Filtered instanton Floer homology and the homology cobordism group
(Joint work with Yuta Nozaki and Kouki Sato)

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This slide is available at
"https://sites.google.com/view/masaki-taniguchis-homepage".

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The homology cobordism group $\Theta^3_{\mathbb{Z}}$

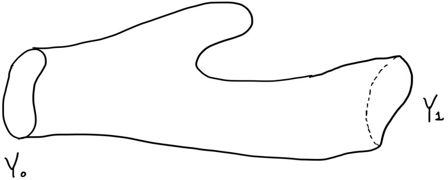
Let Y_0 and Y_1 be oriented homology spheres.

Definition (The homology cobordism group)

We say Y_0 is *homology cobordant* to Y_1 ($Y_0 \sim_{\mathbb{Z}H} Y_1$) if there exists a compact oriented 4-manifold W with $\partial W = Y_0 \amalg (-Y_1)$ such that the maps $H_*(Y_i, \mathbb{Z}) \to H_*(W, \mathbb{Z})$ induced by inclusions $Y_i \to W$ are isomorphisms.

$$\Theta^3_{\mathbb{Z}} := \{ \text{ oriented homology 3-spheres } \} / \sim_{\mathbb{Z}H}$$

The connected sum induces an abelian group structure on $\Theta^3_{\mathbb{Z}}$. The unit element is $[S^3]$.



The knot concordance group $\mathcal{C}$

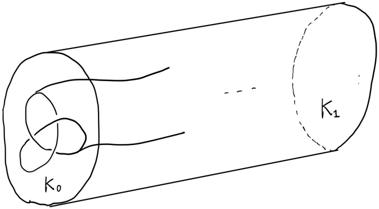
Let K_0 and K_1 be oriented knots in S^3.

Definition (The knot concordance group $\mathcal{C}$)

We say K_0 is *concordant* to K_1 ($K_0 \sim_c K_1$) if there exists an embedding $J : S^1 \times [0,1] \to S^3 \times [0,1]$ such that $J|_{S^1 \times \{i\}} = K_i \times \{i\}$ for $i = 0$ and 1.

$$\mathcal{C} := \{ \text{ all oriented knots } \} / \sim_c$$

The connected sum induces an abelian group structure on $\mathcal{C}$. The unit element is the unknot.



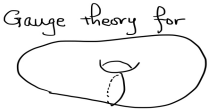
Known results related to $\Theta^3_{\mathbb{Z}}$ and $\mathcal{C}$

- The n-dimensional homology cobordism group $\Theta^n_{\mathbb{Z}}$ (resp. $\mathcal{C}^n$) is completely determined for $n \neq 3$(resp. $n \neq 1$).[Kervaire, Levine]
- Any topological manifold M with $\dim \geq 5$ admits a triangulation $\iff$ $0 = \exists\delta(\Delta(M)) \in H^5(M, \text{Ker } \mu)$, where $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}_2$ is the Roklin homomorphism. [Galewski–Stern, Matumoto]
- The group $\Theta^3_{\mathbb{Q}}$ is defined by replacing $\mathbb{Z}$ with $\mathbb{Q}$ in the definition of $\Theta^3_{\mathbb{Z}}$. The double branched cover gives a homomorphism

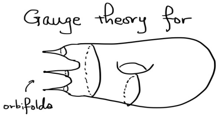
$$\Sigma : \mathcal{C} \to \Theta^3_{\mathbb{Q}}.$$

History of $\Theta^3_{\mathbb{Z}}$ related to our work

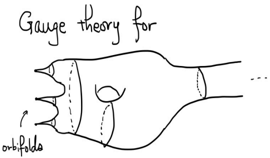
- 1982 Donaldson, Theorem A implies that $\Sigma(2,3,5)$ is not a torsion in $\Theta^3_{\mathbb{Z}}$.



- 1985 Fintushel–Stern, $\Sigma(p,q,pqn-1)$ is not a torsion in $\Theta^3_{\mathbb{Z}}$.



- 1990 Fintushel–Stern, Furuta, $\{\Sigma(p,q,pqn-1)\}_{k=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.



Open questions of $\Theta^3_{\mathbb{Z}}$ and $\mathcal{C}$

Let $T_{p,q}$ be the (p,q)-torus knot. It is known that $S_{1/n}(T_{p,q}) = \Sigma(p,q,pqn-1)$. $(n>0)$

Open question of $\Theta^3_{\mathbb{Z}}$

Is there a nice sufficient condition of K such that $\{S_{1/n}(K)\}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$?

