Lifts of holonomy representations and the volume of a link complement

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§ 1. Introduction

Mostow rigidity $\rightarrow$

a complete hyperbolic 3-mfd $M$ is geometrically determined by its fundamental group $\pi_1(M)$.

$\rightarrow$ the hyperbolic volume $\text{vol}(M)$ should theoretically be computable from a presentation of $\pi_1(M)$.

\[ \begin{align*}
\pi_1(M) & \longrightarrow \text{vol}(M) \\
\downarrow & \quad \downarrow \mathcal{A}_n \\
H_1(M) & \longrightarrow \text{Alexander polynomial}
\end{align*} \]
$2$. Background

Reidemeister torsion twisted by the ajoint representation

associated with an $\text{SL}(2, \mathbb{C})$-rep. [Porti]

\[ \downarrow \]

A relation between the non-acyclic $R$-torsion and a zero of the

acyclic $R$-torsion (Yamaguchi [Y, ’08]) (knot case)

\[ \downarrow \]

Twisted Alexander invariants and non-abelian $R$-torsion (Dubois &

Yamaguchi [DY]) (link case)
A volume formula of a closed hyperbolic 3-manifold using the Ray-Singer analytic torsion (by Müller ['12])

[Müller ’93]

A volume formula of a cusped hyperbolic 3-manifold using R-torsion (by Menal-Ferrer & Porti [MFP, ’14])

[Y]

A volume formula of a hyperbolic knot complement using the twisted Alexander invariant [G]

[DY]

Link case
§3. The Alexander polynomial

\[ K: \text{ a knot in } S^3, \quad E(K) := S^3 - \tilde{N}(K) \]

\[ \pi_1(E(K)) := G(K) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle : \text{ Wirtinger pre.} \]

\[ \alpha : G(K) \longrightarrow H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle : \text{ epimorphism} \]

\[ \mu(\text{meridian}) \longmapsto t \]

That is, \( \alpha(x_1) = \alpha(x_2) = \cdots = \alpha(x_k) = t. \)

\( \alpha \) induces the ring homomorphism between group rings over \( \mathbb{Z} \):

\[ \tilde{\alpha} : \mathbb{Z} G(K) \rightarrow \mathbb{Z}[t^{\pm 1}] \]

\( F_k = \langle x_1, x_2, \cdots, x_k \rangle : \text{ the free group of rank } k \)

\( \phi : F_k \rightarrow G(K) : \text{ epi.} \quad \longrightarrow \quad \tilde{\phi} : \mathbb{Z} F_k \rightarrow \mathbb{Z} G \)
\[ \Phi : \tilde{\alpha} \circ \tilde{\phi} : \mathbb{Z}F_k \to \mathbb{Z}[t^{\pm 1}] : \text{ring homo.} \]

\[ M := \Phi \left( \frac{\partial r_i}{\partial x_j} \right) \quad (\in M_{k-1,k}(\mathbb{Z}[t^{\pm 1}])) : \text{the Alexander matrix,} \]

where \( \frac{\partial x_i}{\partial x_j} = \delta_{ij}, \quad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1}, \quad \frac{\partial (uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j} \)

(Fox’s free differential calculus)

**Example.** \( \frac{\partial}{\partial x} (xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}) \)

\[
= \frac{\partial x}{\partial x} + x \frac{\partial}{\partial x} (y^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}) \\
= 1 + x \left( \frac{\partial y^{-1}}{\partial x} + y^{-1} \frac{\partial}{\partial x} (x^{-1}yxy^{-1}xyx^{-1}y^{-1}) \right) \\
= 1 + xy^{-1} \left( \frac{\partial x^{-1}}{\partial x} + x^{-1} \frac{\partial}{\partial x} (yxy^{-1}xyx^{-1}y^{-1}) \right) \\
= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1} \frac{\partial}{\partial x} (yxy^{-1}xyx^{-1}y^{-1}) = \cdots =
\]
\[
1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}
\]
\[
\tilde{\alpha} \rightarrow 1 - t^{-1} + 1 + 1 - t = -\frac{1}{t} + 3 - t
\]

