Foams, Polytopes, Abstract Tensors, and Homology

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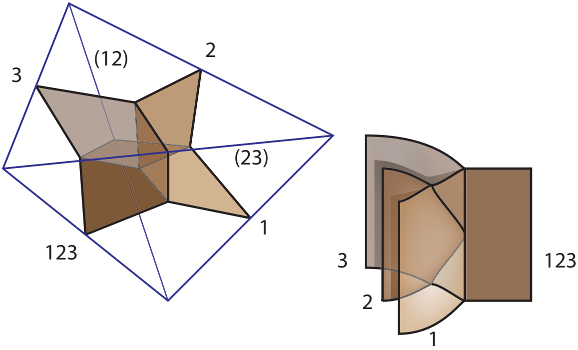
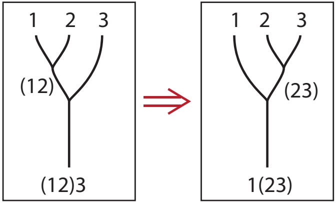
$$Y^n = C\left(\cup_{j=1}^{n+2} Y^{n-1}_j\right).$$

The space Y^2





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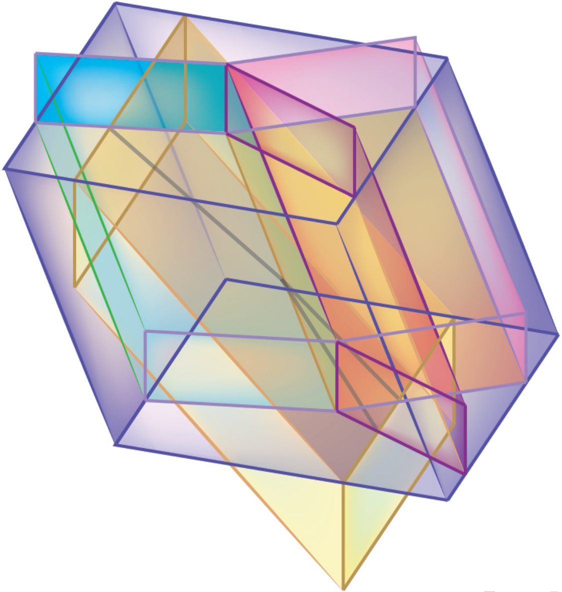

Foam Definition

An n-**dimensional foam** is a compact top. sp. X for which each point $x \in X$ has a nbhd. $N(x)$ that is homeom. to a nbhd. M of a point in Y^n.

Rough Statements

1. foams are worth studying.

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2. There's a method of constructing group and quandle homology from a single point of view.

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4. Moves to foams and critical points grow out of a Morse-type analysis in a categorical context.

Work yet to be complete

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3. Formulate a programmatic method to achieve goals 1 and 2.

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Section 2

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Great invariants for classical knots and knot cobordisms. Not so good for knotted closed surfaces.

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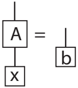
Write A=  . So Ax=b is written  .

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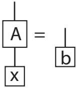
Write A=  . So Ax=b is written  .

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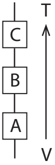

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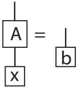
. So Ax=b is written 

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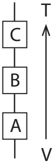

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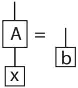
. So Ax=b is written 

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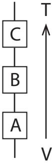

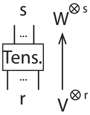

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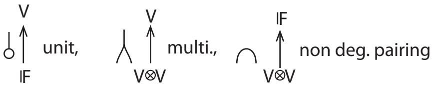

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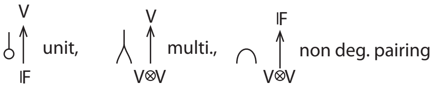

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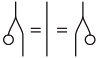

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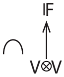

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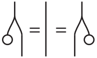

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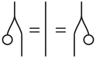

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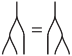

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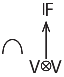

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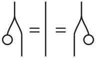
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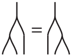
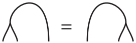

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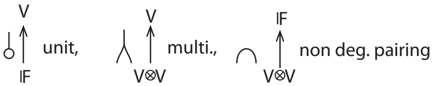

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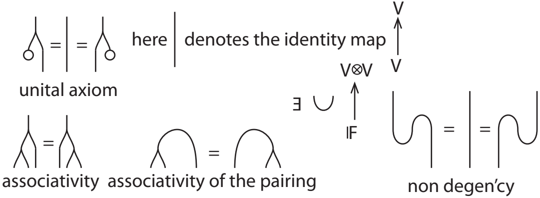
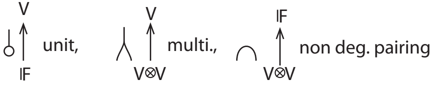


Example: A Frobenius alg. is a v. sp. V tog. w/ linear maps



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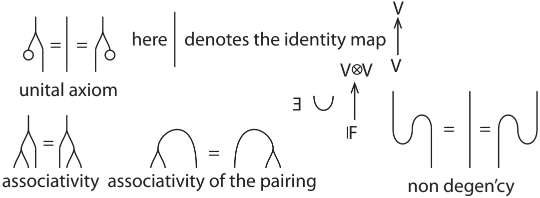

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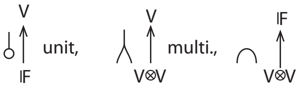
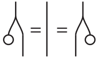


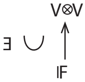
non deg. pairing

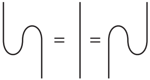
that satisfy:

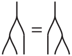


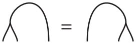
here | denotes the identity map

unital axiom









associativity associativity of the pairing

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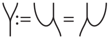

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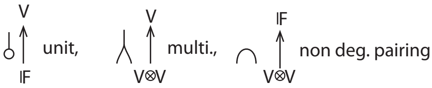

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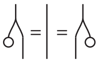
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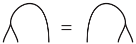
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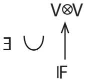

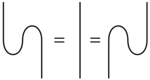
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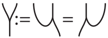

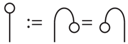

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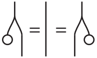

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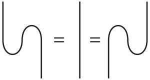
∃

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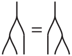

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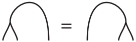

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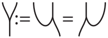

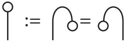

It's remarkable that most of the axiomatics for the alg.coalg str. follows directly from the diagrammatics.

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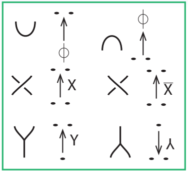
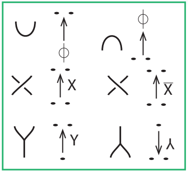


Generating 1-morphs



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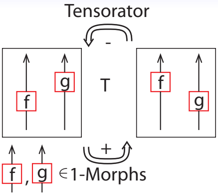

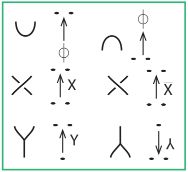

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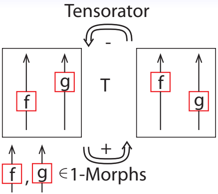

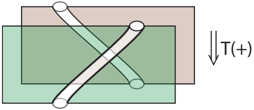

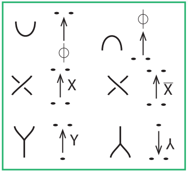

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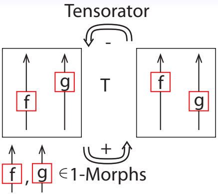

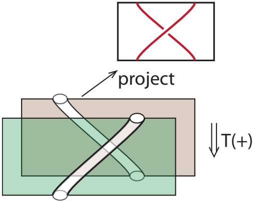

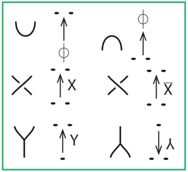

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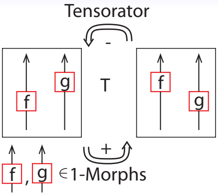
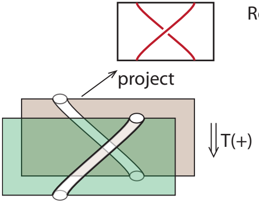
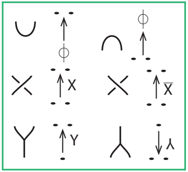


Relations (3-morphism)





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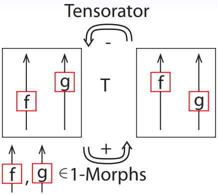

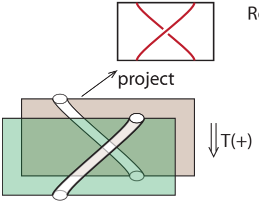

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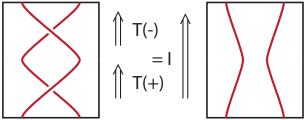
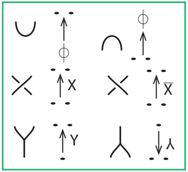
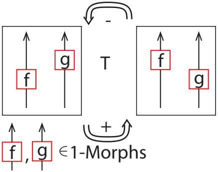
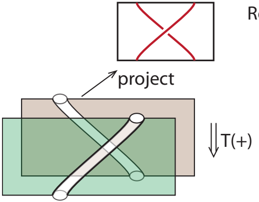




Generating 1-morphs

Tensorator





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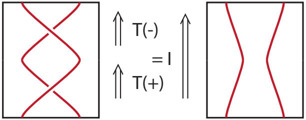
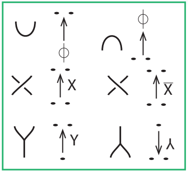


ditto for $T(+)\circ_2 T(-)$



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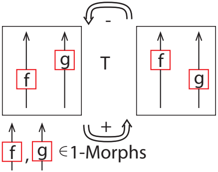
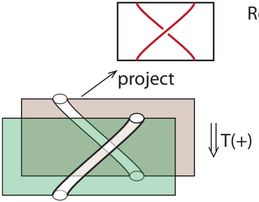
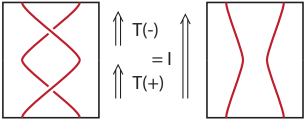




$T(\pm)$ is natural w.r.t. any 2-morph.

Relations (3-morphism)



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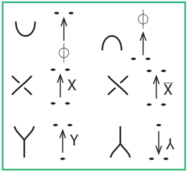

Generating 1-morphs

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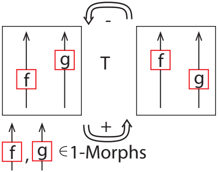

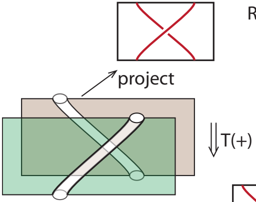

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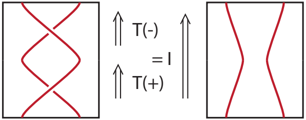

T(±) is natural w.r.t. any 2-morph.

