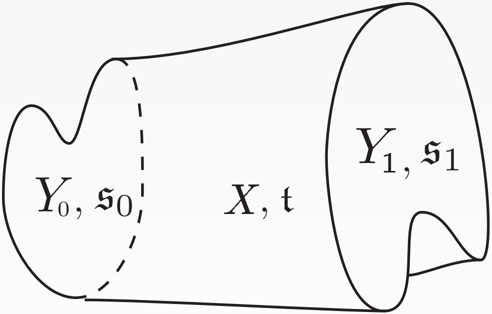
Let Y be a $\mathbb{Q}HS^3$. We define Y to be L-space

$$HF_{red}(Y, \mathfrak{s}) \cong \{0\}$$

for any $\mathfrak{s} \in Spin^c(Y)$.

TQFT invariant

- $(Y_i, \mathfrak{s}_i)$: two Spin^c 3-dim manifolds $(i = 0, 1)$
  $(X, \mathfrak{t})$: a 4-dimensional cobordism between $(Y_i, \mathfrak{s}_i)$



$$HF^\circ(Y_0, \mathfrak{s}_0) \stackrel{F_{X,\mathfrak{t}}}{\to} HF^\circ(Y_1, \mathfrak{s}_1) \quad (\textit{homomorphism})$$

- For any torsion spin^c structure $\mathfrak{s}$, HF° admits an absolute $\mathbb{Q}$-grading ($\circ = \pm, \infty$). $\mathbf{x}_i \in HF^\circ(Y_i, \mathfrak{s}_i)$

$$HF^+(Y_0, \mathfrak{s}_0) \stackrel{F_{W,\mathfrak{t}}}{\to} HF^+(Y_1, \mathfrak{s}_1)$$

$$\mathrm{gr}(F_{W,\mathfrak{t}}(\mathbf{x}_0)) - \mathrm{gr}(\mathbf{x}_0) = \frac{c_1^2(\mathfrak{t}) - 2\chi(W) - 3\sigma(W)}{4}$$

- d-invariant (correction term)

$$d(Y, \mathfrak{s}) = \min\{\mathrm{gr}(x) \in \mathbb{Q} | 0 \neq \boldsymbol{x} \in \mathrm{Image}(\pi_*)\}$$

  - $\lambda(Y, \mathfrak{s}) = \chi(HF_{\mathrm{red}}(Y, \mathfrak{s})) - \frac{1}{2} d(Y, \mathfrak{s})$
  - $\lambda(Y) = \sum_{\mathfrak{s}} d(Y, \mathfrak{s})$ (the Casson-Walker invariant formula)

d-invariant (I)

- $d(Y, \mathfrak{s})$ is spin^c rational homology cobordism invariant, i.e.,

$$(Y_0, \mathfrak{s}_0) \overset{(W,\mathfrak{t})}{\to} (Y_1, \mathfrak{s}_1) \quad \text{(Spin}^c \text{ cobordism)}$$

$$H_*(Y_i, \mathbb{Q}) \overset{\text{inclusion}_*}{\cong} H_*(W, \mathbb{Q}), \qquad \mathfrak{t}|_{Y_i} = \mathfrak{s}_i$$

  Then we have

$$d(Y_0, \mathfrak{s}_0) = d(Y_1, \mathfrak{s}_1)$$

- $\Theta^3_{\mathbb{Z}}$: the group of the homology spheres

$$d : \Theta^3_{\mathbb{Z}} \to 2\mathbb{Z} \ \ (homo.)$$

- Y: $\mathbb{Q}HS^3$
  $\exists W$: negative definite bounding with $\partial W = Y$

$$c_1^2(\mathfrak{t}) + b_2(W) \leq 4d(Y, \mathfrak{s})$$

d-invariant (II)

- Y: $\mathbb{Q}HS^3$

$$HF^+(M,s) \cong T^+_{d(M,\mathfrak{s})} \oplus HF_{\mathrm{red}}(M,\mathfrak{s})$$

- $$d(L(p,q),i) = 3s(q,p) - \tfrac{1-p}{2p} - \tfrac{i}{p} + 2\textstyle\sum_{j=1}^{i}\left(\left(\tfrac{q'j}{p}\right)\right)$$
$$= \tfrac{1}{4p}\textstyle\sum_{r=1}^{p-1}\Big(\cot\tfrac{\pi r}{p}\cot\tfrac{q\pi r}{p}$$
$$+2\cos\left((i-\tfrac{q-1}{2})\tfrac{2\pi r}{p}\right)\mathrm{cosec}\tfrac{\pi r}{p}\mathrm{cosec}\tfrac{q\pi r}{p}\Big) \qquad (T.)$$