\(M_\ell\): the sub matrix of \(M\) deleting \(\ell\)-column

**Definition.** The Alexander polynomial: \(\Delta_K(t) = \det M_\ell\).

**Example.** \(K = 4_1\) (figure 8 knot),

\[
G(K) = \langle x, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle
\]

\[
\Delta_K(t) = \det \left( -\frac{1}{t} + 3 - t \right) = -\frac{1}{t} + 3 - t
\]
§4. The twisted Alexander invariant

\( \rho : G(K) \rightarrow \text{SL}(n, \mathbb{C}) : \text{rep.} \)

extend by linearity

\( \tilde{\rho} : \mathbb{Z}G(K) \rightarrow M(n, \mathbb{C}) \)

\( \tilde{\rho} \otimes \bar{\alpha} : \mathbb{Z}G(K) \rightarrow M(n, \mathbb{C}[t^{\pm 1}]) : \text{tensor prod., ring homo.} \)

\( \Phi : (\tilde{\rho} \otimes \bar{\alpha}) \circ \bar{\phi} : \mathbb{Z}F_k \rightarrow M(n, \mathbb{C}[t^{\pm 1}]) : \text{ring homo.} \)

\[ M := \Phi \left( \frac{\partial r_i}{\partial x_j} \right) (\in M_{n(k-1), nk}(\mathbb{C}[t^{\pm 1}])) : \]

The Alexander matrix associated with \( \rho \)

\( M_\ell \): the sub matrix of \( M \) deleting ‘\( \ell \)’-column
**Definition.** The twisted Alexander invariant:

\[
\Delta_{K,\rho}(t) = \frac{\det M_\ell}{\det \Phi(x_\ell - 1)}
\]

**Example.** \(K = 4_1\) (figure 8 knot), \(\exists \rho : G(K) \to \text{SL}(2, \mathbb{C})\) s.t.

\[
\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: X, \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} =: Y, \text{where } u^2 + u + 1 = 0.
\]

\[
\frac{\partial r}{\partial x} = 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}
\]

\[
\Phi\left(\frac{\partial r}{\partial x}\right) = I - \frac{1}{t}XY^{-1}X^{-1} + XY^{-1}X^{-1}Y + XY^{-1}X^{-1}YXY^{-1}
\]

\[- tXY^{-1}X^{-1}YXY^{-1}XYX^{-1} \]

\[
\Phi(y - 1) = tY - I
\]

\[
\Delta_{K,\rho}(t) = \frac{\det \Phi\left(\frac{\partial r}{\partial x}\right)}{\det \Phi(y - 1)} = \frac{1/t^2(t - 1)^2(t^2 - 4t + 1)}{(t - 1)^2} \equiv t^2 - 4t + 1
\]
$K$: a hyperbolic knot in the 3-sphere.

$\rho_n^\pm : G(\bar{K}) \xrightarrow{\text{Hol.}} \text{PSL}(2, \mathbb{C}) \xrightarrow{\pm} \text{SL}(2, \mathbb{C}) \xrightarrow{\text{irr.} \sigma_n} \text{SL}(n, \mathbb{C})$

$\Delta_{K, \rho_n^\pm}(t)$: the twisted Alexander invariant

Set $A_{K, 2k}^\pm(t) := \frac{\Delta_{K, \rho_{2k}^\pm}(t)}{\Delta_{K, \rho_2^\pm}(t)}$ and $A_{K, 2k+1}(t) := \frac{\Delta_{K, \rho_{2k+1}}(t)}{\Delta_{K, \rho_3}(t)}$

$(\Delta_{K, \rho_{2k+1}^+}(t) = \Delta_{K, \rho_{2k+1}^-}(t))$

**Theorem [G].**

$$\lim_{k \to \infty} \log \left| \frac{A_{K, 2k+1}(1)}{(2k+1)^2} \right| = \lim_{k \to \infty} \log \left| \frac{A_{K, 2k}^\pm(1)}{(2k)^2} \right| = \frac{\text{Vol}(K)}{4\pi}$$
§ 5. Lifts of the holonomy representation

\( M \): an oriented, complete, hyperbolic 3-manifolds of finite volume with \( \partial M \cong T_1^2 \cup \cdots \cup T_b^2 \).

