Positive flow-spines and contact 3-manifolds

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Intelligence of Low-dimensional Topology

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RIMS

Main Theorem

M : a closed oriented 3-manifold

$\{\text{positive flow-spines of } M\} \twoheadrightarrow \mathrm{Cont}(M)/\text{isotopy}$

§ 1 Positive flow-spines

§ 2. Contact 3-manifolds

§ 3. Main results

§ 4. Tabulation

§ 1 Positive flow-spines

§ 2. Contact 3-manifolds

§ 3. Main results

§ 4. Tabulation

M : a closed oriented 3-manifold.

$\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$: a non-singular flow on M

(generated by a non-singular vector field X on M).

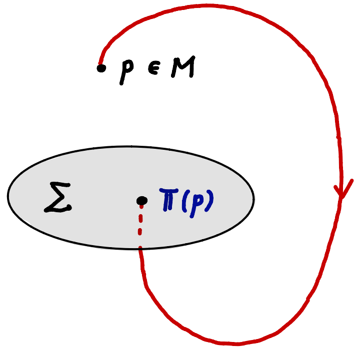
A compact oriented surface $\Sigma \subset M$ is a normal section for Φ

$\overset{\text{def}}{\Longleftrightarrow}$ (i) $\Sigma \pitchfork X$ (positively transverse);

(ii) $\forall p \in M$, $\{\varphi_t(p)\}_{t>0} \cap \Sigma \neq \phi$;

$$\left(\begin{array}{cccc} \hat{\mathbb{T}} : & M & \to & \Sigma \times \mathbb{R} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & p & \mapsto & (\mathbb{T}(p), \underset{\overset{\vee}{0}}{t_p}) \end{array} \right)$$

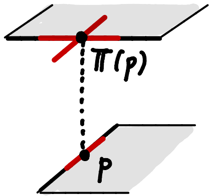
first time to meet Σ



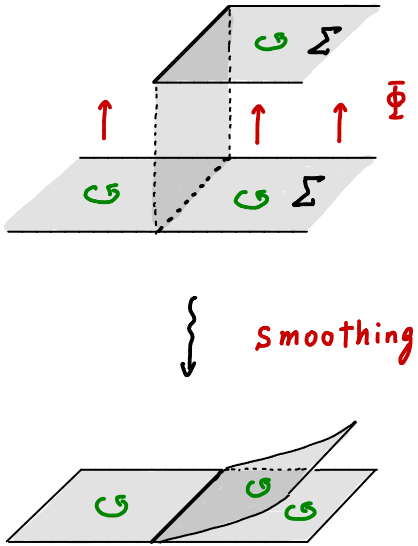
(iii) $p \in \partial\Sigma$, $\mathbb{T}(p) \in \partial\Sigma \Rightarrow \mathbb{T}^2(p) \in \operatorname{Int}\Sigma$; and

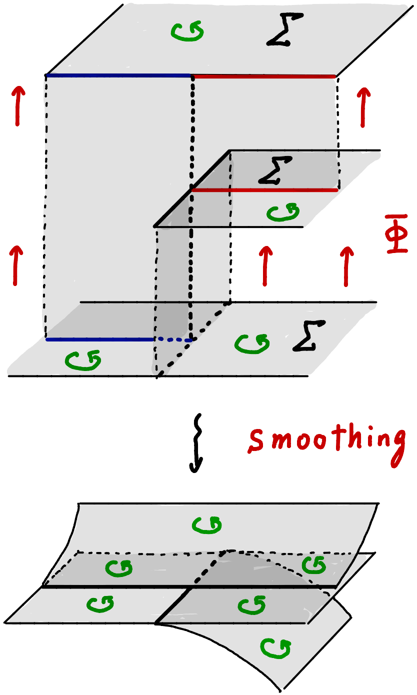
(iv) $p \in \partial\Sigma$, $\mathbb{T}(p) \in \partial\Sigma$

$$\Rightarrow (\varphi_{t_p})_* T_p \partial\Sigma \pitchfork T_{\mathbb{T}(p)} \partial\Sigma .$$



$$P := \text{the discontinuity set of } \hat{\Pi} : M \to \Sigma \times \mathbb{R}$$
$$p \mapsto (\Pi(p), t_p)$$
$$= \Sigma \cup \{ \varphi_t(p) \in M \mid p \in \Sigma,\ \Pi(p) \in \partial\Sigma,\ 0 < t \le t_p \} .$$