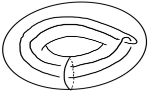
Open question of $\Theta^3_{\mathbb{Z}}$

Is $\Theta^3_{\mathbb{Z}}$ generated by all Seifert homology spheres?

The Whitehead double[1] determines a map $D\colon \mathcal{C} \to \mathcal{C}$.

Hedden–Kirk's conjecture

The map D preserves linear independence.



[1] The satelite knot with respect to the above knot in $S^1 \times D^2$.

Main result

Theorem (2019, Nozaki–Sato–T)

For $s \in \mathbb{R}_{\leq 0} \amalg \{-\infty\}$ and an oriented homology sphere Y, we define $r_s(Y) \in \mathbb{R}_{>0} \amalg \{\infty\}$ satisfying the following conditions:

1. If $s \leq s'$, then $r_{s'}(Y) \leq r_s(Y)$.
2. The values of $r_s(Y)$ are contained in the set of critical values of the Chern–Simons functional of Y.
3. Let Y_0 and Y_1 be $\mathbb{Z}\mathrm{H}S^3$'s and W a negative definite cobordism with $\partial W = Y_0 \amalg -Y_1$. Then $r_s(Y_1) \leq r_s(Y_0)$ holds for any s. If $\pi_1(W) = 1$ and $r_s(Y_0) < \infty$, then $r_s(Y_1) < r_s(Y_0)$ holds.
4. The invariant r_0 satisfies
$$r_0(Y_1 \# Y_2) \geq \min\{r_0(Y_1), r_0(Y_2)\}.$$
5. The condition $r_{-\infty}(Y) < \infty \iff h(Y) < 0$ holds, where $h\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ is the Frøyshov homomorphism.

Remarks for main theorem

- Recently, Daemi(2018) introduced a family of invariants of Y parametrized by $\mathbb{Z}$:

$$\cdots \leq \Gamma_Y(-1) \leq \Gamma_Y(0) \leq \Gamma_Y(1) \leq \dots$$

  using instanton Floer theory. Note that $\Gamma_Y(k)$ also satisfies the conditions 2, 3 and 5 for a positive k. We showed

$$r_0(Y) \leq \cdots \leq r_s(Y) \leq \cdots \leq r_{-\infty}(Y) = \Gamma_Y(1).$$

- $\exists$ an example of Y such that $r_s(Y)$ is not constant w.r.t. s.

Rough definition of r_0

Roughly speaking, $r_0(Y)$ is given by

$$\inf\left\{-\frac{1}{8\pi^2}\int_{Y\times\mathbb{R}}\mathrm{Tr}(F(A)\wedge F(A))\;\middle|\; A\in\Omega^1_{Y\times\mathbb{R}}\otimes\mathfrak{su}(2)\text{ with }(*)\right\}$$

$$=\inf\left\{cs(b)\;\middle|\; A\in\Omega^1_{Y\times\mathbb{R}}\otimes\mathfrak{su}(2)\text{ with }(*), b=\exists\lim_{t\to-\infty}A|_{Y\times\{t\}}\right\}$$

The condition $(*)$ is

- $0=\exists\lim_{t\to\infty}A|_{Y\times\{t\}}$.
- $\exists$ Riemann metric g on Y such that the ASD-equation $\frac{1}{2}(1+*_{g+dt^2})F(A)=0$ is satisfied.
- The Fredholm index of the operator $d_A^+ + d_A^*$ on $Y\times\mathbb{R}$ is 1.

Calculations

Example

$r_s(S^3) = \infty$ for any s.

Theorem

$r_s(-\Sigma(p, q, pqn-1)) = \frac{1}{4pq(pqk-1)}$ for any s.

In general,

$$\bigcup_s r_s(\Theta^3_S) \subset \mathbb{Q}_{>0} \amalg \{\infty\},$$

where Θ^3_S is the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by Seifert homology 3-spheres. We tried to calculate r_s for a hyperbolic manifold obtained by the $1/2$-surgery along the mirror image of 5_2.

Calculations

Theorem

By computer, for any s,

$$r_s(S^3_{1/2}(5_2^*)) \approx 0.0017648904\ 7864885113\ 0739625897\ 0947779330\ 4925308209$$

whose error is 10^{-50}, where $S^3_{1/2}(5_2^*)$ is the $1/2$ surgery on the mirror image of 5_2 in Rolfsen's table.

Our computation is based on Kirk and Klassen's formula (to be explained later).

Our conjecture

$r_s(S^3_{1/2}(5_2^*))$ is irrational.

If the conjecture is true, we can conclude that $\Theta^3_{\mathbb{Z}}/\Theta^3_S$ is non-trivial.