\( \text{Hol}_M : \pi_1(M, p) \rightarrow \text{Isom}^+ \mathbb{H}^3 \cong \text{PSL}(2, \mathbb{C}) (= \text{SL}(2, \mathbb{C})/\{ \pm 1 \}) \)

\( \eta \): a lift of \( \text{Hol}_M \) to \( \text{SL}(2, \mathbb{C}) \). Thus we have a map:

\[ \text{Hol}_{(M, \eta)} : \pi_1(M, p) \rightarrow \text{SL}(2, \mathbb{C}). \]

Definition. (1) \( \eta \) is positive on \( T_i^2 \) if for all \( \gamma \in \pi_1(T_i^2, p) \), we have

\[ \text{trace} \ \text{Hol}_{(M, \eta)}(\gamma) = +2. \]

(2) A lift \( \eta \) is acyclic if \( \eta \) is non-positive on each \( T_i^2 \).
Proposition [MFP]. $M_\gamma$: the mfd obtained by a Dehn filling along $\gamma$. A lift $\eta$ of $\text{Hol}_M$ to $\text{SL}(2, \mathbb{C})$ extends to a lift $\text{Hol}_{M_\gamma}$ to $\text{SL}(2, \mathbb{C})$ if and only if $\text{trace} \text{Hol}_{(M,\eta)}(\gamma) = -2$.

Proposition [MFP]. Assume that, for each boundary component $T_\ell^2$, the map $\iota_* : H_1(T_\ell^2; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z})$ induced by the inclusion $\iota$ has non trivial kernel, then all lifts of $\text{Hol}_M$ are non-positive on each $T_\ell^2$, i.e., all lifts are acyclic.

Example. a knot complement. ($\because \exists$ a Seifert surface), in general any 3-manifold with $\partial M = T^2$ (i.e., $b = 1$) [eg. Hempel 6.8].
§ 6. Results of Dubois & Yamaguchi

$M$: an oriented, complete, hyperbolic 3-manifolds of finite volume with $\partial \overline{M} \cong T_1^2 \cup \cdots \cup T_b^2$.

They investigated a relation between the R-torsion and the twisted Alexander invariants under the following condition:

(∗∗) \[ \text{the hom. } i_* : H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \text{ is onto} \]

and its restriction $(i|_{T_\ell^2})^*$ has rank one for all $\ell$.

(∗) $\iff$ (∗∗) $\iff$ (∗ ∗ ∗)

(∗ ∗ ∗) \[ M \text{ is the complement of a algebraically split link } L \]

(homologically trivial link) in a homology 3-sphere.

i.e., $L = K_1 \cup \cdots \cup K_b$ such that $\ell k(K_i, K_j) = 0$ for all $i, j$.  

13
Example. (algebraically split link) Every knot.

- Figure 8 (4₁)
- Whitehead link
- Borromean link
Suppose $\exists \varphi : \pi_1(M) \to \mathbb{Z}^n$ epi. & $\exists (a_1^{(\ell)}, \ldots, a_n^{(\ell)}) \in \mathbb{Z}^n_{>0}$ such that

$$
\varphi(\pi_1(T_2^{(\ell)})) = \langle t_1^{a_1^{(\ell)}} \cdots t_n^{a_n^{(\ell)}} \rangle \quad \text{for all } T_2^{(\ell)}.
$$