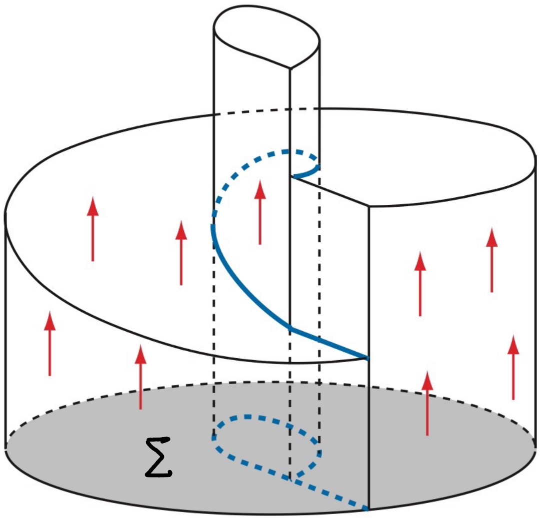




◎ $P \subset M$: a branched polyhedron.

◎ $M - P \cong \mathrm{Int}\,\Sigma \times (0,1)$.

In particular, $\Sigma \cong D^2 \Rightarrow M - P \cong \mathrm{Int}\, D^3$.



$\rightsquigarrow$ P is a <u>flow-spine</u> of (M, Φ). Φ is <u>carried</u> by P.

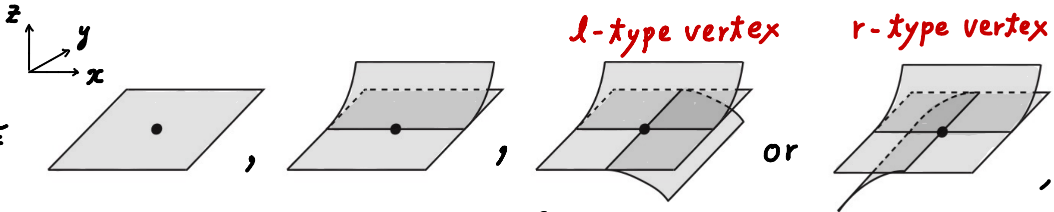
Definition

$P \subset M$: a **flow-spine** of (M, Φ)

$\underset{\text{def}}{\Longleftrightarrow}$ (i) P is a **spine**, i.e. $M - P \cong \mathrm{Int}\, D^3$;

(ii) $\forall p \in P$, $\exists$ a positive chart $(U; x, y, z)$ of M around p s.t.

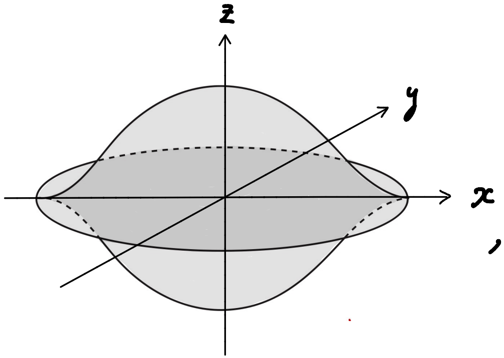
$(U, U \cap P) \cong$



,

where Φ on U is generated by $\dfrac{\partial}{\partial z}$; and

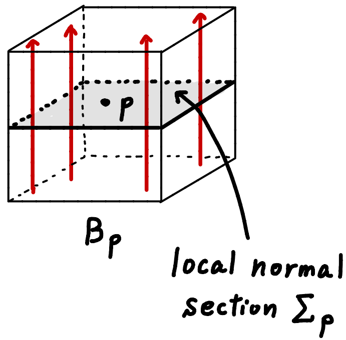
(iii) $M - P \cong$



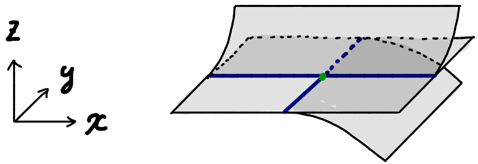
, where Φ on $M - P$ is generated by $\dfrac{\partial}{\partial z}$.