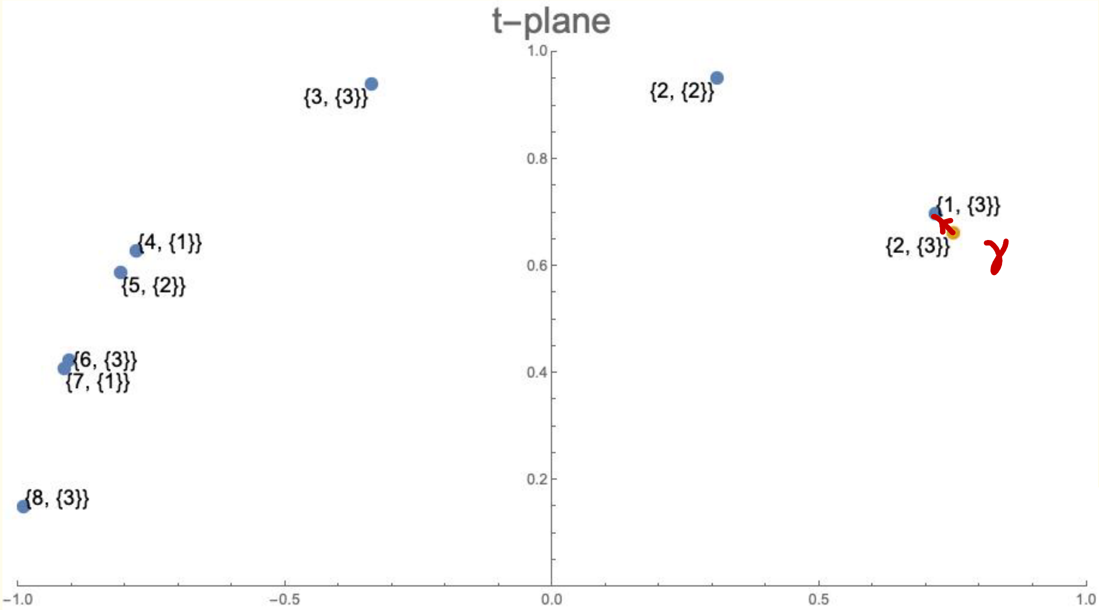
Computation of $r_s(S^3_{1/2}(5^*_2))$

Let ρ_0, ρ_1 be $SU(2)$-representations of $\pi_1 = \pi_1(S^3_{-1/2}(5_2))$ and $\{\rho_s\}_s \subset \mathrm{Hom}(\pi_1, SL(2,\mathbb{C}))$ a path from ρ_0 to ρ_1.
Then Kirk and Klassen gave a fomula of the form

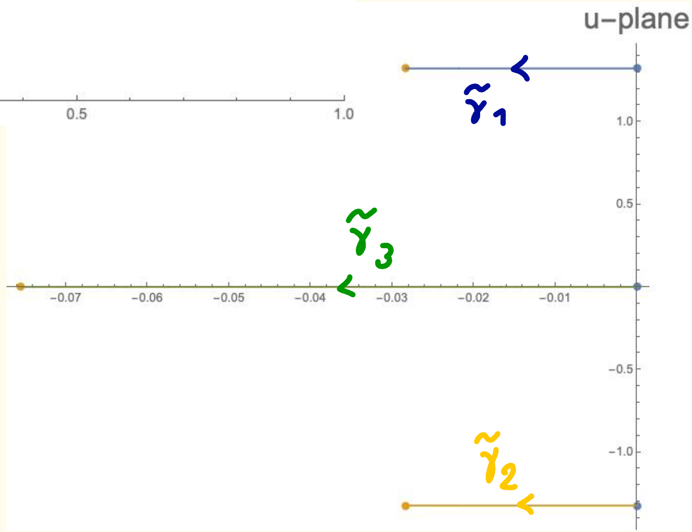
$$cs(\rho_1) - cs(\rho_0) \equiv \int_0^1 \text{``}\rho_s(\lambda) \ \&\ \rho_s(\mu)\text{''} ds \mod \mathbb{Z}.$$

The irreducible representations of π_1 are described by the Riley polynomial $\phi(t,u) = -(t^{-2}+t^2)u + (t^{-1}+t)(2+3u+2u^2) - (3+6u+3u^2+u^3)$.

	t	u	$-cs$
ρ_1	$0.716932 + 0.697143i$	-0.0755806	0.00176489
ρ_2	$0.309017 + 0.951057i$	-1.00000	0.166667
ρ_3	$-0.339570 + 0.940581i$	-2.41421	0.604167
ρ_4	$-0.778407 + 0.627759i$	-1.69110	0.388460
ρ_5	$-0.809017 + 0.587785i$	-1.00000	0.166667
ρ_6	$-0.905371 + 0.424621i$	-2.16991	0.865934
ρ_7	$-0.912712 + 0.408603i$	-3.62043	0.321158
ρ_8	$-0.988857 + 0.148870i$	-2.41421	0.604167