**Theorem** [DY]. Suppose ($**$). Then, we have:

$$
\lim_{t_1, \ldots, t_n \to 1} \frac{\Delta_{M, \varphi \otimes \text{Ad} \circ \rho_2}(t_1, \ldots, t_n)}{\prod_{\ell=1}^b (t_1^{a_1^{(\ell)}} \cdots t_n^{a_n^{(\ell)}} - 1)} = (-1)^b \cdot \mathbb{T}(M; \text{Ad} \circ \rho_2; \{\lambda_\ell\}),
$$

where $\lambda_\ell$ is a longitudinal basis on $T_2^{(\ell)}$. 
§ 7. A volume formula of a link complement

**Proposition.**[MFP’12] Let $M$ be a hyperbolic mfd with $b$ cusps. Then,

\[
\dim_{\mathbb{C}} H_0(M; \rho_n) = 0, \\
\dim_{\mathbb{C}} H_1(M; \rho_n) = a, \\
\dim_{\mathbb{C}} H_2(M; \rho_n) = a,
\]

where $a = b$ if $n$ is odd, and $a = \#$ of cusps for which the lift of the holonomy is positive if $n$ is even,

i.e., $\eta$ is acyclic $\Rightarrow \dim_{\mathbb{C}} H_i(M; \rho_n) = 0$ when $n$ is even.
Proposition. [MFP’14] Let $G_\ell(< \pi_1(M))$ be some fixed realization of $\pi_1(T^2_\ell)$ as a subgroup of $\pi_1(M)$. For each $T^2_\ell$ choose a non-trivial cycle $\theta_\ell \in H_1(T^2_\ell; \mathbb{Z})$, and non-trivial vector $w_\ell \in V_n$ fixed by $\rho_n(G_\ell)$.

1. A basis of $H_1(M; \rho_n)$ is given by $i_{\ell\ast}([w_\ell \otimes \theta_\ell])$, ($\ell = 1, \ldots, b$).

2. A basis of $H_2(M; \rho_n)$ is given by $i_{\ell\ast}([w_\ell \otimes T^2_\ell])$, ($\ell = 1, \ldots, b$).

Set

$$\mathcal{T}_{2k+1}(M, \eta) := \frac{\mathbb{T}(M; \rho_{2k+1}; \{\theta_\ell\})}{\mathbb{T}(M; \rho_3; \{\theta_\ell\})},$$

$$\mathcal{T}_{2k}(M, \eta) := \frac{\mathbb{T}(M; \rho_{2k}; \{\theta_\ell\})}{\mathbb{T}(M; \rho_2; \{\theta_\ell\})}.$$
Theorem [MFP’14]. (1) For any $\eta$,
\[
\lim_{k \to \infty} \frac{\log |T_{2k+1}(M, \eta)|}{(2k + 1)^2} = \frac{\text{Vol}(M)}{4\pi}
\]

(2) If $\eta$ is acyclic, then
\[
\lim_{k \to \infty} \frac{\log |T_{2k}(M, \eta)|}{(2k)^2} = \frac{\text{Vol}(M)}{4\pi}
\]

Applying the idea of the proof of Dubois & Yamaguchi’s Theorem, we have:

Theorem[G]. Let $L$ be an algebraically split link. Then,
\[
\lim_{k \to \infty} \frac{\log |A_{L,2k+1}(1, \ldots, 1)|}{(2k + 1)^2} = \lim_{k \to \infty} \frac{\log |A_{L,2k}(1, \ldots, 1)|}{(2k)^2} = \frac{\text{Vol}(L)}{4\pi}
\]
§8. The irreducible representation $\sigma_n$ of $\text{SL}(2, \mathbb{C})$

$V_n$: the vector space of 2-variables homogeneous polynomials on $\mathbb{C}$ with degree $n - 1$, i.e.,

$$V_n = \{ a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1} \mid a_1, \ldots, a_n \in \mathbb{C} \}$$

$$= \text{span}_{\mathbb{C}} \langle x^{n-1}, x^{n-2} y, x^{n-3} y^2, \ldots, xy^{n-2}, y^{n-1} \rangle$$

