Presentations of (immersed) surface-knots by marked graph diagrams

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A **surface-link** is the image $\mathcal{L}$ of the disjoint union of surfaces in the 4-space $\mathbb{R}^4$ by a smooth embedding. When it is connected, it is called a **surface-knot**.

When a surface-link is oriented, we call it an **oriented surface-link**.

Two surface-links $\mathcal{L}$ and $\mathcal{L}'$ are **equivalent** if there is an orientation preserving homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(\mathcal{L}) = \mathcal{L}'$ orientedly.
Normal forms of surface-links

Theorem (Kawauchi-Shibuya -Suzuki)

For any surface-link $\mathcal{L}$, there is a surface-link $\tilde{\mathcal{L}} \subset \mathbb{R}^3[-1,1]$ satisfying the following conditions:

(0) $\tilde{\mathcal{L}}$ is equivalent to $\mathcal{L}$ and has only finitely many critical points, all of which are elementary.

(1) All maximal points of $\tilde{\mathcal{L}}$ are in $\mathbb{R}^3[1]$.

(2) All minimal points of $\tilde{\mathcal{L}}$ are in $\mathbb{R}^3[-1]$.

(3) All saddle points of $\tilde{\mathcal{L}}$ are in $\mathbb{R}^3[0]$.

We call $\tilde{\mathcal{L}}$ a normal form of $\mathcal{L}$.
A marked graph diagram is a diagram of a finite spatial regular graph with 4-valent rigid vertices such that each vertex has a marker.

An orientation of a marked graph diagram $D$ is a choice of an orientation for each edge of $D$ in such a way that every rigid vertex in $D$ looks like $\begin{array}{c} \uparrow \swarrow \downarrow \nwarrow \end{array}$ or $\begin{array}{c} \nwarrow \downarrow \uparrow \swarrow \end{array}$. A marked graph diagram is said to be orientable if it admits an orientation. Otherwise, it is said to be nonorientable.
A marked graph diagram $D$ is **admissible** if both resolutions $L_+(D)$ and $L_-(D)$ are trivial links.

![Graph Diagrams](image)

**Theorem (Kawauchi-Shibuya-Suzuki, Yoshikawa)**

1. For an admissible marked graph diagram $D$, there is a surface-link $\mathcal{L}$ represented by $D$.

2. Let $\mathcal{L}$ be a surface-link. Then there is an admissible marked graph diagram $D$ such that $\mathcal{L}$ is represented by $D$. 

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Example
Let $\mathcal{L}$ be a surface-link, and $\tilde{\mathcal{L}}$ a normal form of $\mathcal{L}$. Then the cross-section $\tilde{\mathcal{L}} \cap \mathbb{R}^3[0]$ at $t = 0$ is a 4-valent graph in $\mathbb{R}^3[0]$.

We give a marker at each 4-valent vertex that indicates how the saddle point opens up above. Then the diagram $D$ of resulting marked graph represents the surface-link $\mathcal{L}$. We call $D$ a marked graph diagram of $\mathcal{L}$. 

![Diagram of marked graph diagrams](image)
Yoshikawa moves for marked graph diagrams of surface-links

Theorem (Swenton, Kearton-Kurlin, Yoshikawa)

Two surface-links in $\mathbb{R}^4$ are equivalent if and only if their marked graph diagrams can be transformed into each other by a finite sequence of 8 types of moves, called the Yoshikawa moves.
Presentations of (immersed) surface-knots by marked graph diagrams

Γ₁ : 

Γ₂ : 

Γ₃ : 

Γ₄ : 

Γ₄' : 

Γ₅ : 

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\( \Gamma_6 : \) 
\( \Gamma'_6 : \)

\( \Gamma_7 : \) 
\( \Gamma_8 : \)
An immersed surface-link is a closed surface generically immersed in $\mathbb{R}^4$. When $\mathcal{L}$ is connected, it is called an immersed surface-knot.

Two immersed surface-links $\mathcal{L}$ and $\mathcal{L}'$ are equivalent if there is an orientation preserving homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(\mathcal{L}) = \mathcal{L}'$ orientedly.