Take a path γ



Three lifts of γ w.r.t. $\begin{array}{c}\{\phi(t,u)=0\}\\ \downarrow pr_1\\ \mathbb{C}\end{array}$

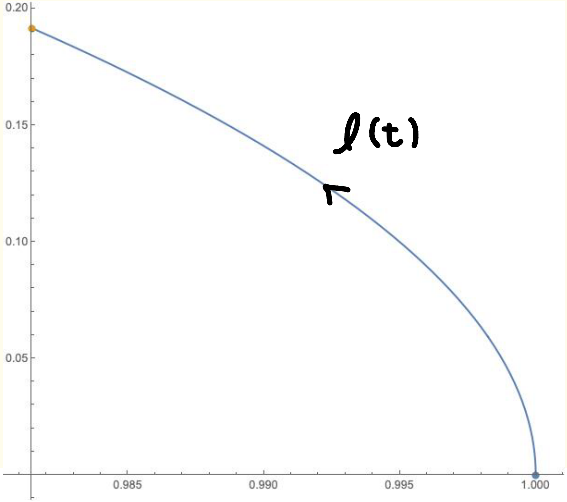
$\tilde{\gamma}_3$ is suitable $\rightsquigarrow \{\rho_t\}_{t\in\gamma}$

$t \in \gamma$

$$P_t^{-1}\rho_t(\mu)P_t = \begin{pmatrix}\sqrt{t} & 0 \\ 0 & 1/\sqrt{t}\end{pmatrix}$$

$$P_t^{-1}\rho_t(\lambda)P_t = \begin{pmatrix}\ell(t) & 0 \\ 0 & 1/\ell(t)\end{pmatrix}$$

$\Big\downarrow$ Kirk - Klassen




```
In[•]:= cs[1] = 2 NIntegrate[
          Log[longitude[patht[s], pathu[s]]] / (2 Pi I) * D[Log[patht[s]] / (4 Pi I ), s], {s, 0, 1},
          WorkingPrecision → 100
        ] + q (Log[longitude[patht[1], pathu[1]]] / (2 Pi I)) ^2 -
       q (Log[longitude[patht[0], pathu[0]]] / (2 Pi I)) ^2

Out[•]= -0.001764890478648851130739625897094777933049253082097123177925233147191085385009681049300589662023
          2403365 + 0. × 10^-118 i
```

irrational ?

```
In[•]:= cs12 =
         -0.164901776188017815535927040769571888733617413584569543488741433519475581281656985617366077004 6
           43425764614462882619`101.09618858068605 + 0``114.32113052110438 i;
        cs[2] = cs[1] + cs12

Out[•]= -0.166666666666666666666666666666666666666666666666666666666666666666666666666666666666666666666666
          66610 + 0. × 10^-115 i
```

$-1/6$?

Useful lemmas

Lemma (I)

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of oriented homology 3-spheres satisfying the following two conditions:

- $r_0(Y_1) > r_0(Y_2) > \dots$ and
- $r_0(-Y_n) = \infty$ for any n.

Then the sequence $\{[Y_n]\}$ are linearly independent in both of $\Theta^3_{\mathbb{Z}}$ and $\Theta^3_{\mathbb{Q}}$.

Lemma (II)

Let Y_0 and Y_1 be $\mathbb{Z}\mathrm{H}S^3$'s and W a negative definite cobordism with $\partial W = Y_0 \amalg -Y_1$. If $\pi_1(W) = 1$ and $r_0(Y_0) < \infty$, then $r_0(Y_1) < r_0(Y_0)$ holds.

Lemma (III)

If Y bounds a negative definite 4-manifold, then $r_0(Y) = \infty$ holds.

Three applications of $\{r_s\}$

Theorem (I)

$\exists$ infinitely many homology spheres $\{Y_k\}$ such that Y_k does not admit any definite bounding.

Set $Y_k := 2\Sigma(2,3,5)\#(-\Sigma(2,3,6k+5))$. $(k \geq 1)$ Then using connected sum formula, we have $r_0(Y_k) = \frac{1}{24(6k+5)} < \infty$. Moreover, the calculation $h(-Y_k) = -1$ implies that $r_0(-Y_k) < \infty$. □

Corollary

$[Y_k]$ does not contain any Seifert homology sphere and homology 3-sphere obtained by a surgery on a knot in S^3.