Proposition

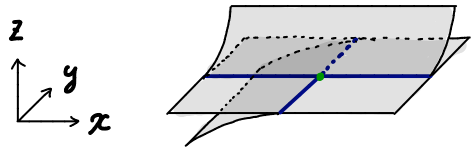
$\forall\, M$: a closed oriented 3-manifold,
$\forall\, \Phi$: a non-singular flow on M,
$\exists$ a flow-spine P of (M, Φ).



Vertices of a flow-spine



ℓ-type



r-type

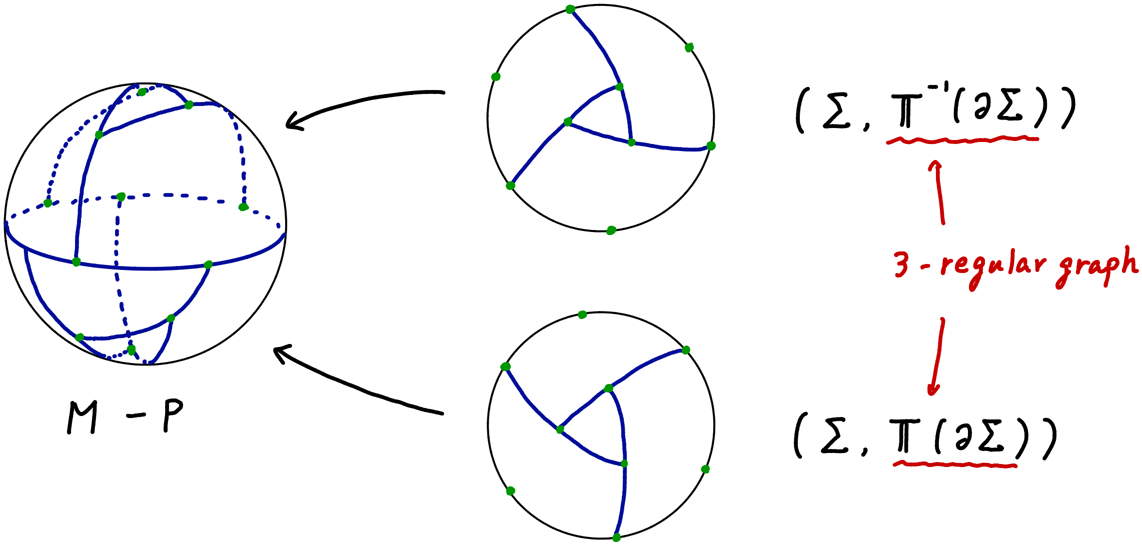
A flow-spine P is **positive** if
- P has at least 1 vertex ; and
- $\forall$ vertex of P is of ℓ-type.

M : a closed oriented 3-manifold

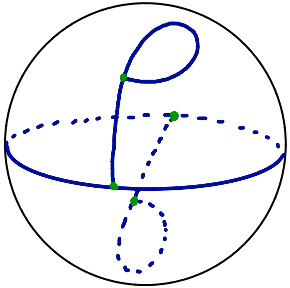
$\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$: a non-singular flow on M

$\Sigma\ (\cong D^2) \subset M$: a normal section for Φ

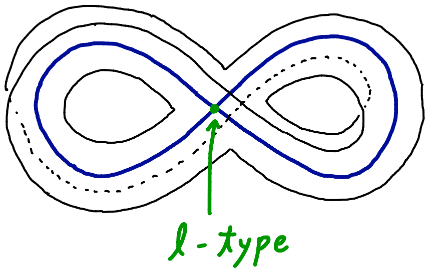
$\rightsquigarrow$ $P \subset M$: a flow-spine of (M, Φ).



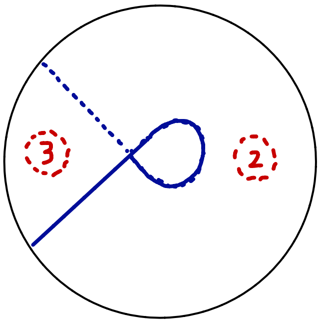
Example Positive abalone $P \subset S^3$



$M - P$



$Nbd(S(P); P)$



P is a flow-spine of the flow on S^3 whose orbits form the Seifert fibration with a regular fiber a positve trefoil.