- Are there relationship between hyperbolic structure and $d(Y,\mathfrak{s})$ and HF_{red}?
- Y: graph manifold (with at most one bad vertex) $HF^+(Y,\mathfrak{s})$ is algorithmically computable (Nemethi's algorighm)

- $HF_{\text{red}}(Y, \mathfrak{s}) \cdots$ Differential topology complexity?
  - If 4-manifolds X_0, X_1 have distinct mixed invariants and

    $$X_1 = (X_0 - W) \cup_Y W,$$

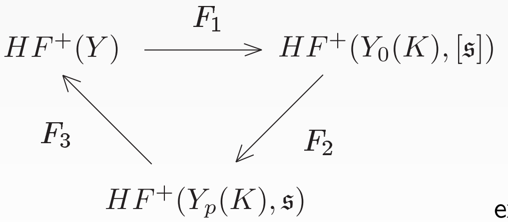
    then $HF_{\text{red}}(Y) \ncong \{0\}$.
  - L-space: $HF_{\text{red}}(Y, \mathfrak{s}) = 0$ (e.g., lens spaces)
    - Y admits no taut foliation on Y.
    - If $S^3_p(K) = Y$, then K is fibered and $\Delta_K(t)$ has flat & alternating.
    - NON Left orderable

Dehn surgery

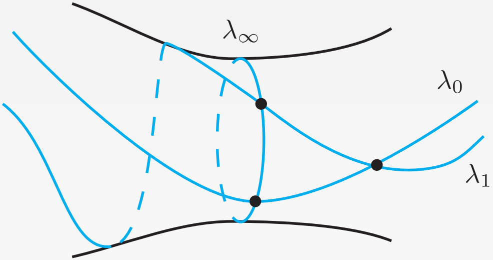
Dehn surgery

$K \subset Y$: null-homologous knot

$$Y_p(K) := [Y - N(K)] \cup_\varphi S^1 \times D^2$$



exact triangle



Y: 3-manifold
$Y = (\Sigma, T, T_\infty)$, $Y_0 = (\Sigma, T, T_0)$, $Y_1 = (\Sigma, T, T_1)$
$Y_0 = Y_0(K)$: 0-surgery
$Y_1 = Y_1(K)$

$$m_2 : CF(\Sigma, T, T_\infty) \otimes CF(\Sigma, T_\infty, T_0) \to CF(\Sigma, T, T_0)$$

Here $(\Sigma, T_\infty, T_0) = \#^{g-1} S^2 \times S^1$

Exact sequence

Applying the m_2-map, we have
$0 \to CF(\Sigma, T, T_\infty) \to CF(\Sigma, T, T_0) \to CF(\Sigma, T, T_0) \to 0$

L-spaces

$\Leftrightarrow HF_{\text{red}}(Y, \mathfrak{s}) = 0 \Leftrightarrow \dim \widehat{HF} = |H_1|$

Example 5 (L-spaces)

- S^3
- *Lens spaces*
- *Elliptic geometry*
- *Double branched cover of (quasi-)alternating knot*
- $S^3_p(K)$ $p >> 0$, K *is L-space knot.*
- *Seifert fibered rational homology spheres with no taut foliation*

taut $\Leftrightarrow \exists$ closed transversal

The cases which two in the exact triangle are L-spaces

$$\begin{array}{ccc} HF^+(Y) & \xrightarrow{F_1} & HF^+(Y_0(K),[\mathfrak{s}]) \\ {\scriptstyle F_3}\nwarrow & & \swarrow{\scriptstyle F_2} \\ & HF^+(Y_p(K),\mathfrak{s}) & \end{array}$$