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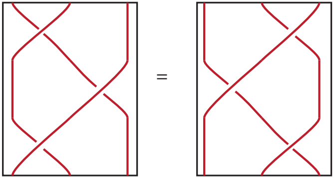
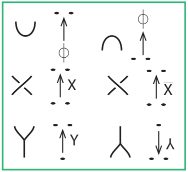
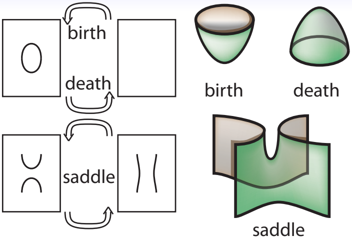
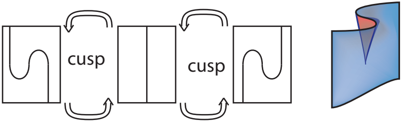
In part.:





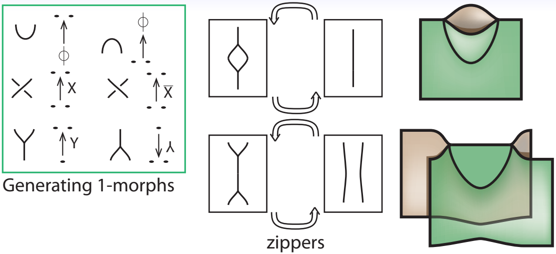
Generating 1-morphs



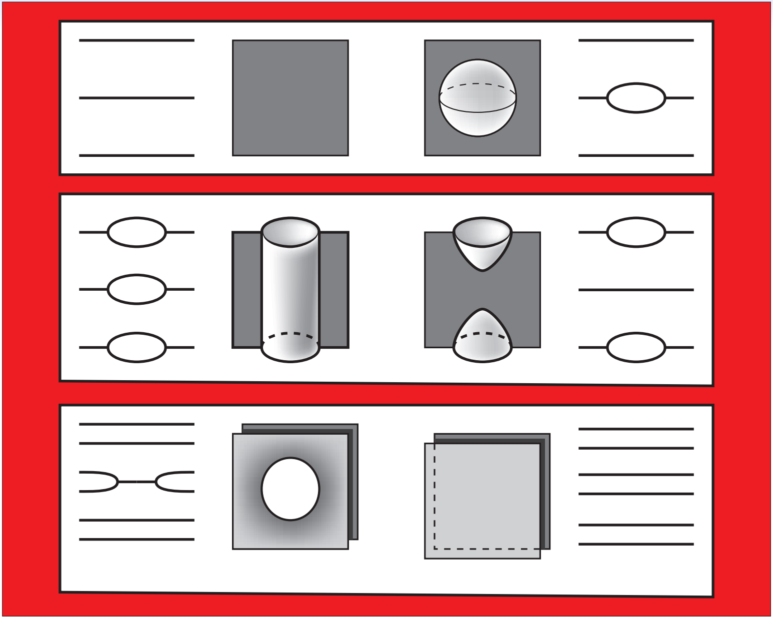


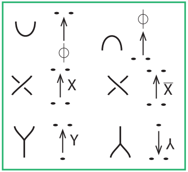
- The critical points evolve to be come folds.
  (co-oriented away from optimal points)

- Several obvious relations hold among these 2-morphisms.
  Including
  - canceling birth/saddle death/saddle
  - lips
  - beak-to-beak
  - swallow-tail
  - horizontal cusp

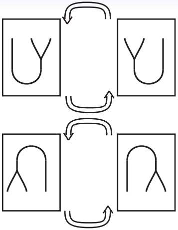


- The vertices of Y or ⅄ evolve to form seams of the foam. (co-oriented towards the single sheet)
- There are zig-zag moves that cancel a pair of zipper 2-morphisms.
- Under some circumstances, one might want to suppose bubble and saddle moves hold.
  But, as is the case with birth followd by death,
  or a pair of opposite saddles,
  it is better to suppose that the moves
  in the next slide are some type of 3-morphisms.

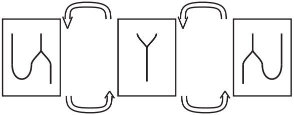




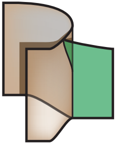
Generating 1-morphs

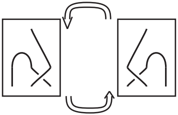


speedometers

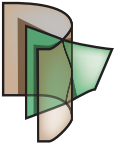


garment adjustments

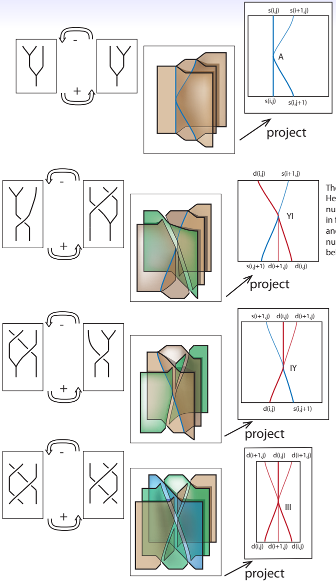




psy-move

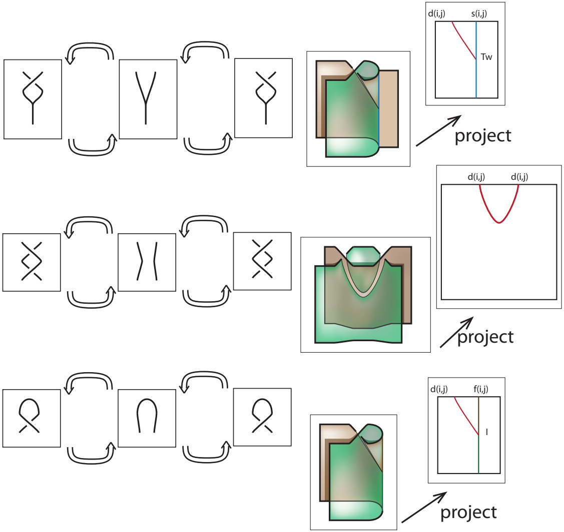


- These 2-morphs indicate how folds, seams, double points can interact.
- Each move is invertible in both directions.



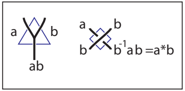
The chart depiction. Here j indicates the number of sheets in front of the point and i indicates the number of sheets behind the point.

The broken surface representation of the resulting foam.

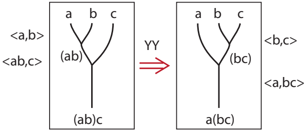

d(i,j)
s(i,j)
Tw
project
d(i,j)
d(i,j)
project
d(i,j)
f(i,j)
I
project

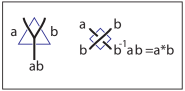
Section 3

Fundamental group

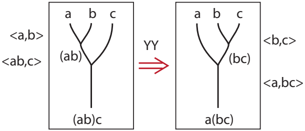


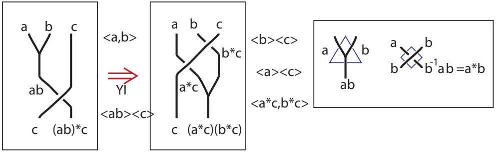
(ab)c = a(bc)





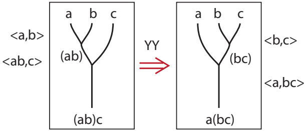
$$(ab)c = a(bc)$$



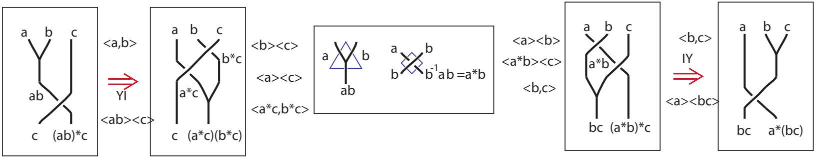


$$(ab)^*c = (a^*c)(b^*c)$$

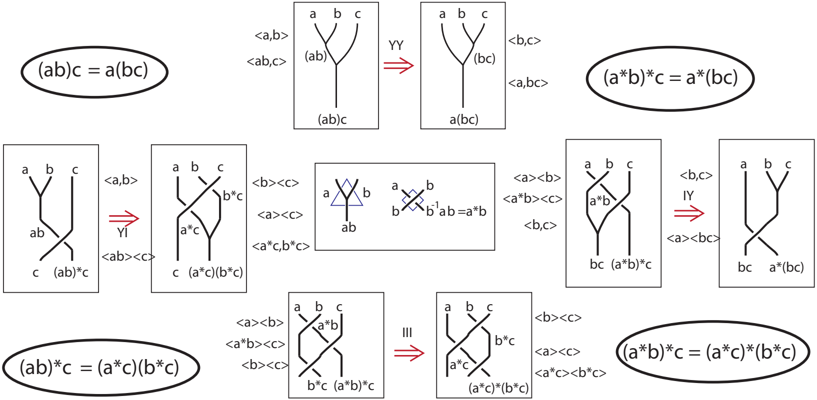
$(ab)c = a(bc)$



$(a*b)*c = a*(bc)$



$(ab)*c = (a*c)(b*c)$


(ab)c = a(bc)
a b c
<a,b>
<ab,c>
(ab)
(ab)c
YY
a b c
(bc)
a(bc)
<b,c>
<a,bc>
(a*b)*c = a*(bc)
a b c
ab
c (ab)*c
<a,b>
YI
<ab><c>
a b c
b*c
a*c
c (a*c)(b*c)
<b><c>
<a><c>
<a*c,b*c>
a b
ab
a b
b b⁻¹ab = a*b
<a><b>
<a*b><c>
<b,c>
a b c
a*b
bc (a*b)*c
<b,c>
IY
<a><bc>
a b c
bc a*(bc)
(ab)*c = (a*c)(b*c)
<a><b>
<a*b><c>
<b><c>
a b c
a*b
b*c (a*b)*c
III
a b c
b*c
a*c
(a*c)*(b*c)
<b><c>
<a><c>
<a*c><b*c>
(a*b)*c = (a*c)*(b*c)

$$\begin{array}{rrcl}
\text{YY} & (ab)c & = & a(bc), \\
\text{IY}: & (ab) \triangleleft c & = & (a \triangleleft c)(b \triangleleft c), \\
\text{YI}: & (a \triangleleft b) \triangleleft c & = & a \triangleleft (bc), \\
\text{III}: & (a \triangleleft b) \triangleleft c & = & (a \triangleleft c) \triangleleft (b \triangleleft c).
\end{array}$$

A quandle

satisfies three axioms that correspond to the Reidemeister moves:

$$
\begin{array}{lrrcl}
I: & (\forall a): & a \triangleleft a & = & a \\
II: & (\forall a, b)(\exists c): & c \triangleleft b & = & a \\
III: & (\forall a, b, c): & (a \triangleleft b) \triangleleft c & = & (a \triangleleft c) \triangleleft (b \triangleleft c).
\end{array}
$$

A quandle

satisfies three axioms that correspond to the Reidemeister moves:

$$\begin{array}{lrrcl} I: & (\forall a): & a \triangleleft a & = & a \\ II: & (\forall a,b)(\exists c): & c \triangleleft b & = & a \\ III: & (\forall a,b,c): & (a \triangleleft b) \triangleleft c & = & (a \triangleleft c) \triangleleft (b \triangleleft c). \end{array}$$

We are interested in how the group G and its associated quandle $\mathrm{Conj}(G)$ interact.

Remark

There are related concepts for which the homology sketched below applies, *e.g.*:

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- G-families of quandles (IIJO)

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Remark

There are related concepts for which the homology sketched below applies, *e.g.*:

- G-families of quandles (IIJO)
- MCQ multiple conjugation quandles (Ishii, See also CIST)
- Lebed's qualgebras

Homology of G-families of quandles was defined to study handle-body knots.

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Here we use, YY, YI, IY, and III to define the homological conditions.

Slicing

Cut the interval $[0, n]$ into integral pieces.