It is known that every double point singularity is constructed by a cone over a Hopf link.
Definition

A link $L$ is **H-trivial** if $L$ is a split union of a finite number of trivial knots and Hopf links.

\[
\begin{align*}
\text{\text{Diagram 1}} & \quad m \geq 0 \\
\text{\text{Diagram 2}} & \quad n \geq 0
\end{align*}
\]
Trivial knot cones $\hat{O}[a,b] \& \check{O}[a,b]$, and Hopf link cones $\hat{P}[a,b] \& \check{P}[a,b]$
H-trivial link cones $H \land [a, b]$ & $H \lor [a, b]$
Theorem (Kamada-Kawamura)

For any immersed surface-link \( L \), there is an immersed surface-link \( \tilde{L} \subset \mathbb{R}^3[-2,2] \) satisfying the following conditions:

(0) \( \tilde{L} \) is equivalent to \( L \) and has only finitely many critical points, all of which are elementary.

(1) The cross-sections \( H = \tilde{L} \cap \mathbb{R}^3[1] \) and \( H' = \tilde{L} \cap \mathbb{R}^3[-1] \) of \( \tilde{L} \) are H-trivial links.

(2) All maximal points of \( \tilde{L} \) are in \( \mathbb{R}^3[2] \).

(3) All minimal points of \( \tilde{L} \) are in \( \mathbb{R}^3[-2] \).

(4) All saddle points of \( \tilde{L} \) are in \( \mathbb{R}^3[0] \).

(5) \( \tilde{L} \cap \mathbb{R}^3[1,2] = H \wedge [1,2] \) and \( \tilde{L} \cap \mathbb{R}^3[-2,-1] = H' \vee [-2,-1] \).

We call \( \tilde{L} \) a normal form of \( L \).
A marked graph diagram $D$ is **H-admissible** if both resolutions $L_+(D)$ and $L_-(D)$ are H-trivial links.
Theorem (Kamada-Kawauchi-K.-Lee)

(1) For an H-admissible marked graph diagram $D$, there is an immersed surface-link $L$ represented by $D$.

(2) Let $L$ be an immersed surface-link. Then there is an H-admissible marked graph diagram $D$ such that $L$ is represented by $D$. 
Construction of immersed surface-links from H-admissible marked graph diagrams

\[ R^3[0] \subset R^3[-1,1] \]

- \(0 < t \leq 1\)
- \(t = 0\)
- \(-1 \leq t < 0\)
$t = 2$

$1 < t < 2$

$0 < t \leq 1$

$t = 0$

$-1 \leq t < 0$

$-2 < t < -1$

$t = -2$
Let $\mathcal{L}$ be an immersed surface-link, and $\tilde{\mathcal{L}}$ a normal form of $\mathcal{L}$. Then the cross-section $\tilde{\mathcal{L}} \cap \mathbb{R}^3[0]$ at $t = 0$ is a 4-valent graph in $\mathbb{R}^3[0]$.

We give a marker at each 4-valent vertex that indicates how the saddle point opens up above. Then the diagram $D$ of a resulting marked graph presents the surface-link $\mathcal{L}$. We call $D$ a marked graph diagram of $\mathcal{L}$.
Further moves for immersed surface-links

Definition

A crossing point \( p \) (in a marked graph diagram \( D \)) is an **upper singular point** if \( p \) is an unlinking crossing point of a Hopf link diagram in the resolution \( L_+(D) \), and a **lower singular point** if \( p \) is an unlinking crossing point in the resolution \( L_-(D) \), resp.

Example

\[ D \quad L_+(D) \quad D' \quad L_-(D') \]

\( p \) : upper singular point,  \( p' \) : lower singular point.
The following moves are new entries on marked graph diagrams.

\[ \Gamma_9 : \]

- In \( \Gamma_9 \), the component containing \( l^+ \) in \( L_+(D) \) is a trivial knot.
- In \( \Gamma_9 \), \( p \) is an upper singular point.
- In \( \Gamma'_9 \), the component containing \( l^- \) in \( L_-(D) \) is a trivial knot.
- In \( \Gamma'_9 \), \( p \) is a lower singular point.
The following move is a new entry on marked graph diagrams.

\[ \Gamma_{10} : \quad \text{Diagram 1} \iff \text{Diagram 2} \]

**Note**

Let \( D \) be an H-admissible marked graph diagram. Let \( h_+(D) \) and \( h_-(D) \) be the numbers of Hopf-links in \( L_+(D) \) and \( L_-(D) \), resp.