§ 1 Positive flow-spines

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M : an oriented 3-manifold

Definition

(1) α : a 1-form on M.

α is a (positive) contact form on M $\underset{\text{def}}{\iff}$ $\alpha \wedge d\alpha > 0$.

(2) ξ : a 2-plane field on M.

ξ is a (positive) contact structure

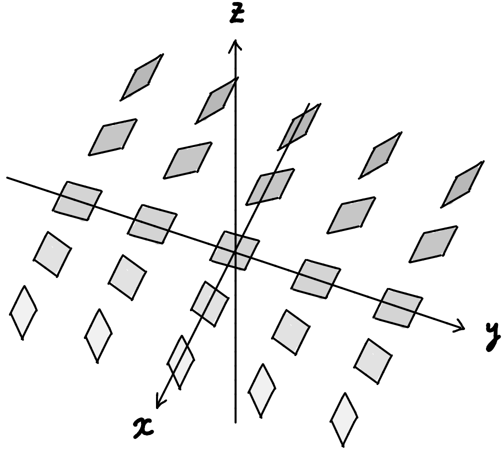
$\underset{\text{def}}{\iff}$ ${}^{\exists}\alpha$: a (positive) contact form on M

s.t. $\xi = \ker \alpha$.

Example

(1) $\alpha_{std} := dz + x\,dy.$

$$(\mathbb{R}^3, \underbrace{\ker \alpha_{std}}_{\substack{\| \\ \xi_{std}}})$$

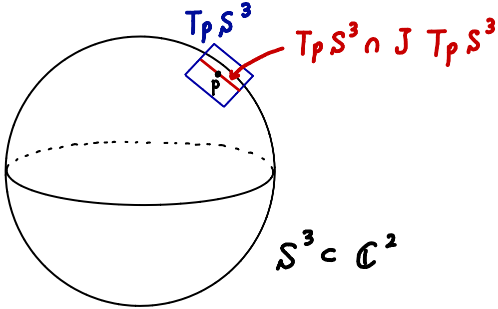


$$(\xi_{std})_{(x,y,z)} = \mathrm{Span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}.$$

(2) $S^3 = \{(x_1, y_1, x_2, y_2) \mid x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\} \subset \mathbb{R}^4 \ (= \mathbb{C}^2).$

$$\alpha_{std} := \sum_{i=1}^{2} (x_i\,dy_i - y_i\,dx_i)\Big|_{S^3}.$$

$$(S^3, \underbrace{\ker \alpha_{std}}_{\substack{\| \\ \xi_{std}}}).$$



$$(\xi_{std})_p = T_pS^3 \cap J\,T_pS^3.$$

<u>Definition</u> (M_1, ξ_1), (M_2, ξ_2) : contact 3-manifolds

$(M_1, \xi_1) \cong (M_2, \xi_2)$ <u>contactomorphic</u>

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists f : M_1 \to M_2$: a diffeomorphism s.t. $f_*(\xi_1) = \xi_2$

<u>Contactomorphism</u>

<u>Theorem</u> (Darboux 1882)

$\forall \alpha$: a contact form on an oriented 3-manifold M, $\forall p \in M$,
$\exists$ coordinates x, y, z on an open neighborhood $U \subset M$ of p s.t.
$p = (0,0,0)$ and $\alpha|_U = dz + x\,dy$.

In particular,

$$(U, \ker \alpha|_U) \cong (V, \xi_{std}|_V), \quad \text{where } V \subset \mathbb{R}^3 : \text{an open set}$$

$$p \in U \longleftrightarrow 0 \in V$$

Theorem (Gray 1959)

M : a closed oriented 3-manifold

$\{\alpha_t\}_{t\in[0,1]}$: a smooth family of contact forms on M

$\Rightarrow \exists$ an isotopy $\{\psi_t\}_{t\in[0,1]}$ s.t. $(\psi_t)_*(\xi_0) = \xi_t \quad \forall t \in [0,1]$.