Slicing

Cut the interval $[0, n]$ into integral pieces.

$$\langle 1, 2, \ldots, \ell_1 \rangle \langle \ell_1 + 1, \ldots, \ell_1 + \ell_2 \rangle \cdots$$

$$\left\langle \sum_{i=1}^{j-1} \ell_i + 1, \ldots, \sum_{i=1}^{j} \ell_i \right\rangle \ldots \left\langle \sum_{i=1}^{k} \ell_i + 1, \ldots, n \right\rangle .$$

Slicing

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$$\left\langle \sum_{i=1}^{j-1} \ell_i + 1, \ldots, \sum_{i=1}^{j} \ell_i \right\rangle \cdots \left\langle \sum_{i=1}^{k} \ell_i + 1, \ldots, n \right\rangle.$$

Such a slice corresponds to a decomposition of the n-ball into a product of simplices. There are 2^{n-1} ways to cut.

Boundaries

$\partial\langle j+1, j+2, \ldots, j+k\rangle$

Boundaries

$$\begin{aligned} &\partial\langle j+1, j+2, \ldots, j+k\rangle \\ &\quad = \ \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \end{aligned}$$

Boundaries

$$
\begin{aligned}
&\partial\langle j+1, j+2, \ldots, j+k\rangle \\
&\quad = \; \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \\
&\quad + \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle
\end{aligned}
$$

Boundaries

$$
\begin{aligned}
&\partial\langle j+1, j+2, \ldots, j+k\rangle \\
&\quad = \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \\
&\quad + \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle \\
&\qquad +(-1)^{k}\langle j+1, \ldots, j+k-1\rangle .
\end{aligned}
$$

Boundaries

$$
\begin{aligned}
&\partial\langle j+1, j+2, \ldots, j+k\rangle \\
&\quad = \;\triangleleft(j+1)\langle j+2, \ldots, j+k\rangle \\
&\quad + \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle \\
&\qquad +(-1)^{k}\langle j+1, \ldots, j+k-1\rangle.
\end{aligned}
$$

$$
\partial(PQ) = (\partial P)Q + (-1)^{\dim P} P(\partial Q).
$$

Boundaries

$$\partial\langle j+1, j+2, \ldots, j+k\rangle$$
$$= \triangleleft(j+1)\langle j+2, \ldots, j+k\rangle$$
$$+ \sum_{\ell=1}^{k-1}(-1)^{\ell}\langle j+1, \ldots, (j+\ell)\cdot(j+\ell+1), \ldots, j+k\rangle$$
$$+(-1)^{k}\langle j+1, \ldots, j+k-1\rangle.$$

$$\partial(PQ) = (\partial P)Q + (-1)^{\dim P}P(\partial Q).$$

In part,

$$\partial\langle j+1\rangle = \triangleleft(j+1)\textvisiblespace - \textvisiblespace.$$

Following Przytycki, one can observe that $\partial \circ \partial = 0$ in this context, if and only if

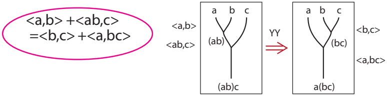
- $a(bc) = (ab)c$
- $a \triangleleft (bc) = (a \triangleleft b) \triangleleft c$
- $(ab) \triangleleft c = (a \triangleleft c)(b \triangleleft c)$
- $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

Sample computations 1

$$\begin{aligned}\partial\langle 1,2,3\rangle &= \langle 2,3\rangle - \langle 1\cdot 2,3\rangle \\ &\quad + \langle 1,2\cdot 3\rangle - \langle 1,2\rangle\end{aligned}$$

Sample computations 1

$$\begin{aligned}\partial\langle 1,2,3\rangle &= \langle 2,3\rangle - \langle 1\cdot 2,3\rangle \\ &\quad + \langle 1,2\cdot 3\rangle - \langle 1,2\rangle\end{aligned}$$

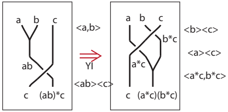


Sample computations 2

$$\begin{aligned}\partial\langle 1,2\rangle\langle 3\rangle &= \partial\langle 1,2\rangle\langle 3\rangle + \langle 1,2\rangle \triangleleft (3) - \langle 1,2\rangle \\ &= \langle 2\rangle\langle 3\rangle - \langle 1\cdot 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\ &\quad +\langle 1 \triangleleft 3, 2 \triangleleft 3\rangle - \langle 1,2\rangle\end{aligned}$$

Sample computations 2

$$
\begin{aligned}
\partial\langle 1,2\rangle\langle 3\rangle &= \partial\langle 1,2\rangle\langle 3\rangle + \langle 1,2\rangle \triangleleft (3) - \langle 1,2\rangle \\
&= \langle 2\rangle\langle 3\rangle - \langle 1\cdot 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\
&\quad + \langle 1 \triangleleft 3, 2 \triangleleft 3\rangle - \langle 1,2\rangle
\end{aligned}
$$



<a,b> +<ab><c>
=<b><c> +<a><c>
+<a*c,b*c>

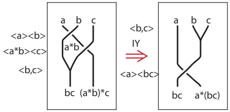
Sample computations 3

$$\begin{aligned}\partial\langle 1\rangle\langle 2,3\rangle &= \langle 2,3\rangle - \langle 2,3\rangle - \langle 1\rangle\partial\langle 2,3\rangle \\ &= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 2\cdot 3\rangle - \langle 1\rangle\langle 2\rangle\end{aligned}$$

Sample computations 3

$$\begin{aligned}\partial\langle 1\rangle\langle 2,3\rangle &= \langle 2,3\rangle - \langle 2,3\rangle - \langle 1\rangle\partial\langle 2,3\rangle \\ &= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 2\cdot 3\rangle - \langle 1\rangle\langle 2\rangle\end{aligned}$$

<a><b> +<a*b><c>
+<b,c> =<b,c> +<a><bc>

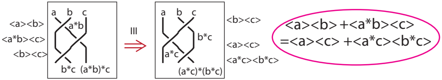


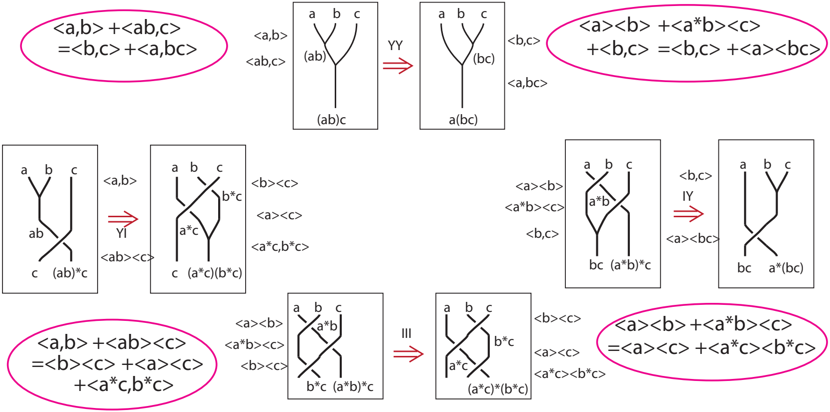
Sample computations

$$\begin{aligned}
\partial\langle 1\rangle\langle 2\rangle\langle 3\rangle &= \langle 2\rangle\langle 3\rangle - \langle 2\rangle\langle 3\rangle - \langle 1\rangle\partial(\langle 2\rangle\langle 3\rangle) \\
&= -\langle 1 \triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\
&\quad +\langle 1 \triangleleft 3\rangle\langle 2 \triangleleft 3\rangle - \langle 1\rangle\langle 2\rangle
\end{aligned}$$

Sample computations

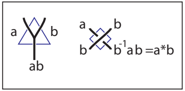
$$\begin{aligned}\partial\langle 1\rangle\langle 2\rangle\langle 3\rangle &= \langle 2\rangle\langle 3\rangle - \langle 2\rangle\langle 3\rangle - \langle 1\rangle\partial(\langle 2\rangle\langle 3\rangle) \\ &= -\langle 1\triangleleft 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle \\ &\quad +\langle 1\triangleleft 3\rangle\langle 2\triangleleft 3\rangle - \langle 1\rangle\langle 2\rangle\end{aligned}$$



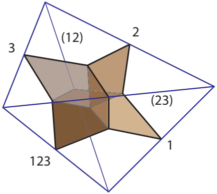


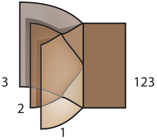
<a,b> +<ab,c>
=<b,c> +<a,bc>
<a,b>
<ab,c>
a b c
(ab)
(ab)c
YY
a b c
(bc)
a(bc)
<b,c>
<a,bc>
<a><b> +<a*b><c>
+<b,c> =<b,c> +<a><bc>
a b c
ab
c (ab)*c
<a,b>
YI
<ab><c>
a b c
b*c
a*c
c (a*c)(b*c)
<b><c>
<a><c>
<a*c,b*c>
<a><b>
<a*b><c>
<b,c>
a b c
a*b
bc (a*b)*c
<b,c>
IY
<a><bc>
a b c
bc a*(bc)
<a,b> +<ab><c>
=<b><c> +<a><c>
+<a*c,b*c>
<a><b>
<a*b><c>
<b><c>
a b c
a*b
b*c (a*b)*c
III
a b c
b*c
a*c
(a*c)*(b*c)
<b><c>
<a><c>
<a*c><b*c>
<a><b> +<a*b><c>
=<a><c> +<a*c><b*c>