- The ordered pair \((h_+(D), h_-(D))\) is an invariant except \( \Gamma_{10} \).
- If \( D \) and \( D' \) are related by a single \( \Gamma_{10} \) move, then \((h_+(D'), h_-(D')) = (h_+(D) + \varepsilon, h_-(D) - \varepsilon)\) for \( \varepsilon \in \{1, -1\} \).
Definition

The generalized Yoshikawa moves for marked graph diagrams are the deformations $\Gamma_1, \ldots, \Gamma_8, \Gamma_9, \Gamma'_9,$ and $\Gamma_{10}$.

Theorem (Kamada-Kawauchi-K.-Lee)

Let $L$ and $L'$ be immersed surface-links, and $D$ and $D'$ their marked graph diagrams, resp. If $D$ and $D'$ are related by a finite sequence of generalized Yoshikawa moves, then $L$ and $L'$ are equivalent.
Sketch of Proof. The moves $\Gamma_9$ (or $\Gamma'_9$) can be generated by $\Omega_9$ (or $\Omega'_9$) and $\Gamma_2$, resp.

We need to show that if two marked graphs are related by $\Omega_9, \Omega'_9$, and $\Gamma_{10}$, then their immersed surface-links are equivalent.
\( \Omega_9 : \)
\[
\begin{array}{c}
\text{(a)} \\
\text{(b)}
\end{array}
\]

\( \Omega'_9 : \)
\[
\begin{array}{c}
\text{(a)} \\
\text{(b)}
\end{array}
\]

\( \Gamma_{10} : \)
\[
\begin{array}{c}
\text{(a)} \\
\text{(b)}
\end{array}
\]
The following marked graph diagrams $D$ and $D'$ are related by a finite sequence of generalized Yoshikawa moves.
$L_+(D)$  $L_-(D)$  $L_+(D')$  $L_-(D')$

H-admissibility
$\Gamma_9 : \quad p \quad \rightarrow \quad$
$\Gamma_4'$ :  

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{array} \]
$\Gamma_{10}: \quad \rightarrow \quad \rightarrow$
\[ \Gamma_1 : \quad \begin{array}{c} \infty \\ \end{array} \rightarrow \quad \begin{array}{c} \cup \\ \end{array} \]
\[ \Gamma_1 : \quad \longrightarrow \quad \longrightarrow \]
$\Gamma_4$:
\( \Gamma_4 : \) \hspace{1cm} \text{Diagram 1} \hspace{1cm} \rightarrow \hspace{1cm} \text{Diagram 2}
\[ \Gamma'_9 : \quad \text{Diagram} \quad \rightarrow \quad \text{Diagram} \]
Well-definedness of the move $\Gamma'_9$:
Well-definedness of the move $\Gamma'_9$:
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**Definition**

A **positive** (or **negative**) standard singular 2-knot, denoted by \( S(+) \) (or \( S(-) \)) is the immersed 2-knot of \( D \) (or \( D' \)), resp. An **unknotted immersed sphere** is defined to be the connected sum \( mS(+) \# nS(-) \) for \( m, n \in \mathbb{Z}_{\geq 0} \) with \( m + n > 0 \).

\[ \begin{align*}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{D}\quad \includegraphics[width=0.3\textwidth]{D'}
\end{array}
\end{align*} \]

**Definition**

A double point singularity \( p \) of an immersed 2-knot \( S \) is **inessential** if \( S \) is the connected sum of an immersed 2-knot and an unknotted immersed sphere such that \( p \) belongs to the unknotted immersed sphere. Otherwise, \( p \) is **essential**.
I answer the following question.

**Question**
For any integer $n \geq 1$, is there an immersed 2-knot with $n$ double point singularities every of which is essential?
I answer the following question.