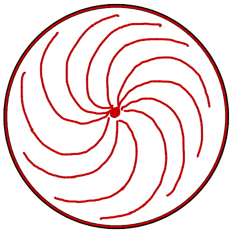
Theorem (Martinet 1971, Thurston - Winkelnkemper 1975)

Every closed oriented 3-manifold admits a contact structure.

Problem M : a closed oriented 3-manifold

Classify $\{$contact structures on $M\}$ / isotopy or contactomorphism.

$$\begin{gathered} \| \\ \mathrm{Cont}(M) \\ \| \\ \mathrm{Tight}(M) \sqcup \mathrm{OT}(M) \end{gathered}$$



Theorem (Eliashberg 1989)

M : a closed oriented 3-manifold.

coori. $OT(M)/\text{isotopy} \overset{1:1}{\longleftrightarrow} \{\text{coori. 2-plane field on } M\}/\text{homotopy}$.

Theorem $\text{Tight}(M)/\text{isotopy}$ is classified when

- $M = S^3$ (Eliashberg 1989)
- $M = T^3$ (Kanda 1997)
- $M = L(p,q)$ (Giroux 2000, Honda 2000)
- $M = -\Sigma(2,3,5)$ (Etnyre-Honda 2001). etc.

Theorem (Giroux 2002)

M : a closed oriented 3-manifold.

$\text{Cont}(M)/\text{isotopy} \overset{1:1}{\longleftrightarrow} \{\text{open book decomp.'s of } M\}/\text{positive Hopf plumbing}$.

Definition

(1) α : a contact form on a closed oriented 3-manifold.

R_α : the **Reeb vector field** on M

$\overset{\text{def}}{\iff}$ (i) $d\alpha(R_\alpha, \cdot) \equiv 0$; and

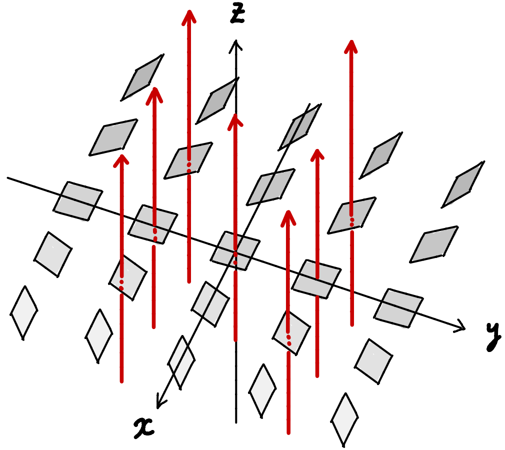
(ii) $\alpha(R_\alpha) \equiv 1$.

(2) The **Reeb flow** Φ_α of α is a flow generated by R_α.

Example

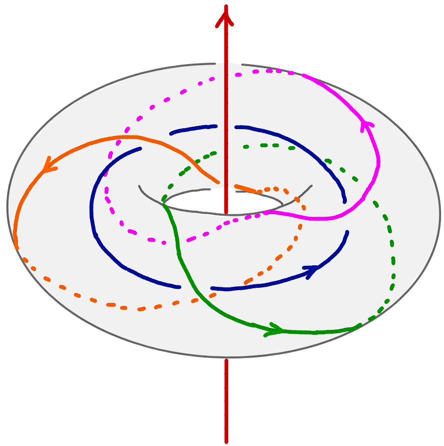
(1) $(\mathbb{R}^3, \alpha_{std} := dz + x\,dy)$.

$$R_{\alpha_{std}} = \frac{\partial}{\partial z}.$$



(2) (S^3, α_{std})

The orbits of $\Phi_{\alpha_{std}}$ form the Hopf fibration of S^3.



Lemma

M : a closed oriented 3-manifold.

α_0, α_1 : contact forms on M.

$R_{\alpha_0} \equiv R_{\alpha_1} \Rightarrow \ker \alpha_0 \underset{\text{isotopic}}{\approx} \ker \alpha_1$.

(Proof) Gray stability. $\square$

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M : a closed oriented 3-manifold

Definition

A contact structure ξ on M is **supported** by a flow-spine $P \subset M$ if $\exists$ a contact form α on M s.t.