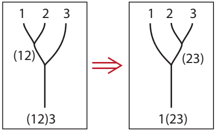
Triangles and Squares



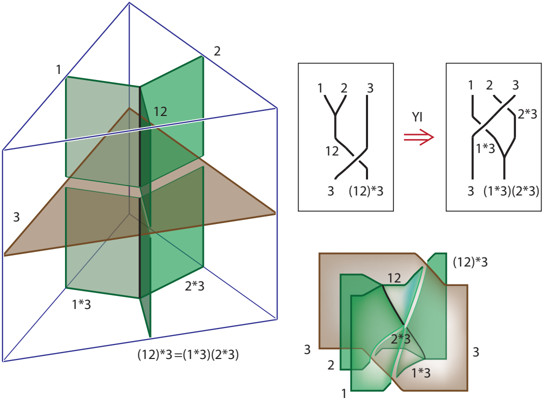
Tetrahedron



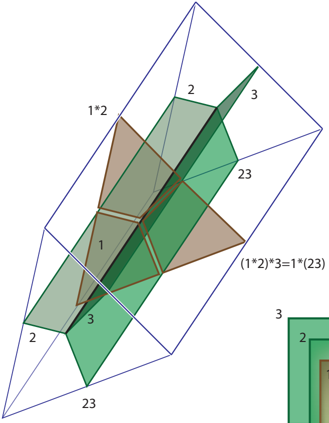


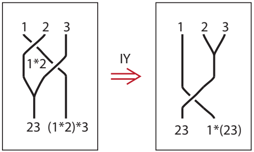


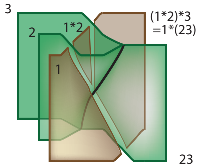
First Prism


1
2
12
3
1*3
2*3
(12)*3=(1*3)(2*3)
1 2 3
12
3 (12)*3
YI
1 2 3
2*3
1*3
3 (1*3)(2*3)
(12)*3
12
2*3
1*3
3
2
1
3

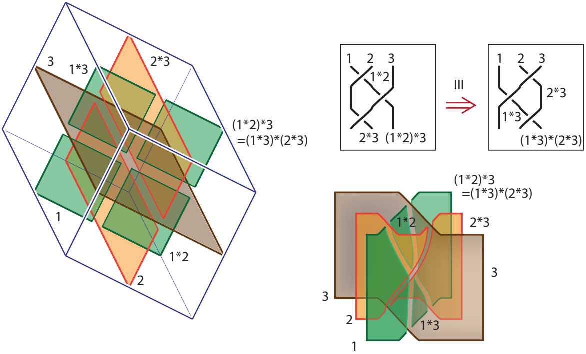
Second Prism



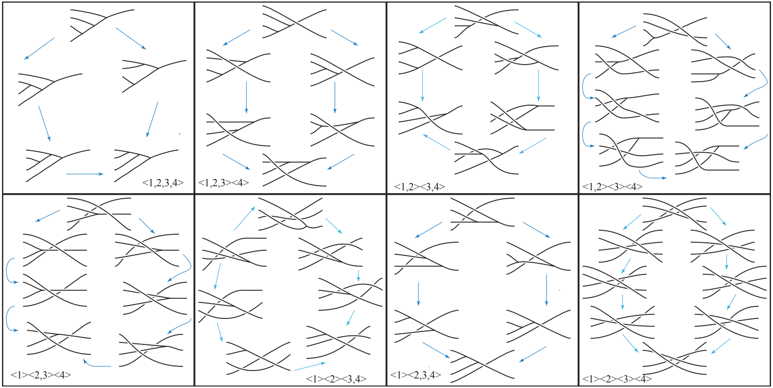




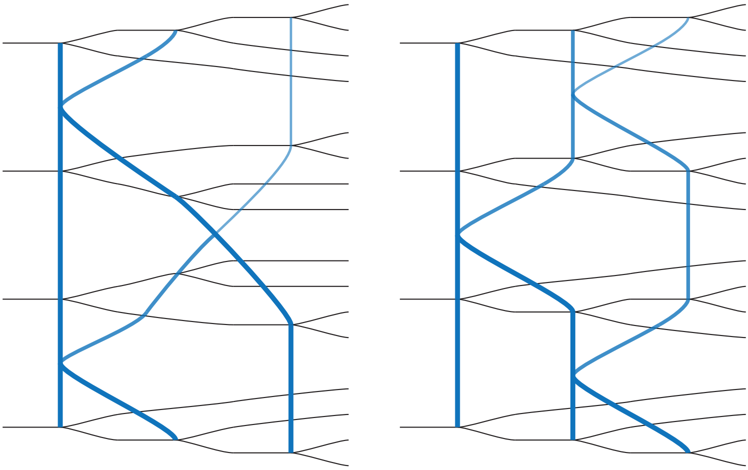
Cube


3
1*3
2*3
(1*2)*3
=(1*3)*(2*3)
1
1*2
2
1 2 3
1*2
2*3 (1*2)*3
III
1 2 3
2*3
1*3
(1*3)*(2*3)
(1*2)*3
=(1*3)*(2*3)
1*2
2*3
3
3
2
1*3
1

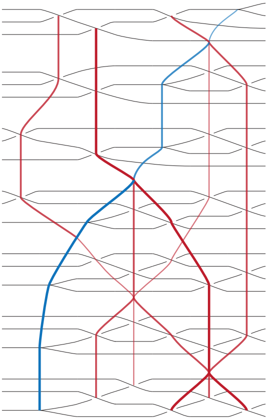
8 interesting moves

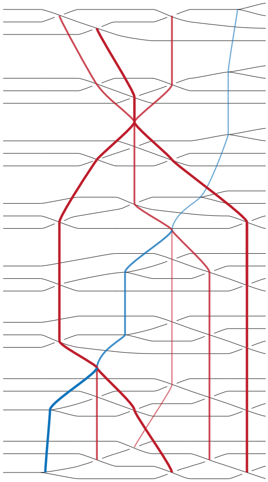


The YYY-move (Stasheff polytope)