Question
For any integer $n \geq 1$, is there an immersed 2-knot with $n$ double point singularities every of which is essential?

Yes. There are infinitely many immersed 2-knots with $n$ double point singularities every of which is essential.
Example
Example

$D$
The knot group is \(< x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15} | x_1 = x_2^{-1} x_3 x_2, x_2 = x_3^{-1} x_5 x_3, x_1 = x_3^{-1} x_4 x_3, x_2 = x_1^{-1} x_3 x_1, x_6 = x_2^{-1} x_1 x_2, x_6 = x_1^{-1} x_7 x_1, x_1 = x_7^{-1} x_8 x_7, x_2 = x_7^{-1} x_9 x_7, x_{10} = x_2^{-1} x_7 x_2, x_{10} = x_1^{-1} x_{11} x_1, x_1 = x_{11}^{-1} x_{12} x_{11}, x_2 = x_{11}^{-1} x_{13} x_{11}, x_{14} = x_2^{-1} x_{11} x_2, x_{14} = x_1^{-1} x_2 x_1, x_1 = x_2^{-1} x_{15} x_2 > .\)

The first elementary ideal \(\varepsilon(D)\) is \(< 1 - 2t, 4 - 3t > \) and it is equivalent to the ideal \(< 2t - 1, 5 > \). Since \(< 2t - 1, 5 > \) is not equivalent to the ideal \(< t - 2, 5 > \), it is non-symmetric. (\(: \mathbb{Z}_5[t, t^{-1}]\) is a principal ideal domain.)
We have $\varepsilon(D_n) = < 2t - 1, n + 2 >$, $\varepsilon(D'_n) = < 2t - 1, n - 1 >$. 
Denote the first Alexander module \( H_1(\tilde{E}(K)) \) of a 2-knot \( K \) by \( H(K) \). Let

\[
DH(K) = \{ x \in H(K) \mid \exists \{ \lambda_i \}_{1 \leq i \leq m} : \text{coprime (} m \geq 2 \text{) with } \lambda_i x = 0, \ \forall i \},
\]
called the annihilator \( \Lambda \)-submodule. The following lemma is used in our argument.

**Lemma**

If \( K \) is a 2-knot such that the dual \( \Lambda \)-module

\[
DH(K)^* = \text{hom}(DH(K), \mathbb{Q}/\mathbb{Z})
\]
is \( \Lambda \)-isomorphic to \( DH(K) \), then the first elementary ideal \( \varepsilon(K) \) is symmetric.
Lemma (Kawauchi-K.)

The following statements are equivalent:

1. The ideal \( < 2t - 1, m > \) is symmetric.
2. An integer \( m \) is \( \pm 2^r \) or \( \pm 2^r 3 \) for any integer \( r \geq 0 \).

Lemma (Kawauchi-K.)

There are infinitely many immersed 2-knots with one essential double point singularity.

**Sketch of Proof.** Let \( K_n \) and \( K'_n \) be immersed 2-knots represented by \( D_n \) and \( D'_n \), resp. Suppose that \( K_n = K \# S(\pm) \), where \( K \) is a 2-knot and \( S(\pm) = S(+) \) or \( S(-) \). Then the ideal \( \mathcal{E}(K_n) = < 2t - 1, n + 2 > \) is symmetric. There is a contradiction if \( n \) isn’t \( 2^{r+2} - 2 \) nor \( 2^r 3 - 2 \) \((r \geq 0)\). Hence \( K_n \) is an immersed 2-knot with essential singularity except that \( n \) is \( 2^{r+2} - 2 \) or \( 2^r 3 - 2 \) \((r \geq 0)\). So is \( K'_n \) except that \( n \) is 1, \( 2^r + 1 \) or \( 2^r 3 + 1 \) \((r \geq 0)\).
Theorem (Kawauchi-K.)

Let $K = nK_m^*$ be the connected sum of $n$ copies of an immersed 2-knot $K_m^*$ with one essential double point singularity whose first elementary ideal is $<2t-1,m>$ for any integer $m \geq 5$ without factors 2 and 3. Then $K$ gives infinitely many immersed 2-knots with $n$ double point singularities every of which is essential.
Thank you