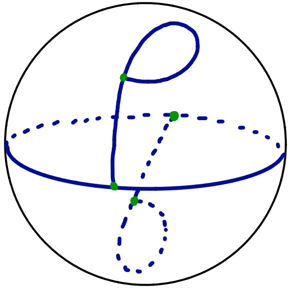
(i) $\xi = \ker\alpha$; and

(ii) P is a flow-spine of (M, Φ_α) .

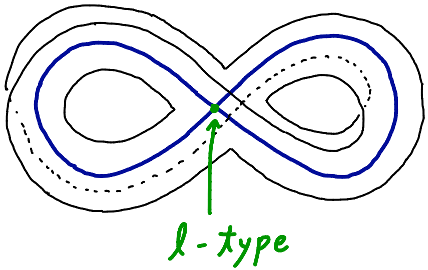
the Reeb flow of α

Remark We consider ξ modulo isotopy .

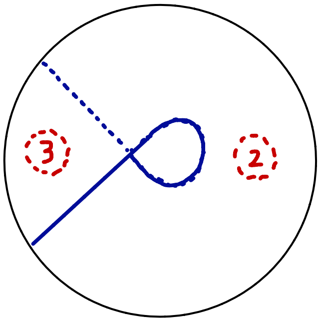
Example Positive abalone $P \subset S^3$



$M - P$



$Nbd(S(P); P)$



P is a flow-spine of the flow on S^3 whose orbits form the Seifert fibration with a regular fiber a trefoil.

$\rightsquigarrow$ P supports ξ_{std}.

Theorem (Ishii - Ishikawa - K. - Naoe)

1. $\forall$ positive flow spine $P \subset M$,
   $\exists$ a contact structure ξ supported by P.

2. $P \subset M$: a positive flow-spine.
   ξ_0, ξ_1 : contact structures on M supported by P.
   $\Rightarrow \xi_0 \underset{\text{isotopic}}{\approx} \xi_1$.

3. $\forall$ contact structures ξ on M.
   $\exists$ a positive flow-spine $P \subset M$ supporting ξ.

$\rightsquigarrow$ M : a closed oriented 3-manifold.

$\{\text{positive flow-spines of } M\} \twoheadrightarrow \mathrm{Cont}(M) / \text{isotopy}$

Theorem (Ishii-Ishikawa-K.-Naoe)

1. $\forall$ positive flow spine $P \subset M$,
   $\exists$ a contact structure ξ supported by P.

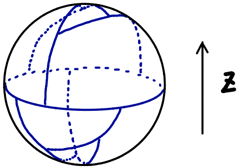
(Idea)

Constructive. (Similar to Thurston-Winkelnkemper's argument.)

$P \subset M$: a positive flow spine.

- Using the stratification $V(P) \subset S(P) \subset P \subset M$,
  we define a 1-form η on M s.t. $\eta \wedge d\eta \geq 0$.
  (We use the positivity of P here.)

$\eta \doteqdot dz$



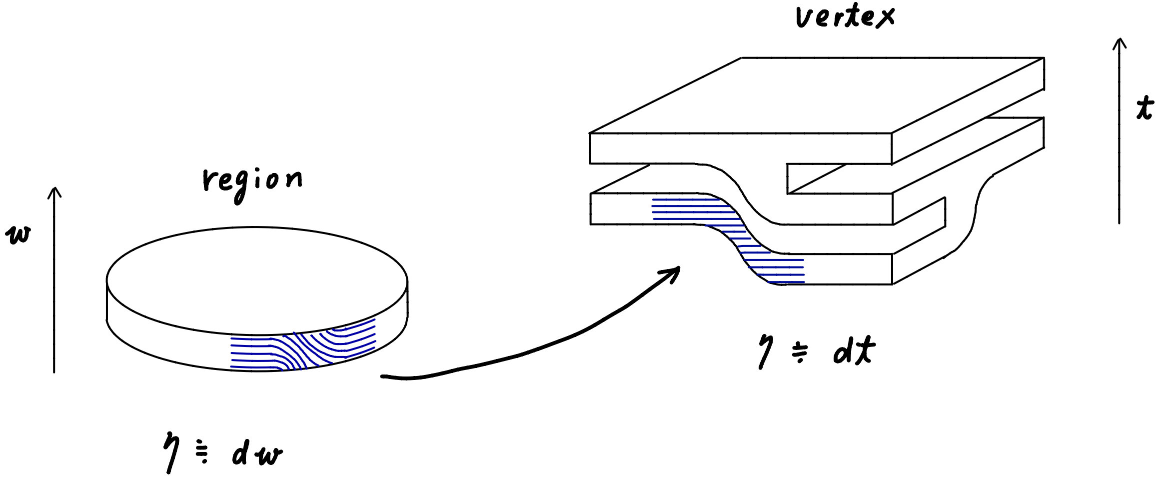
- $\exists \beta$: a "1-form" on P w/ $d\beta > 0$ on P.