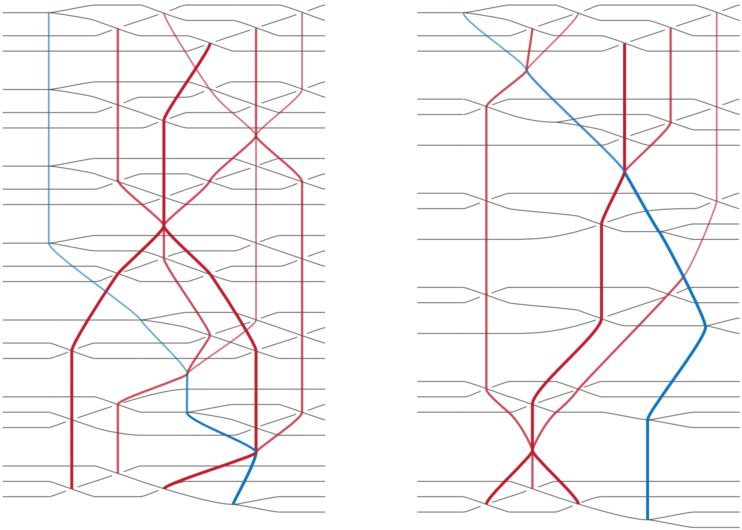


The YII-move

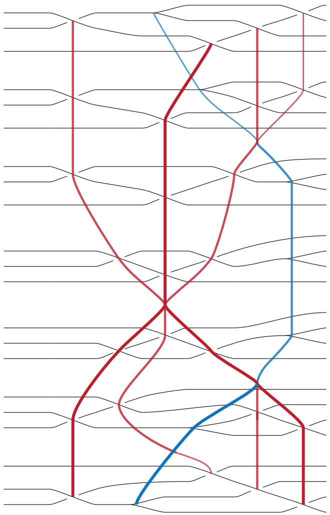


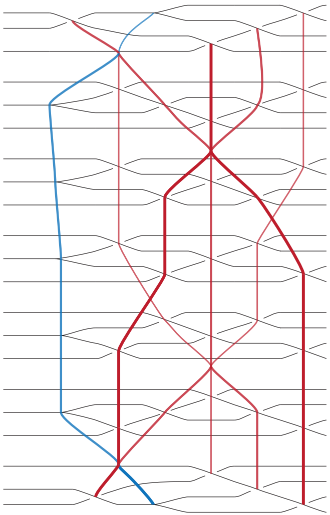


The IIY-move

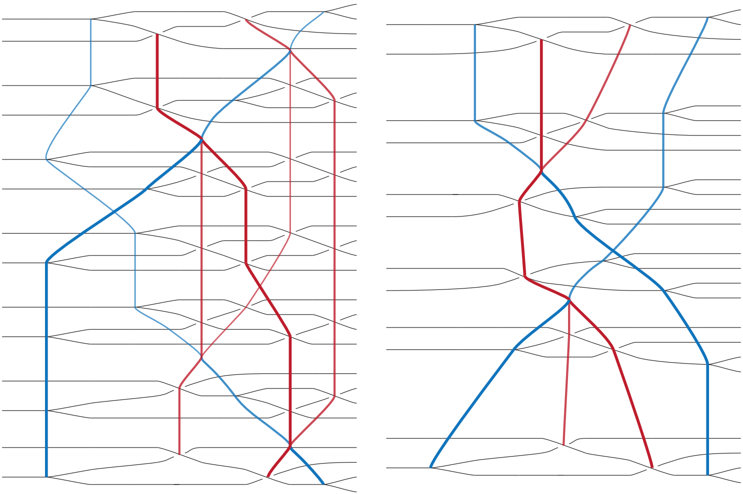


The IYI-move

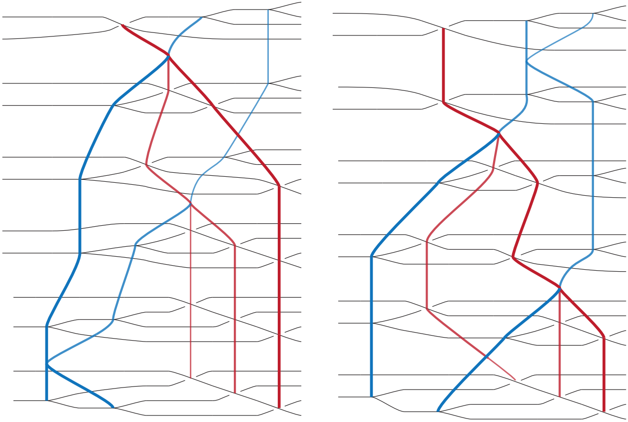




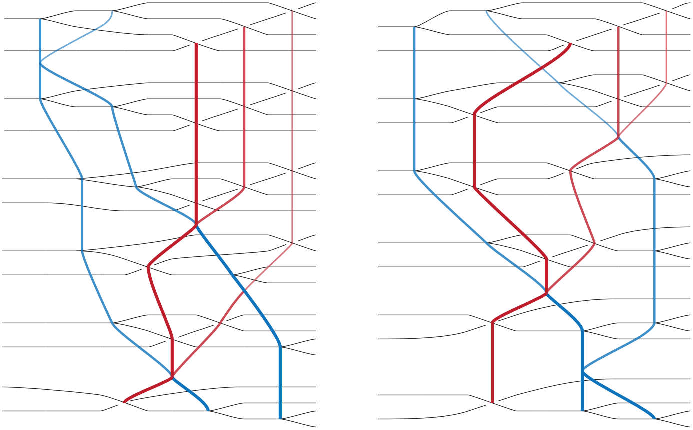
The YY-move



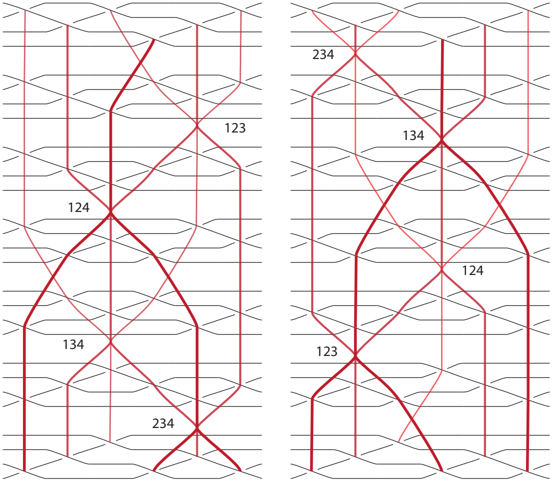
The YYI-move



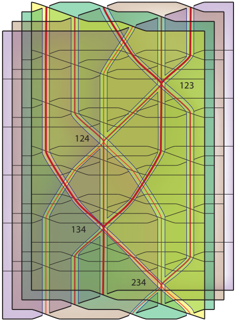
The IYY-move

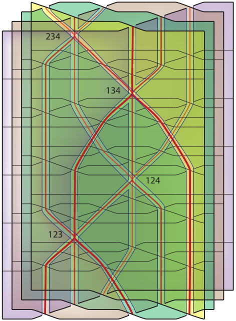


The tetrahedral-move

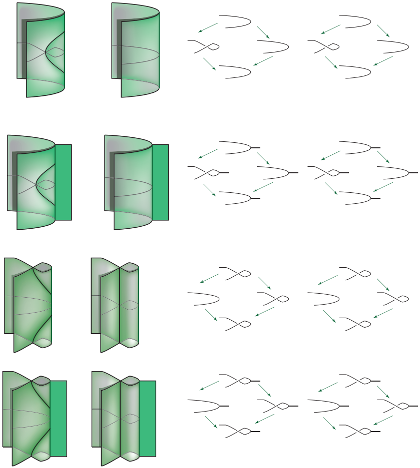


The tetrahedral-move

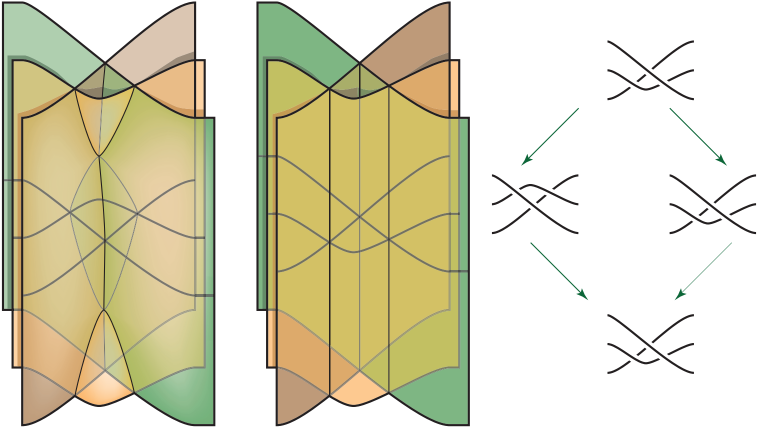




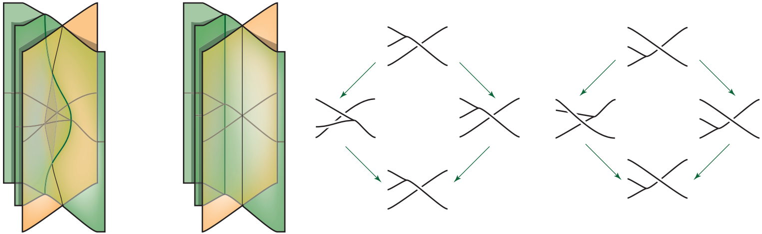
Critical points of the branch point set



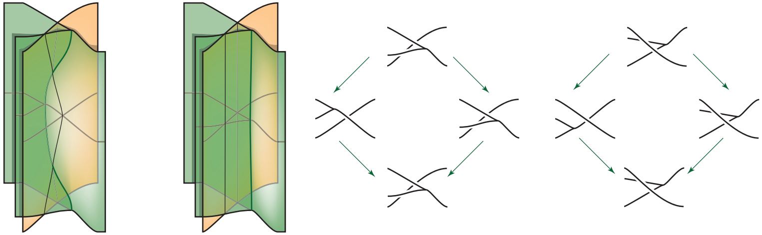
Critical points of the triple point set



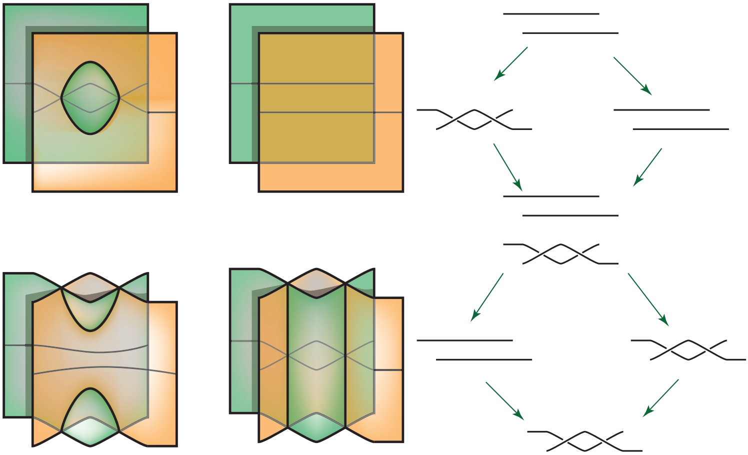
Critical points of the intersection set 1



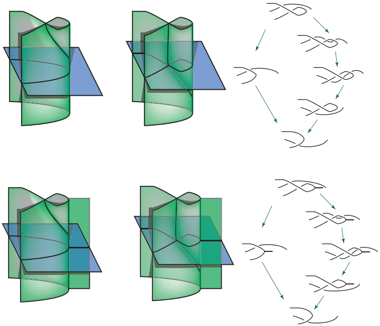
Critical points of the intersection set 2



Critical points of the double point set



Int. pts. b/2 branch/twist set and trnsvs. sheet



Not all 3-morphisms (or identities among 2 morphisms) are listed here.

Not all 3-morphisms (or identities among 2 morphisms) are listed here. Some missing moves are due to considerations on charts.

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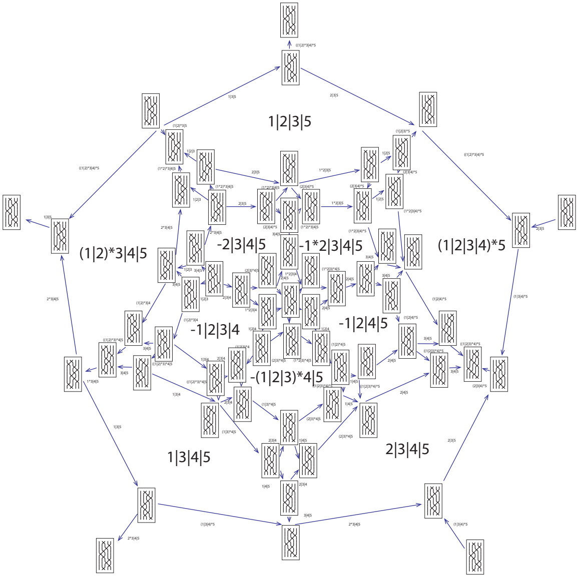
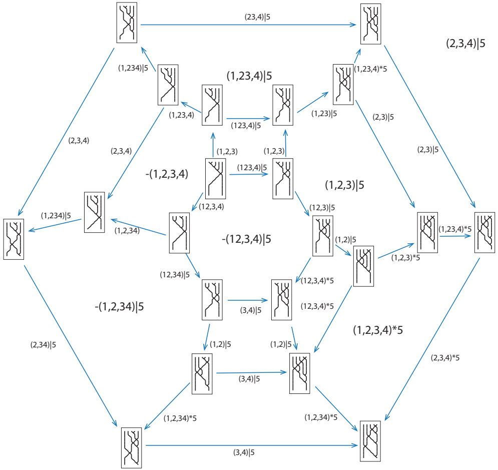
Not all 3-morphisms (or identities among 2 morphisms) are listed here. Some missing moves are due to considerations on charts. I just haven't drawn them yet. Others are not listed here for spacial considerations.

The analogues in one higher dimensions

1|2|3|4|5



1234|5



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(1,234)|5
(1,23,4)|5
(1,23,4)*5
(1,23,4)
(123,4)|5
(1,23)|5
(2,3)|5
(2,3,4)
(2,3,4)
(1,2,3)
(1,2,3)
(2,3)|5
-(1,2,3,4)
(123,4)|5
(1,2,3)|5
(12,3,4)
(12,3)|5
(1,234)|5
(1,2,34)
-(12,3,4)|5
(1,2)|5
(1,23,4)*5
(1,2,3)*5
(12,34)|5
(12,3,4)*5
-(1,2,34)|5
(3,4)|5
(12,3,4)*5
(1,2,3,4)*5
(2,34)|5
(1,2)|5
(1,2)|5
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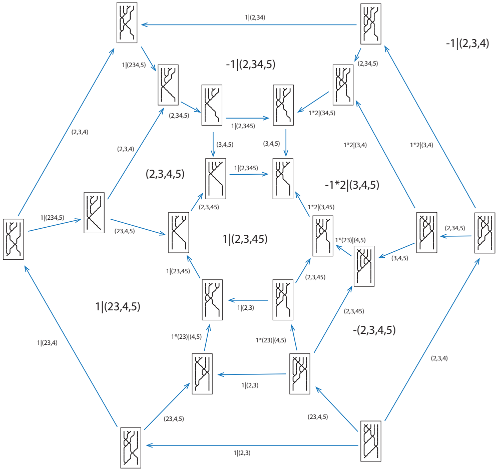
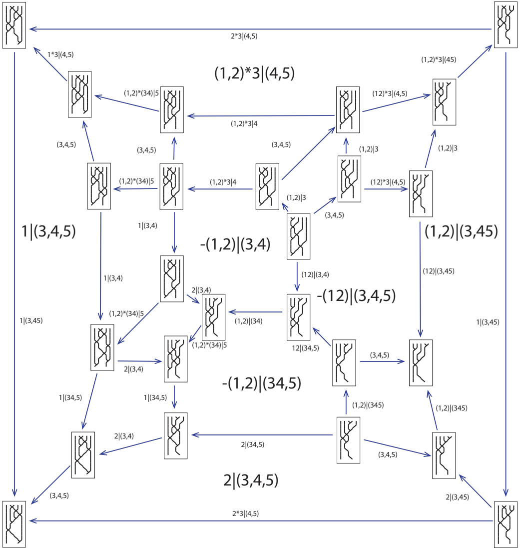
(3,4)|5
(1,2,34)*5
(1,2,34)*5
(3,4)|5

1|2345



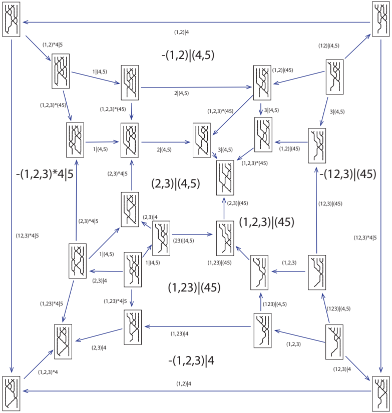
1|(2,34)
-1|(2,3,4)
1|(234,5)
-1|(2,34,5)
(2,34,5)
(2,34,5)
1|(2,345)
1*2|(34,5)
(2,3,4)
(2,3,4)
(3,4,5)
(3,4,5)
1*2|(3,4)
1*2|(3,4)
(2,3,4,5)
1|(2,345)
-1*2|(3,4,5)
1|(234,5)
(2,3,45)
1*2|(3,45)
(23,4,5)
1|(2,3,45)
1*(23)|(4,5)
(2,34,5)
(3,4,5)
1|(23,45)
(2,3,45)
1|(23,4,5)
1|(2,3)
(2,3,45)
-(2,3,4,5)
1*(23)|(4,5)
1*(23)|(4,5)
1|(23,4)
(2,3,4)
1|(2,3)
(23,4,5)
(23,4,5)
1|(2,3)

12|345

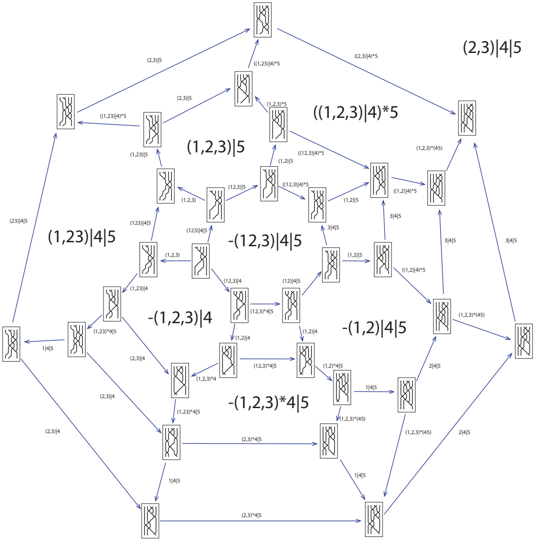


2*3|(4,5)
1*3|(4,5)
(1,2)*3|(4,5)
(1,2)*3|(45)
(1,2)*(34)|5
(12)*3|(4,5)
(1,2)*3|4
(3,4,5)
(3,4,5)
(3,4,5)
(1,2)|3
(1,2)|3
(1,2)*(34)|5
(1,2)*3|4
(12)*3|(4,5)
(1,2)|3
(3,4,5)
1|(3,4,5)
1|(3,4)
-(1,2)|(3,4)
(1,2)|(3,45)
1|(3,4)
(12)|(3,4)
(12)|(3,45)
2|(3,4)
-(12)|(3,4,5)
(1,2)*(34)|5
1|(3,45)
(1,2)|(34)
1|(3,45)
(1,2)*(34)|5
12|(34,5)
(3,4,5)
2|(3,4)
-(1,2)|(34,5)
1|(34,5)
1|(34,5)
(1,2)|(345)
(1,2)|(345)
2|(3,4)
2|(34,5)
(3,4,5)
2|(3,4,5)
(3,4,5)
2|(3,45)
2*3|(4,5)