$\mathfrak{s} \leftrightarrow i \in \mathbb{Z}/pZ$

$$(I) \quad \begin{cases} Y : \text{L-space } \mathbb{Z}HS^3 \\ Y_p(K) \text{ are L-space} \end{cases} \quad \Rightarrow HF^+_{\mathrm{red}}(Y_0, i) \cong T^+[n_i]$$

$$(II) \quad \begin{cases} Y : \text{L-space } \mathbb{Z}HS^3 \\ Y_0(K) \text{ L-space} \Leftrightarrow HF_{\mathrm{red}}(Y, s) = \{0\} \end{cases}$$

$$\Rightarrow HF^+_{\mathrm{red}}(Y_p(K), i) \cong T^+[n_i]$$

$$(III) \quad \begin{cases} Y_0(K) \text{ an L-space} \\ Y_p(K) \text{ an L-space} \end{cases} \quad \Rightarrow HF^+_{\mathrm{red}}(Y, i) \cong \mathbb{Z}[U]/U^n$$

(I)$\Leftrightarrow$ Y, Y_p L-spaces

$$HF^+(Y) \xrightarrow{F_1} HF^+(Y_0(K),[\mathfrak{s}]) \xrightarrow{F_2} HF^+(Y_p(K),\mathfrak{s}) \xrightarrow{F_3} HF^+(Y)$$

$\Rightarrow$

Theorem 6 (Ozsváth-Szabó)

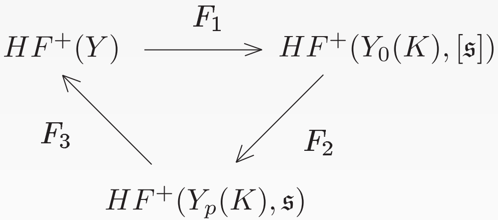
$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{m-j}(t^{n_j} + t^{-n_j})$$

$$0 < n_1 < n_2 < \cdots < n_m$$

- In $Y = S^3$ case, Greene classified a simple lens space knot.
- In Y general L-space $\mathbb{Z}HS^3$ case, T. classified a simple lens space knot (TBA).

Other cases

(II)$\Leftrightarrow$ Y, Y_0 L-spaces
(III)$\Leftrightarrow$ Y_0, Y_p L-spaces



$\Rightarrow g(K) = 1$

Problem 7

Classify these surgeries.

Gordon conjecture

Theorem 8 (Kronheimer-Mrowka-Ozsváth-Szabó (2003))

Let U be the unknot.
If $S^3_{\frac{p}{q}}(K) = S^3_{\frac{p}{q}}(U)$, then $K = U$.

$\because$ Seiberg-Witten Floer

Problem 9

Prove it by using the Heegaard Floer homology.

Geometric aspect

Hyperbolic Dehn surgery (W.Thurston)

Geom. of Knots "$\leftrightarrow$" Geom. of 3-manifolds

Torus knots Satellite knots Hyperbolic knots	"$\leftrightarrow$"	Spherical manifolds Elliptic manifolds $\vdots$ Hyperbolic manifolds

In fact some hyperbolic knots K yield lens spaces.

Fundamental group

Y is an L-space.

Theorem 10

If Y is L-space, then Y does not admit taut foliation.

Conjecture 11

If Y is irreducible rational homology sphere, then this converse is true.

Theorem 12 (Gabai)

If Y admit taut foliation, then the universal cover of Y is homeomorphic to $\mathbb{R}^3$.

L-space condition gives some kind of the restriction to the fundamental group.

Left-orderability

Definition 13 (Left-orderable)

$G \neq \{e\}$: group
$\exists$ order $<$ on G $g, h, k \in \pi_1(Y)$

$$g < h \Rightarrow kg < kh$$

If $b_1(Y) > 0$ then $\pi_1(Y)$ is left-orderable.

In particular left-orderable G is non-torsion.

Conjecture 14

Y: $\mathbb{Q}HS^3$.
$\pi_1(Y)$ is not left-orderable $\Leftrightarrow$ Y: L-space.

Evidences

Theorem 15 (Many topologists)

Well-known L-spaces admit no left-orderable.

Theorem 16 (Strong L-space)

If Y is strong L-space then Y is not left-orderable.

Strong L-space dim $\widehat{CF} = |H_1|$.

Theorem 17

If Y is an L-space Seifert fibered $\mathbb{Q}HS^3$

$\Leftrightarrow$

Y is not left-orderable.

$\Leftrightarrow$

Y is not taut foliation.