$$\rightsquigarrow \quad \alpha := \hat{\beta} + R\eta \quad : \quad \text{a required contact form on } M. \qquad \square$$

extension of β to M

sufficiently large

Why positive flow-spine (ℓ-type) ?



Theorem (Ishii-Ishikawa-K.-Naoe)

2. $P \subset M$: a positive flow-spine.

ξ_0, ξ_1 : contact structures on M supported by P.

$\Rightarrow \xi_0 \underset{\text{isotopic}}{\approx} \xi_1$.

(Idea)

$\xi_0 = \ker \alpha_0, \quad \xi_1 = \ker \alpha_1$.

Using η, we find a 1-parameter family $\{\alpha_t\}_{t \in [0,1]}$ of contact forms on M.

The conclusion follows from Gray stability. $\square$

Theorem (Ishii - Ishikawa - K. - Naoe)

3. $\forall$ contact structures ξ on M,

$\exists$ a positive flow-spine $P \subset M$ supporting ξ.

(Idea)

$\xi \underset{\text{[Giroux 2000]}}{\rightsquigarrow}$ an open book decomp. of M supporting ξ.

$\rightsquigarrow$ a positive flow-spine supporting ξ. $\square$

The positivity condition for flow-spines is really essential ?

… Yes !

Theorem (Ishii-Ishikawa-K.-Naoe)

M : a 3-manifold admitting a tight contact structure.

$\Rightarrow$ $^\exists$ a non-positive flow-spine P of M s.t.

(i) P does not support any contact structure ; or

(ii) P supports two non-isotopic contact structures.

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M : a closed oriented 3-manifold