The action of $A \in \text{SL}(2, \mathbb{C})$ is expressed as:

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}) \quad \text{for} \ p \begin{pmatrix} x \\ y \end{pmatrix} \in V_n$$

$(V_n, \sigma_n)$: the rep. given by this action of $\text{SL}(2, \mathbb{C})$ where $\sigma_n$ means the homomorphism from $\text{SL}(2, \mathbb{C})$ to $\text{GL}(V_n)$.
(The fact is, $\sigma_n(A) \in \text{SL}(n, \mathbb{C})$.)

**Theorem.** Every irreducible $n$-dim. representation of $\text{SL}(2, \mathbb{C})$ is unique and equivalent to $(V_n, \sigma_n)$. 
Example. Set $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (\(\in \text{SL}(2, \mathbb{C})\)). \(X^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\).

\[
X^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ y \end{pmatrix}
\]

\[
p(X^{-1} \begin{pmatrix} x \\ y \end{pmatrix}) = p \begin{pmatrix} x - y \\ y \end{pmatrix}
\]

\[
(x - y)^2 = 1x^2 - 2xy + 1y^2, \ (x - y)y = 1xy - y^2, \ y^2 = 1y^2
\]

\[
\sigma_3(X) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^T
\]
\[(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3,\]
\[(x - y)^2y = x^2y - 2xy^2 + y^3,\]
\[(x - y)y^2 = xy^2 - y^3,\]
\[y^3 = y^3\]

\[\sigma_4(X) = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T\]

In the same way, we have:
\[ \sigma_n(X) = \begin{pmatrix}
1 & -n-1C_1 & n-1C_2 & \cdots & (-1)^{n-2}n-1C_{n-2} & (-1)^{n-1} \\
0 & 1 & -n-2C_1 & \cdots & (-1)^{n-3}n-2C_{n-3} & (-1)^{n-2} \\
0 & 0 & 1 & \cdots & (-1)^{n-4}n-3C_{n-4} & (-1)^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & \cdots & 1 & -1 \\
0 & 0 & \cdots & \cdots & 0 & 1 
\end{pmatrix}^T \]

**Example.** \( K = 4_1, \ G(K) = \langle x, y | xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle. \)

\[ \rho(x) = \begin{pmatrix} 1 & 1 \\
0 & 1 \end{pmatrix} = X, \quad \rho(y) = \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix} = Y \]

: holonomy rep. where \((u^2 + u + 1 = 0)\).
\[ \sigma_n(Y) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ u & 1 & 0 & \cdots & 0 \\ u^2 & 2u & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u^{n-2} & n-2C_1u^{n-3} & \cdots & 1 & 0 \\ u^{n-1} & n-1C_1u^{n-2} & n-1C_2u^{n-3} & \cdots & n-1C_{n-2}u & 1 \end{pmatrix}^T \]
§9. Experimentation using a computer \( K = 4_1, \text{Vol}(K) = 2.0298832 \cdots \)

\[
\text{Vol}(K) = 4\pi \lim_{k \to \infty} \frac{\log |A_{K,2k}(1)|}{(2k)^2}
\]

\[
\Delta_{K,\rho_2}(t) = \frac{1/t^2(t - 1)^2(t^2 - 4t + 1)}{(t - 1)^2} = t^2 - 4t + 1
\]

\[
\Delta_{K,\rho_4}(t) = \frac{1}{t^4}(t^2 - 4t + 1)^2 = (t^2 - 4t + 1)^2
\]

\[
A_{K,4}(t) = \frac{\Delta_{K,\rho_4}(t)}{\Delta_{K,\rho_2}(t)} = \frac{(t^2 - 4t + 1)^2}{t^2 - 4t + 1} = t^2 - 4t + 1
\]