Mixed 4-manifold invariant

X: closed 4-manifold
$X = W_1 \cup_N W_2$:$b_2^+(W_i) > 1$
N: admissible cut. ($\Leftrightarrow \delta H^1(N) \subset H^2(X)$ is trivial)

$$\begin{array}{ccccc}
1 \in HF^-(S^3) & \xrightarrow{F_{W_-}} & HF^-(N) & & \\
& \searrow^{p} & \uparrow & & \\
& & HF_{\text{red}}(N) & & \\
& & \uparrow & \searrow^{p} & \\
& & HF^+(N) & \xrightarrow{F^+_{W_+}} & HF^+(S^3) \ni \Phi_{X,\mathfrak{s}} \in \mathbb{Z}/2\mathbb{Z}
\end{array}$$

$HF^\infty(S^3) \stackrel{F_W}{\to} HF^\infty(N)$
$HF^\infty(N) \stackrel{F_W}{\to} HF^\infty(S^3)$ are 0-map.

$$OS_{X,\mathfrak{s}} = \sum_{\mathfrak{s} \in \text{Spin}^c(X)} \Phi_{X,\mathfrak{s}} \mathfrak{s}$$

Theorem 18

$OS_{X,\mathfrak{s}}$ is smooth 4-manifold invariant.

Conjecture 19

Let X be a closed oriented 4-manifold. Then

$$[SW_{X,\mathfrak{s}}] = OS_{X,\mathfrak{s}}$$

Contact invariant

Definition 20 ((Y, ξ) contact structure)

(Y, ξ): *everywhere non-integrable plane field*

Definition 21

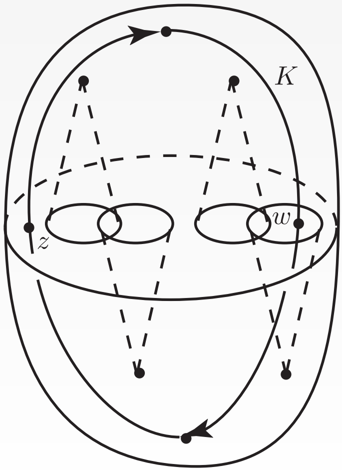
$$\exists c(\xi) \in HF^{+}(Y, \mathfrak{s}(\xi))$$

isotopy invariant of (Y, ξ).
Furthermore, if ξ *is over-twisted, then* $c(\xi) = 0$.

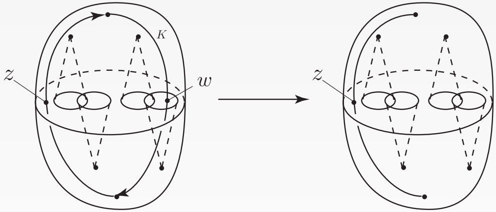
Thus we have
$c(\xi) \neq 0 \Rightarrow (Y, \xi)$ is tight.

§4 Knot Floer homology

Heegaard decomposition of K



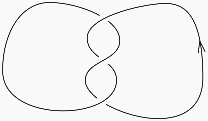
Knot filtration

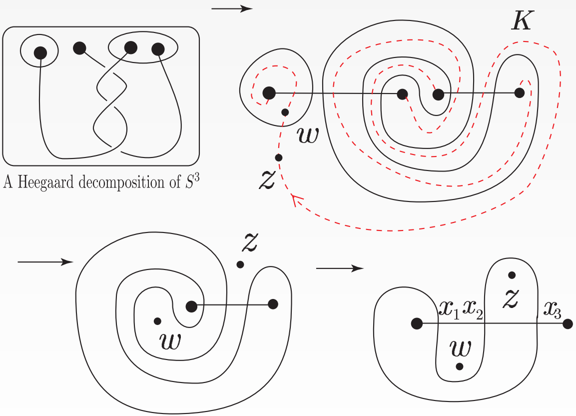


$$\widehat{CFK}(S^3, K) \to \widehat{CF}(S^3)$$

$CFK^\infty(\Sigma, \alpha, \beta, z, w)$: Filtered chain complex of $CF^\infty(\Sigma, \alpha, \beta, z)$
For generators it is the identity.