123|45

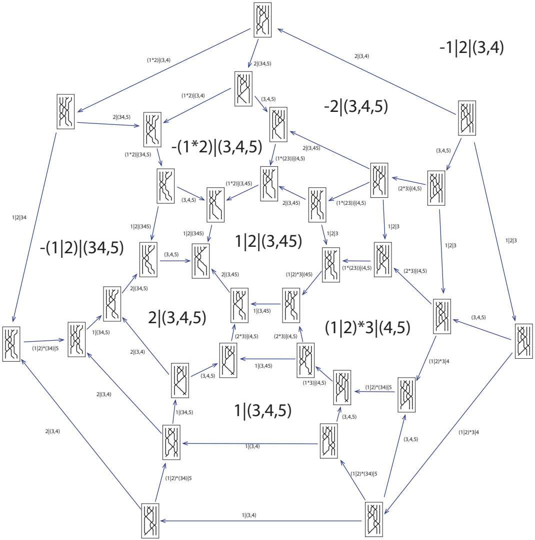

(1,2)|4
(1,2)*4|5
-(1,2)|(4,5)
(12)|(4,5)
1|(4,5)
(1,2)|(45)
2|(4,5)
(1,2,3)*(45)
(1,2,3)*(45)
(1,2,3)*(45)
3|(4,5)
3|(4,5)
1|(4,5)
2|(4,5)
3|(4,5)
(1,2)|(45)
(1,2,3)*(45)
-(1,2,3)*4|5
(2,3)*4|5
(2,3)|(4,5)
-(12,3)|(45)
(2,3)|(45)
(12,3)|(45)
(2,3)|4
(1,2,3)|(45)
(2,3)*4|5
(12,3)*4|5
(23)|(4,5)
(12,3)*4|5
1|(4,5)
1|(4,5)
(1,23)|(45)
(1,2,3)
(2,3)|4
(1,23)|(45)
(1,23)*4|5
(1,23)*4|5
(123)|(4,5)
(123)|(4,5)
(1,23)|4
(2,3)|4
(1,2,3)
(1,2,3)*4
-(1,2,3)|4
(12,3)|4
(1,2)|4

123|4|5

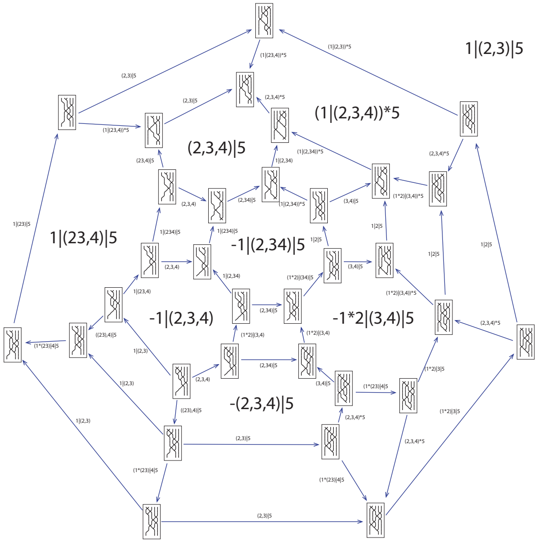


(2,3)|4|5
((1,2,3)|4)*5
(1,2,3)|5
(1,23)|4|5
-(12,3)|4|5
-(1,2,3)|4
-(1,2)|4|5
-(1,2,3)*4|5
(2,3)|5
((1,23)|4)*5
((2,3)|4)*5
(2,3)|5
((1,23)|4)*5
(1,2,3)*5
(1,23)|5
((12,3)|4)*5
(1,2)|5
(1,2,3)*(45)
(12,3)|5
((12,3)|4)*5
(1,2,3)
((1,2)|4)*5
(1,2)|5
(23)|4|5
(123)|4|5
(123)|4|5
3|4|5
3|4|5
3|4|5
3|4|5
(1,2,3)
(1,2)|5
((1,2)|4)*5
(1,23)|4
(12,3)|4
(12)|4|5
(12,3)*4|5
(1,2,3)*(45)
(1,23)*4|5
1|4|5
(1,2)|4
(1,2)|4
(2,3)|4
(12,3)*4|5
(1,2,3)*4
(1,2)*4|5
1|4|5
2|4|5
(2,3)|4
(1,23)*4|5
(1,2,3)*(45)
(2,3)|4
(2,3)*4|5
(1,2,3)*(45)
2|4|5
1|4|5
1|4|5
(2,3)*4|5

1|2|345


-1|2|(3,4)
-2|(3,4,5)
-(1*2)|(3,4,5)
-(1|2)|(34,5)
1|2|(3,45)
2|(3,4,5)
(1|2)*3|(4,5)
1|(3,4,5)
(1*2)|(3,4)
2|(34,5)
2|(3,4)
(3,4,5)
(1*2)|(34,5)
2|(3,45)
(1*(23))|(4,5)
(1*2)|(3,45)
(2*3)|(4,5)
1|2|34
1|2|(345)
1|2|3
(1|2)*3|(45)
1|(3,45)
1|(34,5)
(1|2)*(34)|5
(1*3)|(4,5)
(1|2)*3|4
1|(3,4)

1|234|5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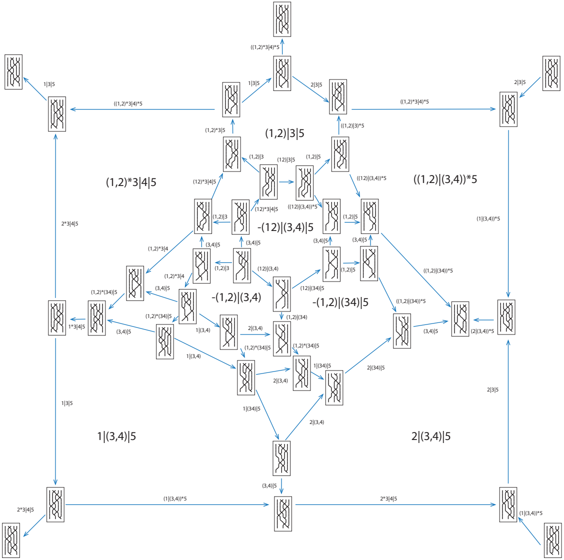
(1|(2,3,4))*5
(2,3,4)|5
1|(23,4)|5
-1|(2,34)|5
-1|(2,3,4)
-1*2|(3,4)|5
-(2,3,4)|5
(1|(23,4))*5
(1|(2,3))*5
(2,3)|5
(2,3,4)*5
(1|(2,34))*5
1|(2,34)
(23,4)|5
(2,3,4)
(2,34)|5
(3,4)|5
(1*2)|(3,4))*5
1|(234)|5
1|2|5
1|(23)|5
1|(23,4)
(1*2)|(34)|5
1|(2,3)
((23),4)|5
(1*2)|(3,4)
(1*(23)|4|5
(1*2)|3|5

12|34|5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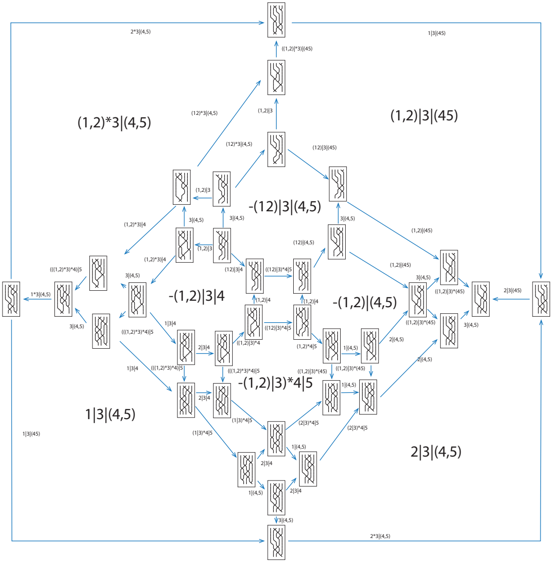
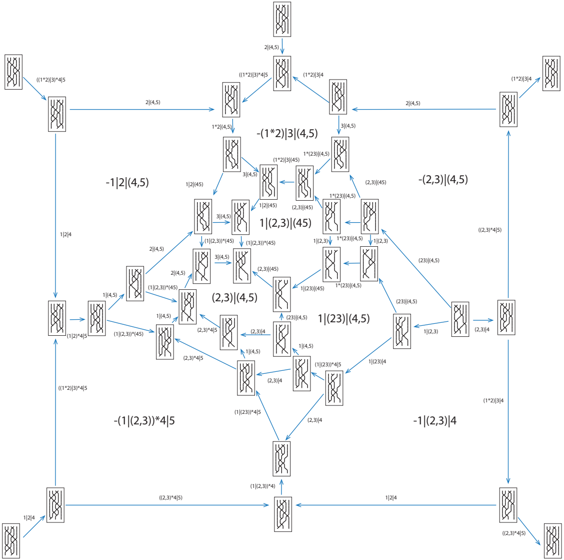
1|3|5
2|3|5
((1,2)*3|4)*5
((1,2)*3|4)*5
(1,2)*3|5
((1,2)|3)*5
(1,2)|3|5
(1,2)|3
(12)|3|5
(1,2)|5
(1,2)*3|4|5
(12)*3|4|5
((12)|(3,4))*5
((1,2)|(3,4))*5
(1,2)|3
(12)*3|4|5
((12)|(3,4))*5
(1,2)|5
2*3|4|5
-(12)|(3,4)|5
(1|(3,4))*5
(3,4)|5
(3,4)|5
(3,4)|5
(3,4)|5
(1,2)*3|4
(1,2)|3
(12)|(3,4)
(1,2)|5
((1,2)|(34))*5
(1,2)*3|4
(1,2)*(34)|5
(3,4)|5
-(1,2)|(3,4)
(12)|(34)|5
-(1,2)|(34)|5
((1,2)|(34))*5
1*3|4|5
(1,2)*(34)|5
(1,2)|(34)
(3,4)|5
1|(3,4)
2|(3,4)
(3,4)|5
(2|(3,4))*5
(1,2)*(34)|5
(1,2)*(34)|5
1|(3,4)
1|(34)|5
2|(34)|5
2|(3,4)
1|3|5
2|3|5
1|(34)|5
2|(3,4)
1|(3,4)|5
2|(3,4)|5
(3,4)|5
2*3|4|5
(1|(3,4))*5
2*3|4|5
(1|(3,4))*5