\[
4\pi \frac{\log |A_{K,4}(1)|}{4^2} = \frac{\pi \log 2}{4} \approx 0.54440 \cdots \]
\[ \text{Vol}(K) = 4\pi \lim_{k \to \infty} \log |A_{K,2k+1}(1)| \]

\[ \Delta_{K,\rho_3}(t) = -\frac{1}{t^3} (t - 1)(t^2 - 5t + 1) \]

\[ \Delta_{K,\rho_5}(t) = -\frac{1}{t^5} (t - 1)(t^4 - 9t^3 + 44t^2 - 9t + 1) \]

\[ \Delta_{K,\rho_3}(t) = (t - 1)(t^4 - 9t^3 + 44t^2 - 9t + 1) \]

\[ A_{K,5}(t) = \frac{\Delta_{K,\rho_5}(t)}{\Delta_{K,\rho_3}(t)} = \frac{t^4 - 9t^3 + 44t^2 - 9t + 1}{t^2 - 5t + 1} \]

\[ 4\pi \frac{\log |A_{K,5}(1)|}{5^2} \approx 1.12273 \cdots \]
\[ n(\text{even}) \quad \frac{4\pi \log |\mathcal{A}_{K,n}^+(1)|}{n^2} \quad \frac{4\pi \log |\mathcal{A}_{K,n}^-(1)|}{n^2} \quad n(\text{odd}) \quad \frac{4\pi \log |\mathcal{A}_{K,n}(1)|}{n^2} \]

| \( n(\text{even}) \) | \( \frac{4\pi \log |\mathcal{A}_{K,n}^+(1)|}{n^2} \) | \( \frac{4\pi \log |\mathcal{A}_{K,n}^-(1)|}{n^2} \) | \( n(\text{odd}) \) | \( \frac{4\pi \log |\mathcal{A}_{K,n}(1)|}{n^2} \) |
|---|---|---|---|---|
| 4 | 0.54439 \ldots | 1.40724 \ldots | 5 | 1.12273 \ldots |
| 8 | 1.66441 \ldots | 1.84668 \ldots | 9 | 1.76436 \ldots |
| 12 | 1.86678 \ldots | 1.94781 \ldots | 13 | 1.90158 \ldots |
| 16 | 1.93822 \ldots | 1.98381 \ldots | 17 | 1.95494 \ldots |
| 20 | 1.97121 \ldots | 2.00039 \ldots | 21 | 1.98076 \ldots |
| 24 | 1.98914 \ldots | 2.00940 \ldots | 25 | 1.99522 \ldots |
| 28 | 1.99994 \ldots | 2.01483 \ldots | 29 | 2.00412 \ldots |
| 32 | 2.00696 \ldots | 2.01836 \ldots | 33 | 2.00999 \ldots |
§10. On Evaluation by Root of unity (in progress)

Theorem [DY’12]. $M_q$: $q$-fold cyclic covering of $M$, and $s = t^q$

$$\Delta_{M_q}^\hat{\rho \otimes \hat{\alpha}}(s) = \prod_{k=0}^{q-1} \Delta_M^\rho \otimes \alpha(e^{2\pi k \sqrt{-1}/q_t})$$

Corollary.

$$\lim_{k \to \infty} \frac{\log |\mathcal{A}_{K,2k}(-1)|}{(2k)^2} = \frac{\text{Vol}(K)}{4\pi}.$$ (∵)

Suppose $t = 1$, $q = 2$, then $s = 1$.

$$\Delta_{M_2}^\hat{\rho \otimes \hat{\alpha}}(1) = \Delta_{M}^\rho \otimes \alpha(1) \cdot \Delta_{M}^\rho \otimes \alpha(-1)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$2\text{Vol}(M) \quad \text{Vol}(M) \quad \text{Vol}(M)$$
3-rd root of unity $\Rightarrow$ OK

4-th root of unity $\Rightarrow$ OK

5-th root of unity $\Rightarrow$ ????
Thank you very much for your attention!