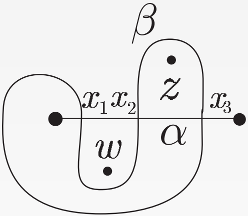
Example 22 (The trefoil knot)

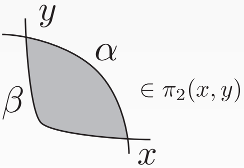




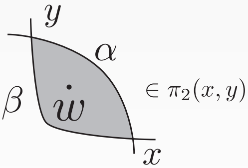
x_1, x_2, x_3 generators of $\widehat{CF}(\Sigma, \alpha, \beta, z, w)$

Differentials





$\partial x = y$



$\partial x = Uy$

$$\partial^\infty x_1 = 0, \partial^\infty x_2 = Ux_1 + x_3, \partial^\infty x_3 = 0$$

$$\hat{\partial} x_1 = 0, \hat{\partial} x_2 = x_3, \hat{\partial} x_3 = 0$$

Alexander Filtration

$$\partial^\infty x_1 = 0, \partial^\infty x_2 = Ux_1 + x_3, \partial^\infty x_3 = 0$$
$$\hat{\partial} x_1 = 0, \hat{\partial} x_2 = x_3, \hat{\partial} x_3 = 0$$

Alexander filtration

$$Alex : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$$

For a disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$

$$Alex(\mathbf{x}) - Alex(\mathbf{y}) = n_z(\phi) - n_w(\phi)$$

$$Alex(x_2) - Alex(x_1) = 0 - 1 = -1$$

$$Alex(x_2) - Alex(x_3) = 1 - 0 = 1$$

Alexander filtration

$$\partial^\infty x_1 = 0, \partial^\infty x_2 = Ux_1 + x_3, \partial^\infty x_3 = 0$$
$$\hat{\partial} x_1 = 0, \hat{\partial} x_2 = x_3, \hat{\partial} x_3 = 0$$

Thus,

$$Alex(x_1) = 1, Alex(x_2) = 0, Alex(x_3) = -1$$

Actually generators are **Symmetric** via this Alexander filtration:
$(\because)$
Generators $\leftrightarrow$ Kauffman's states
Poincaré polynomial of $\widehat{CFK} \leftrightarrow$ Alexander polynomial

Another grading

$$\partial^\infty x_1 = 0, \partial^\infty x_2 = Ux_1 + x_3, \partial^\infty x_3 = 0$$
$$\hat{\partial} x_1 = 0, \hat{\partial} x_2 = x_3, \hat{\partial} x_3 = 0$$

Maslov index

$$M(x) - M(y) = \mu(\phi) - 2n_w(\phi)$$

$$M(x_2) - M(x_1) = 1 - 2 \cdot 1 = -1$$

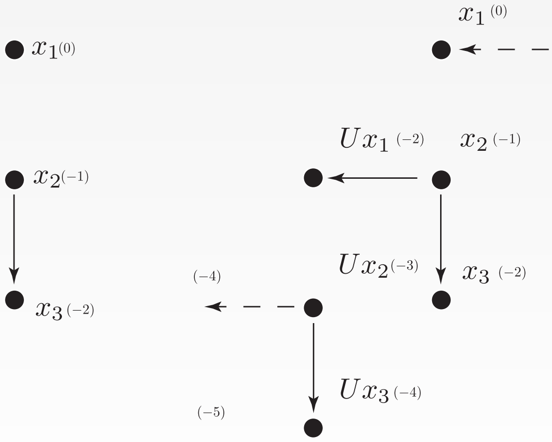
$$M(x_2) - M(x_3) = 1 - 2 \cdot 0 = 1$$

Thus,

$$M(x_1) = 0, M(x_2) = -1, M(x_3) = -2$$

(The grading of $\widehat{CF}(S^3)$ is zero.)

$$\partial^\infty x_1 = 0, \partial^\infty x_2 = Ux_1 + x_3, \partial^\infty x_3 = 0$$
$$\hat{\partial} x_1 = 0, \hat{\partial} x_2 = x_3, \hat{\partial} x_3 = 0$$