12|3|45

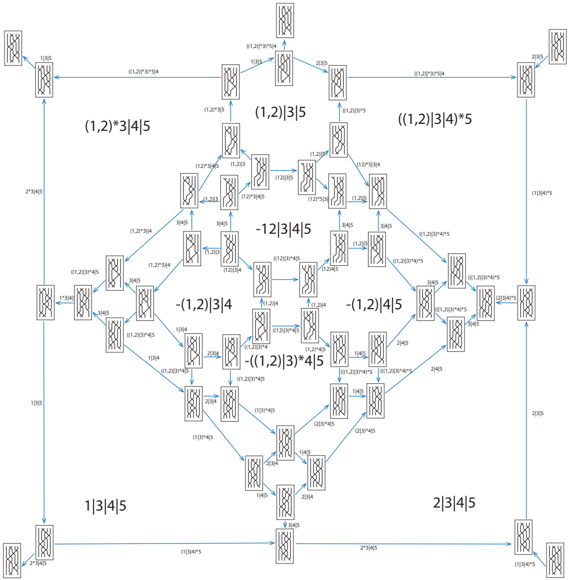


1|23|45

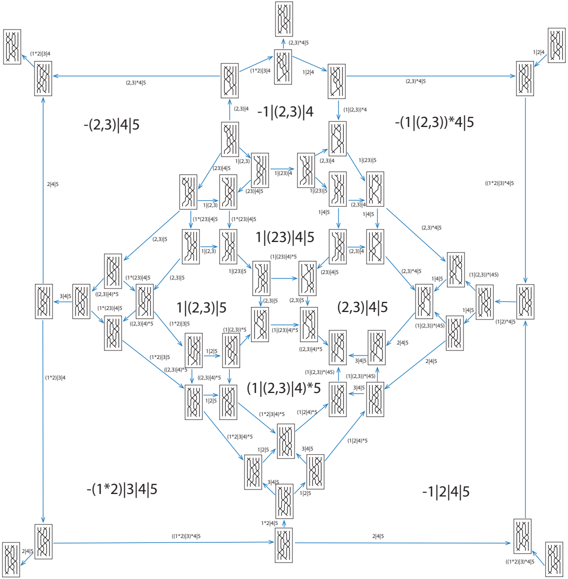


-(1*2)|3|(4,5)
-1|2|(4,5)
-(2,3)|(4,5)
1|(2,3)|(45)
(2,3)|(4,5)
1|(23)|(4,5)
-(1|(2,3))*4|5
-1|(2,3)|4
2|(4,5)
((1*2)|3)*4|5
(1*2)|3|4
1|2|4
(1|(2,3))*4
((2,3)*4|5)

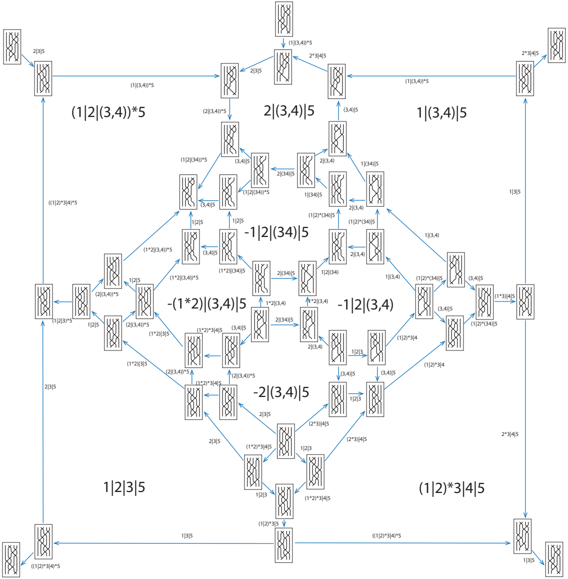
12|3|4|5



1|23|4|5



1|2|34|5



(1|(3,4))*5
2*3|4|5
2|3|5
(1|(3,4))*5
(1|(3,4))*5
2*3|4|5
(1|2|(3,4))*5
2|(3,4)|5
1|(3,4)|5
(2|(3,4))*5
(3,4)|5
(1|2|(34))*5
(3,4)|5
2|(34)|5
2|(3,4)
1|(34)|5
((1|2)*3|4)*5
(1|2|(34))*5
1|(34)|5
2|(3,4)
1|3|5
(3,4)|5
(1|2)*(34)|5
(1|2)*(34)|5
1|2|5
1|2|5
-1|2|(34)|5
(1*2|(3,4))*5
(3,4)|5
2|(3,4)
1|(3,4)
(1*2)|(34)|5
2|(34)|5
1|2|(34)
1|(3,4)
(1|2)*(34)|5
(3,4)|5
1|2|5
(1*2|(3,4))*5
(2|(3,4))*5
-(1*2)|(3,4)|5
1*2|(3,4)
1*2|(3,4)
-1|2|(3,4)
(1*3)|4|5
(1|2|3)*5
1|2|5
(2|(3,4))*5
2|(34)|5
(3,4)|5
(1|2)*(34)|5
(1*2)|3|5
(1*2)*3|4|5
(3,4)|5
2|(3,4)
(1|2)*3|4
1|2|3
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(2|(3,4))*5
(2|(3,4))*5
(1*2)*3|4|5
(3,4)|5
(3,4)|5
(1|2)*3|4
2|3|5
-2|(3,4)|5
1|2|3
2|3|5
2|3|5
(2*3)|4|5
(1*2)*3|4|5
1|2|3
(2*3)|4|5
2*3|4|5
1|2|3|5
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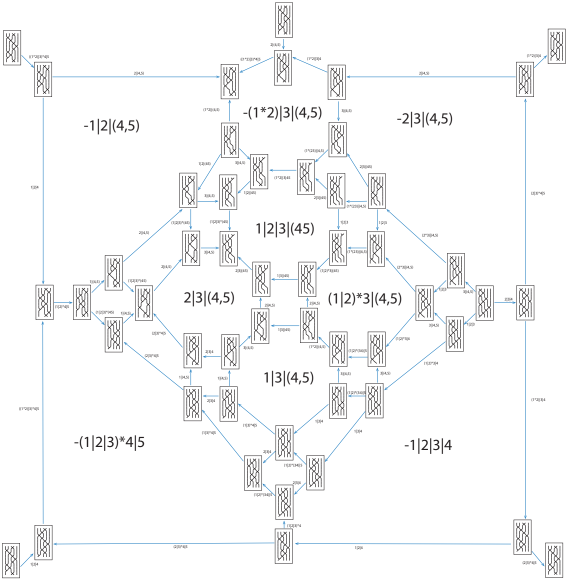
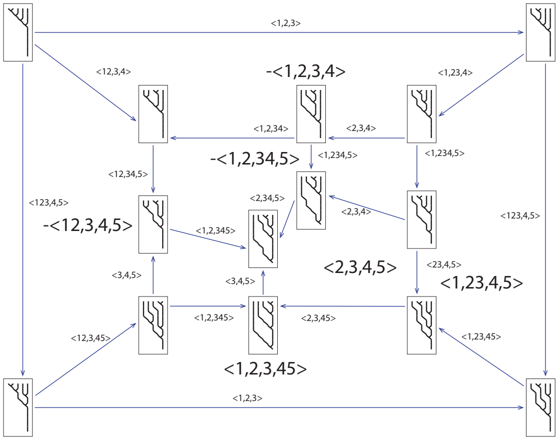
1|2|3
(1*2)*3|4|5
(1|2)*3|5
1|3|5
((1|2)*3|4)*5
((1|2)*3|4)*5
1|3|5

1|2|3|45



12345



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-<1,2,3,4>
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<1,2,34>
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-<1,2,34,5>
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<1,2,345>
<2,3,45>
<12,3,45>
<1,23,45>
<1,2,3,45>
<1,2,3>

Section 4

Section 4

The polytopes.

In this last section, I am going to review something that is

In this last section, I am going to review something that is (sort-of)

In this last section, I am going to review something that is (sort-of) well-known.

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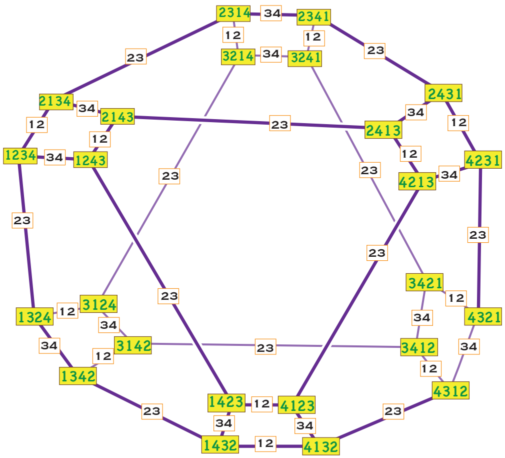
$\boldsymbol{i}(\boldsymbol{i}+1)\boldsymbol{i}(\boldsymbol{i}+1)\boldsymbol{i}(\boldsymbol{i}+1)$. *etc.*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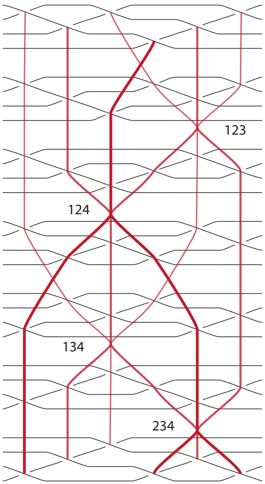
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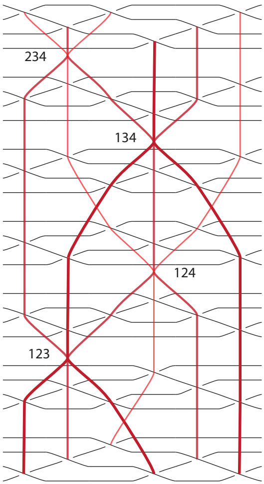
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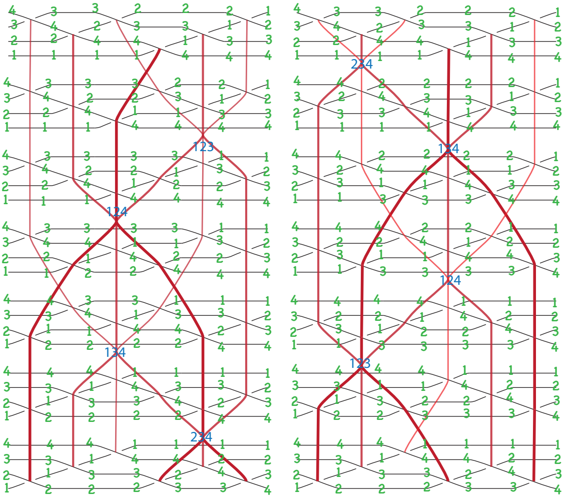