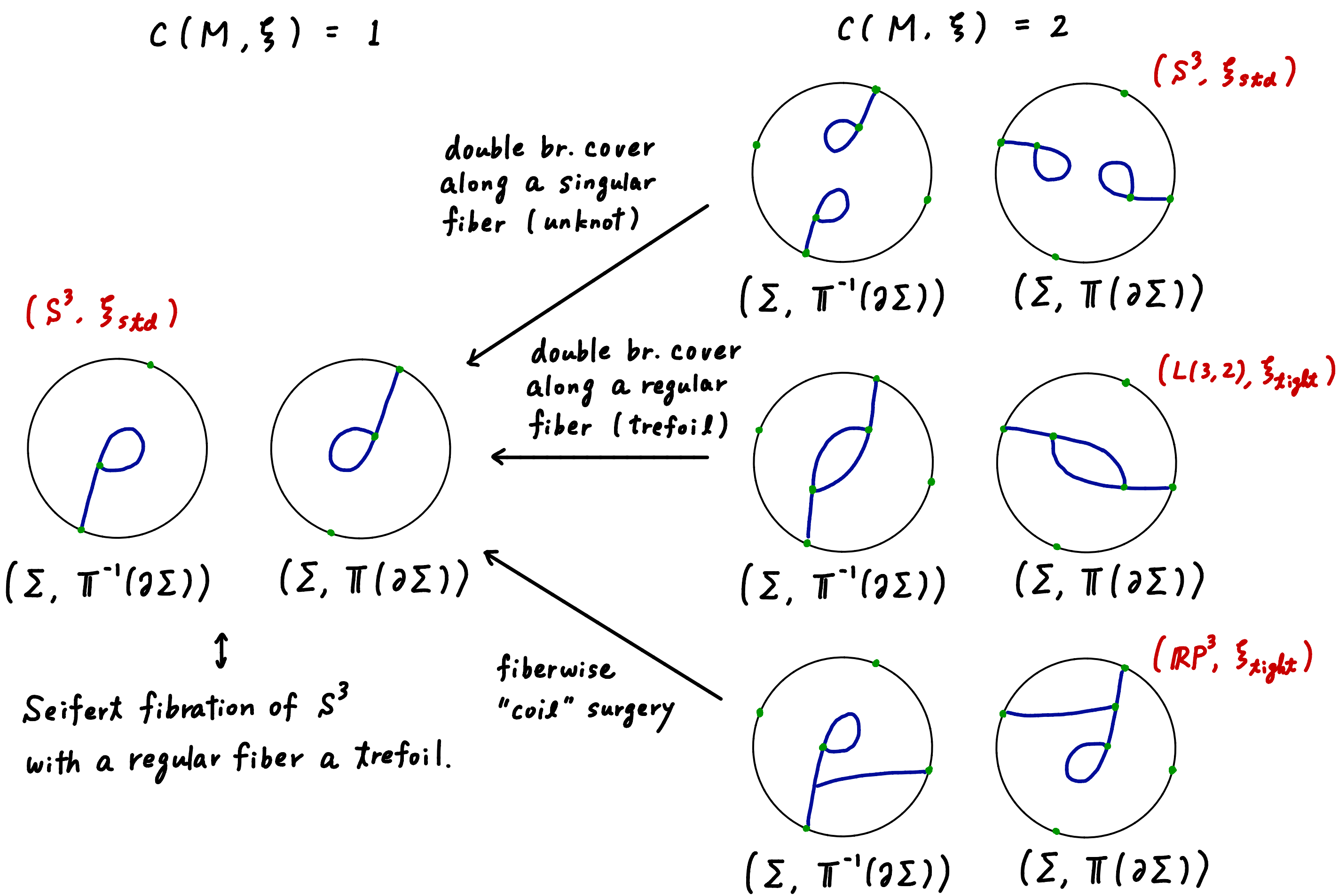
ξ : a contact structure on M

<u>Definition</u>

$$c(M,\xi) \underset{\text{def}}{:=} \min \left\{ \#\text{ of vertices of } P \;\middle|\; \begin{array}{l} P \text{ is a positive flow-spine} \\ \text{supporting } \xi \end{array} \right\}$$

<u>Recall</u>

$$c(M,\xi) = 1 \iff (M,\xi) = (S^3, \xi_{std})$$


$c(M,\xi) = 1$
$c(M,\xi) = 2$
(S^3, ξ_{std})
double br. cover along a singular fiber (unknot)
$(\Sigma, \pi^{-1}(\partial\Sigma))$
$(\Sigma, \pi(\partial\Sigma))$
(S^3, ξ_{std})
$(L(3,2), \xi_{tight})$
double br. cover along a regular fiber (trefoil)
$(\Sigma, \pi^{-1}(\partial\Sigma))$
$(\Sigma, \pi(\partial\Sigma))$
$(\Sigma, \pi^{-1}(\partial\Sigma))$
$(\Sigma, \pi(\partial\Sigma))$
$(\mathbb{R}P^3, \xi_{tight})$
fiberwise "coil" surgery
Seifert fibration of S^3 with a regular fiber a trefoil.
$(\Sigma, \pi^{-1}(\partial\Sigma))$
$(\Sigma, \pi(\partial\Sigma))$

$c(M,\xi)$	
1	(S^3, ξ_{std})
2	$(\mathbb{R}P^3, \xi_{tight})$ $(L(3,2), \xi_{tight})$
3	$(L(3,1), \xi_{tight})$ $(L(4,3), \xi_{tight})$ $(L(5,2), \xi_{tight})$ $(\Sigma(2,3,3), \xi_{tight})$

* to be confirmed

Remark

\# of tight contact structures on lens spaces (up to isotopy) :

S^3	$\mathbb{R}P^3$	$L(3,1)$	$L(3,2)$	$L(4,1)$	$L(4,3)$	$L(5,1)$	$L(5,2)$	$L(5,4)$
1	1	2	1	3	1	4	2	1

Theorem (Ishii-Ishikawa-K.-Naoe)

The complexity of the link of singularity of

$$f(x, y, z) = x^2 + y^3 + z^n$$

with the contact structure given by the complex tangency is at most n.

It is exactly n if $n \leq 5$.