$\widehat{CFK}(S^3, K)$ $\quad$ $CFK^\infty(S^3, K)$

$$(\widehat{CKF}, \hat{\partial}^K)$$

$$\partial^\infty x_1 = 0, \partial^\infty x_2 = Ux_1 + x_3, \partial^\infty x_3 = 0$$

$$\hat{\partial} x_1 = 0, \hat{\partial} x_2 = x_3, \hat{\partial} x_3 = 0$$

Definition 23 (Differential operator $\hat{\partial}^K$ of $\widehat{CFK}$)

$$\hat{\partial}^K x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \nu(\phi)=1, n_z(\phi)=n_z(\phi)=0} \#(\frac{\mathcal{M}(\phi)}{\mathbb{R}}) y$$

$$\hat{\partial}^K x_1 = 0, \hat{\partial}^K x_2 = 0, \hat{\partial}^K x_3 = 0$$

$$\widehat{HFK}(3_1, j) = \begin{cases} \mathbb{Z}_{(0)} & j = 1 \\ \mathbb{Z}_{(-1)} & j = 0 \\ \mathbb{Z}_{(-2)} & j = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \sum_{j=-1}^{1} \chi(\widehat{HFK}(3_1, j)) t^j &= 1 \cdot t + (-1) \cdot 1 + 1 \cdot t^{-1} \\ &= t^{-1} - 1 + t = \Delta_K(t) \end{aligned}$$

$(\widehat{CKF}, \hat{\partial})$

$z, w \in \Sigma_g - \alpha_1 \cdots - \beta_g$
$z \times \mathrm{Sym}^{g-1}(\Sigma_g), w \times \mathrm{Sym}^{g-1}(\Sigma_g) \subset \mathrm{Sym}^g(\Sigma_g)$

$$(CFK^\infty(\Sigma, \alpha, \beta, z, w), \partial^\infty)$$

Definition 24 ($(\widehat{CFK}, \hat{\partial}^K)$)

- *Generators: a lift of $T_\alpha \cap T_\beta$.*
- *Differential: Counting holomorphic disks*

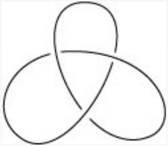
$$\partial^\infty \mathbf{x} = \sum_{\mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi) = 0} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) U^{n_w(\phi)} \mathbf{y}$$

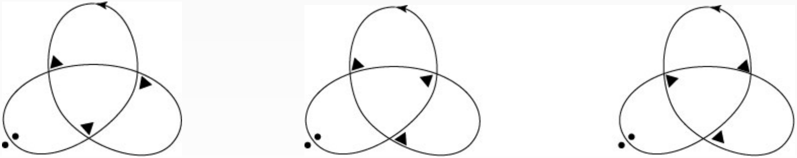
Euler number: $\sum_s \chi(\widehat{HFK}(K, s)) t^s = \Delta_K(t)$

Alexander polynomial

Interpretation of $(\widehat{CKF}, \hat{\partial}^K)$ in low-dimensional topology

Generators: $\widehat{CFK}(K) \leftrightarrow$ {Kauffman states}





Differential ∂^∞: clockwise exchange

Alternating knot

Definition 25 (Alternating knot)

K admits an alternating knot diagram.

Theorem 26 (Ozsváth-Szabó)

Let K be a alternating knot.

- $\widehat{HFK}(K, s) \cong \mathbb{Z}^{|a_s|}_{(s+\frac{\sigma}{2})}$
- $d(S^3_1(K)) = 2\min(0, -\lceil \frac{-\sigma(K)}{2} \rceil)$

Theorem 27 (Ozsváth-Szabó)

$$\widehat{HFK}(KinoTera, s) \not\cong \widehat{HFK}(Conw, s) \quad \text{for some } s$$

Problem 28

Clarify the mutant effect to $\widehat{HFK}$.

§5 Knot Concordance

$\sim_c$:concordant equivalent

$$\mathcal{C} := \{\text{knots}\}/\sim_c$$

$$K_0 \sim_c K_1 \Leftrightarrow \exists \varphi : S^1 \times I \hookrightarrow S^3 \times I,$$
$$K_i = \varphi(S^1 \times i)$$

$\mathcal{C}$ is an abelian group.

$$\mathcal{C} \ni [K] = 0 \Leftrightarrow K \text{ is slice knot}$$

Definition 29 (Zero element)

K is slice knot if $\exists$ embedding $D^2 \hookrightarrow B^4$ with $\partial D^2 = K$.

Concordance invariants

$g_4(K) = \min\{g(\Sigma) | \partial\Sigma = K, \Sigma \hookrightarrow B^4\}$ $\qquad$ $g_4 : \mathcal{C} \to \mathbb{Z}_{\geq 0}$

$\omega \in S^1$
$\sigma_\omega(K) = \sigma((1-\omega)V + (1-\bar{\omega})^t V)$ $\qquad$ $\sigma_\omega : \mathcal{C} \to \mathbb{Z}$ (homo.)
$|\sigma(X)| \leq 2g_4(K)$

$dS_1(K) = d(S^3_1(K))$ $\qquad$ $\mathcal{C} \to \mathbb{Z}$(not homo.)