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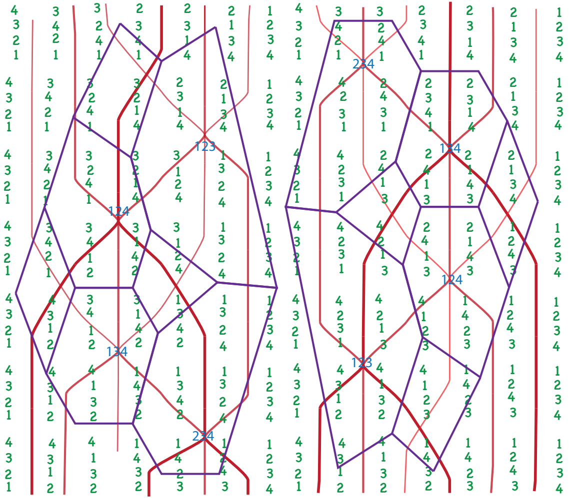
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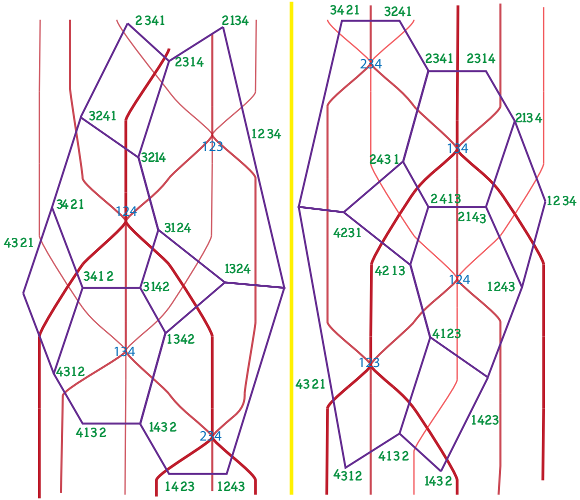
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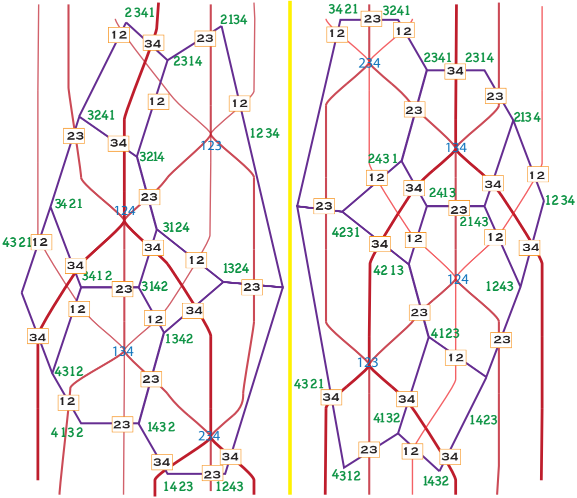
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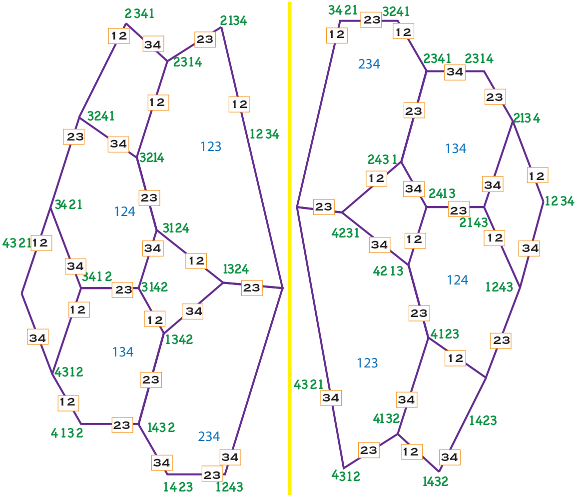

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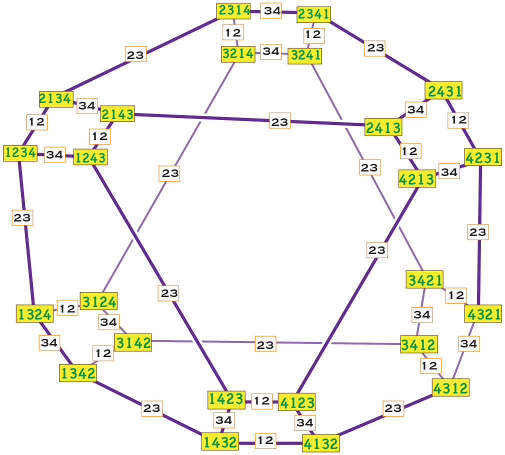

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In case of the IIY-move, I'll demonstrate the associated polytope.

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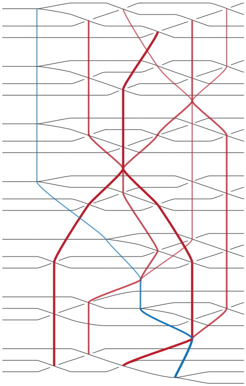
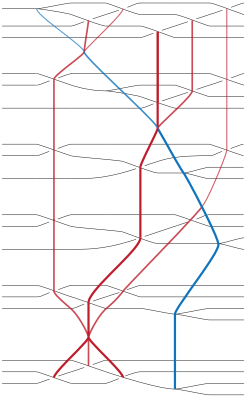
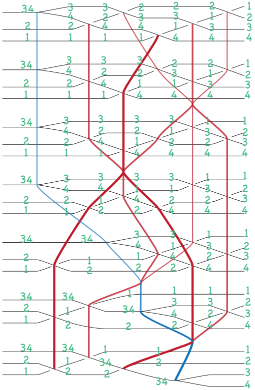
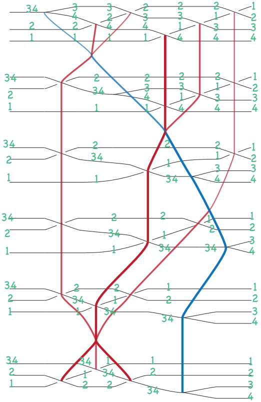
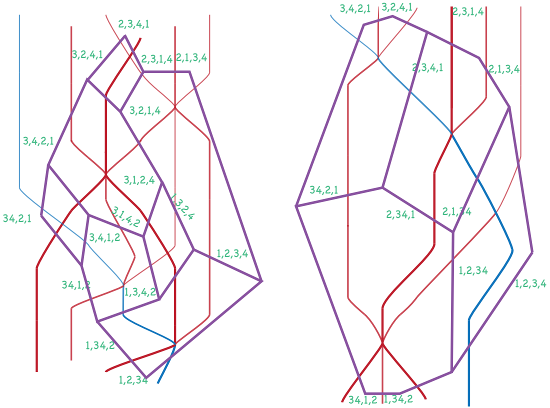
In fact, there is such a polytope for each of the YYY (1234), YYI 123|4, YY (12|34), YII 12|3|4 , IYY (1|234), IYI (1|23|4), IIY (1|2|34), and IIII (1|2|3|4)







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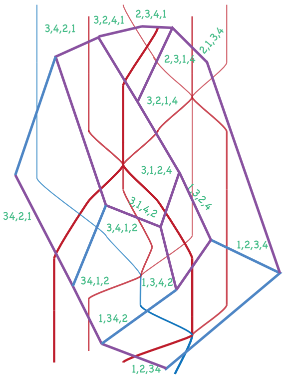
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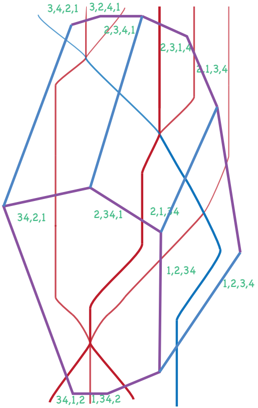
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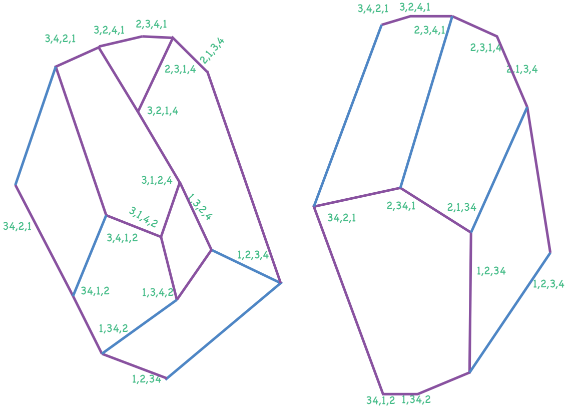
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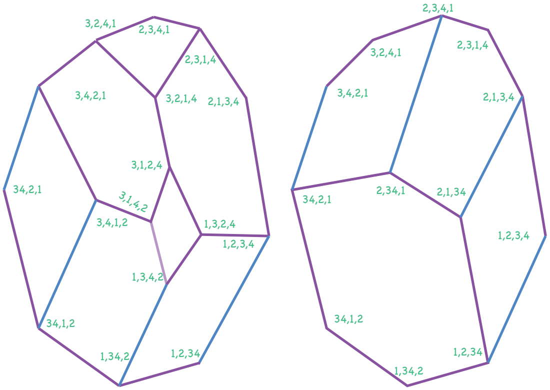
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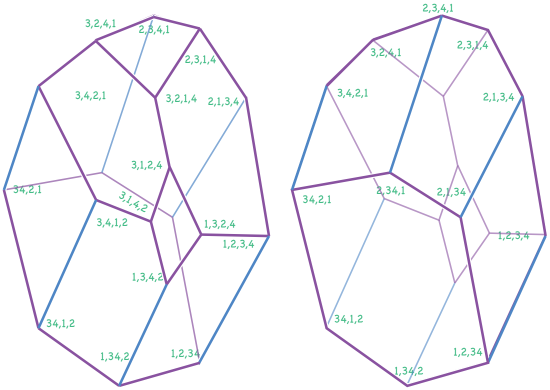
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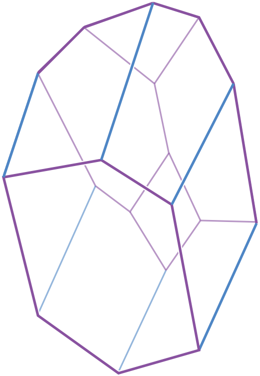

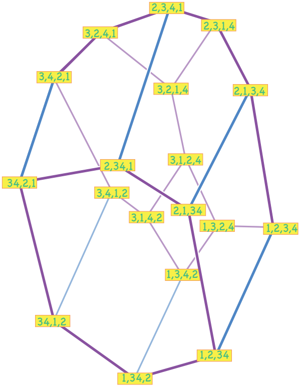
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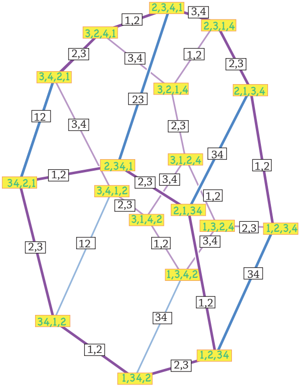
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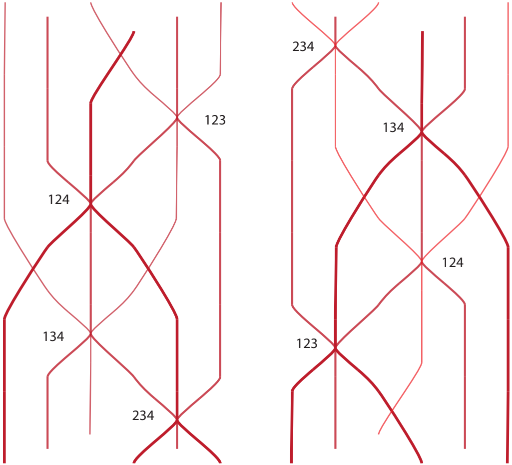
Clearly, graphical structure can be used to formulate a series of Abstract tensor equations.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In fact, one version of these equations is the Zamolodchikov tetrahedral equation.

$$S_{123}S_{124}S_{134}S_{234} = S_{234}S_{134}S_{124}S_{123}.$$

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I believe that these additional equations will also find physical applications as has the YBE.

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Finally, observe that the assoc. (co)hom thy. gives a nice parameter space in which to cast these equations.

Thanks

Thank you for your attention!