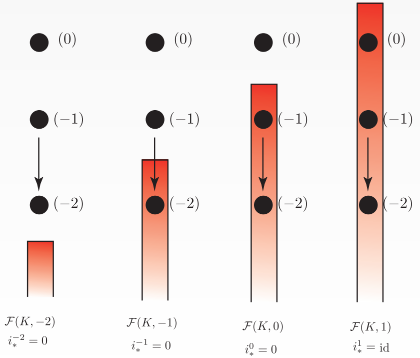
Heegaard Floer concordance invariant

$\delta(K) = 2d(\Sigma_2(K), \mathfrak{s}_0) \in \mathbb{Z}$ (homo.)

$$\cdots \subset \mathcal{F}(K,s) \subset \mathcal{F}(K,s+1) \subset \mathcal{F}(K,s+2) \subset \cdots$$

$\tau : \mathcal{C} \to \mathbb{Z}$ (Ozsváth-Szabó's τ-invariant (homo.))

$$\tau(K) = \min\{s \mid i_*^s : H_*(\mathcal{F}(K,s)) \to \widehat{HF}(S^3) \text{ non-trivial}\}$$



Theorem 30

$|\tau(K)| \leq g_4(K)$
The τ is independent of $\sigma(K)$ and Rasmussen's $s(K)$.

c.f. $|\sigma(K)| \leq 2g_4(K)$ (Murasugi)

Theorem 31 (Ozsváth-Szabó)

If K is alternating knot, then

$$\tau(K) = -\frac{1}{2}\sigma(K)$$

Theorem 32 (Another formula (T.))

$$\tau(K) = \sum_{s=-g}^{g} \chi(\mathcal{F}(K, s))$$

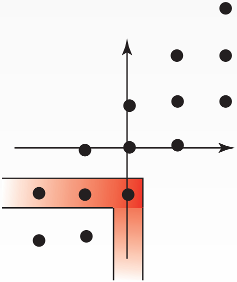
Dehn surgery formula

$g = g(K)$, $n \leq 2g - 1$

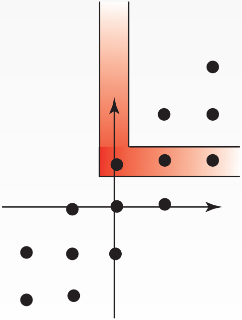
Theorem 33 (Ozsváth-Szabó)

$$\widehat{CF}(S^3_n(K), [s]) \cong \widehat{CFK}(K)\{\max\{i, j - s\} = 0\}$$

$$\widehat{CF}(S^3_{-n}(K), [s]) \cong \widehat{CFK}(K)\{\min\{i, j - s\} = 0\}$$



$\max\{i, j - s\} = 0$



$\min\{i, j - s\} = 0$

$F_s : \widehat{HF}(S^3) \to \widehat{HF}(S^3_{-N}(K), [s]),\ G_s : \widehat{HF}(S^3_N(K), [s]) \to \widehat{HF}(S^3)$

Definition 34 (Hom's ϵ)

$\tau = \tau(K)$.

$$\epsilon(K) = \begin{cases} 1 & F_\tau \text{ trivial} \\ -1 & G_\tau \text{ trivial} \\ 0 & F_\tau, G_\tau \text{ non-trivial} \end{cases}$$

Theorem 35 (Hom)

- *$\epsilon(K)$ is concordant invariant and stronger than τ.*
- *$[K] \in \mathcal{C}$ then $\epsilon(K) = 0$.*
- *$\epsilon(K) = 0$ then $\tau(K) = 0$*
- *$\epsilon(-K) = \epsilon(K)$*
- *The cable $\tau(K_{p,q})$ is computable by $\tau(K)$ and $\epsilon(K)$.*
- *$\mathbb{Z}^\infty \subset \mathcal{C}_{top}$*

Other invariants in terms of $\widehat{CFK}$

- V_k-invariant
- $\nu(K)$
- $\nu^-(K) = \nu^+(K)$
- $g_c(K)$
- $g_c(C)$

Theorem 36 (Wu-Hom)

Strictly sharper bounds:

$$\tau(K) \leq \nu^+(K) \leq g_4(K)$$

Υ-invariant

Livingston's interpretation.

$t, s \in \mathbb{R}$
$C = CFK^\infty$

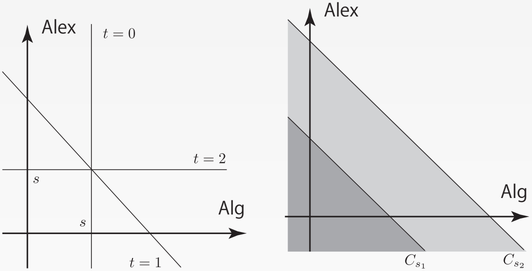
$$(C, \mathcal{F}_t)_s = \left\{x \in C | \frac{t}{2} Alex(x) + \left(1 - \frac{t}{2}\right) Alg(x) \leq s\right\}$$

$$\nu(C, \mathcal{F}_t) = \min\{s | \mathrm{Im}(H_0((C, \mathcal{F}_t)_s) \to H_0(C) \text{ is surjective}\} \in \mathbb{R}$$

For $s_1 \leq s_2$

$$(C, \mathcal{F}_t)_{s_1} \subset (C, \mathcal{F}_t)_{s_2}$$

$0 \leq t \leq 2$

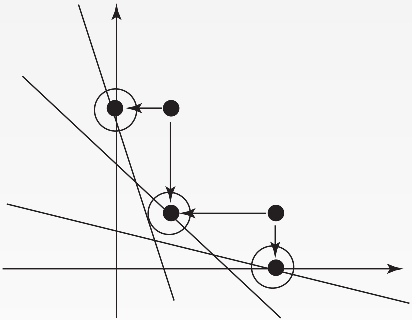


Definition 37 (Ozsváth-Stipsicz-Szabó)

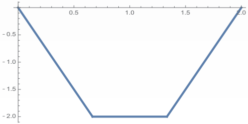
$\Upsilon(K) : \mathcal{C} \to C([0,2])$ (*homo.*)

$$\Upsilon_K(t) = -2\nu(C, \mathcal{F}_t)$$

(3, 4)-torus knot



$$\Upsilon_K(t) = \begin{cases} -3t & 0 \leq t \leq \frac{2}{3} \\ -2 & \frac{2}{3} \leq t \leq \frac{4}{3} \\ 3t - 6 & \frac{4}{3} \leq t \leq 2 \end{cases}$$



Properties

- Knot concordance invariant
- $\Upsilon_K(0) = 0$
- $\Upsilon'_K(0) = -\tau(K)$
- $\Upsilon_K(t) = \Upsilon_K(2-t)$
- $\Upsilon_K(\frac{m}{n}) \in \frac{1}{n}\mathbb{Z}$
- $0 < t < 1$, $|\frac{\Upsilon_K(t)}{t}| \leq g_4(K)$
- $\max\{\text{slopes of } |\Upsilon_K(t)|\} \leq g_c(K)$
- K is alternating: $\Upsilon_K(t) = (1 - |t-1|)\sigma$

Theorem 38 (Ozsváth-Szabó)

K: a torus knot.

$$\Delta_K(t) = \sum_{i=0}^{n} (-1)^i t^{n_i}$$

$$\delta_i = \begin{cases} 0 & j = n \\ \delta_{i+1} - 2(n_{i+1} - n_i) + 1 & n - i : odd \\ \delta_{i+1} - 1 & n - i : even \end{cases}$$

Then

$$\Upsilon_K(t) = \max_{\{i \mid 0 \le 2i \le n\}} \{\delta_{2i} - t n_{2i}\}$$

Problem 39

Let K be a torus knot. Give an explicit formula of $\Upsilon_K(t)$.

§6 Future's problem

- For given chain complex C, construct a knot with $C = CFK^{\infty}$.
- Give combinatorial description of knot (link) Floer homology $\mathbb{Z}$-coefficient.
- Consider torsion part of $HF(Y, K)$ with $\mathbb{Z}$-coefficient.
- Classify the categorical action on Bordered Floer homology.
- Connection with Khovanov homology.
- Deal (branched) covering.
- Deal hyperbolic structure.
- L-space & Left-orderable.
- Relate to Quantum invariants.
- Relate to Twisted Alexander polynomial.
- Increase people who study Heegaard Floer homology in Japan.