It is known that all Seifert homology spheres and homology 3-spheres obtained by surgeries on knots admit a definite bounding. □

Three applications of $\{r_s\}$

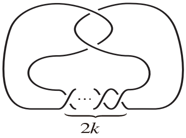
Theorem (II)

For any knot K in S^3 with $h(S_1(K)) < 0$, $\{S^3_{1/n}(K)\}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.

If we take $K = T_{p,q}$, this theorem recover the results of Furuta, Fintushel–Stern in '90.

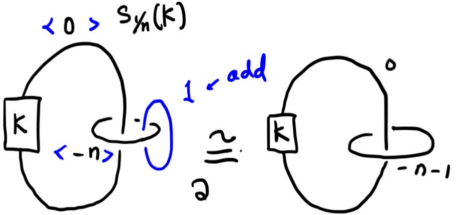
Proposition

All positive k-twisted knots ($k \geq 1$) and $(2, q)$-cable knots ($g \geq 3$) satisfy $h(S_1(K)) < 0$.



Sketch of proof of II

Set $Y_n := S_{1/n}(K)$. The fifth and third property of r_0 implies $r_0(Y_1) < \infty$ and $r_0(-Y_n) = \infty$. On the other hand, we have a positive definite cobordism W_n with $\partial(W_n) = -Y_n \amalg (Y_{n+1})$ described by



One can see that W_n is simply connected for each n. Therefore the third property of r_0 implies that

$$r_0(Y_1) > r_0(Y_2) > \dots .$$

$\square$

Three applications of $\{r_s\}$

Let $T_{p,q}$ be the (p,q)-torus knot. We denote $D(T_{p,q})$ by $D_{p,q}$.

Theorem (Hedden–Kirk, 2012)

$\{D(T_{2,2^n-1})\}_{n=2}^{\infty}$ are lineary independent in $\mathcal{C}$.

Theorem (III)

$\{D(T_{p,np+q})\}_{n=1}^{\infty}$ are lineary independent in $\mathcal{C}$ for any relative prime numbers (p,q).

Sketch of proof of III

Since

$$\Sigma\colon \mathcal{C} \to \Theta^3_{\mathbb{Q}}$$

is a homomorphism, it is sufficient to prove $\{\Sigma(D_{p,kp+q})\}_{k=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Q}}$. Moreover, since $\Sigma(D_{p,q}) = S^3_{1/2}(T_{p,q} \# T_{p,q})$ is $\mathbb{Z}HS^3$, this is followed by:

Lemma

- $r_0(\Sigma(D_{p,q})) < \infty$.
- $r_0(\Sigma(D_{p,q})) > r_0(\Sigma(D_{p,p+q}))$.

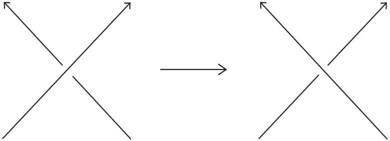
To prove the above lemma, we construct

- neg. defn. cob. with boundary $\Sigma(p, q, 2pq - 1)) \amalg (-\Sigma(D_{p,q}))$
- simp. conn. neg. defn. cob. with boundary $\Sigma(D_{p,q}) \amalg (-\Sigma(D_{p,p+q}))$

Sketch of proof of III

Lemma

If $K_0 \to \cdots \to K_1$ by a seq. of pos. crossing changes, then $\exists$ neg. defn. cob. with boundary $S^3_{1/n}(K_1) \amalg (-S^3_{1/n}(K_0))$ for $\forall n$.



,

- $T_{p,q} \# T_{p,q} \overset{\text{pos. c.c.}}{\longrightarrow} T_{p,q} \rightsquigarrow r_0(\Sigma(D_{p,q})) < \infty$
- $T_{p,q+p} \# T_{p,q+p} \overset{\text{pos. c.c.}}{\longrightarrow} T_{p,q} \# T_{p,q} \rightsquigarrow r_0(\Sigma(D_{p,q})) > r_0(\Sigma(D_{p,p+q}))$

$\square$

History of instanton homology related to our work

Let Y be an oriented homology 3-sphere.

- 1987, Floer, Instanton homology $I_*(Y)$ with $* \in \mathbb{Z}/8\mathbb{Z}$.
- 1992, Fintushel–Stern, Filtered version of instanton homology $I_*^{[r,r+1]}(Y)$ with $* \in \mathbb{Z}$ for $r \in \mathbb{R}$.
- 2002, Donaldson, The obstruction class $[\theta_Y] \in I^1(Y)$. If Y admits a negative definite bounding with non-standard intersection form, then $0 \neq [\theta_Y] \in I^1(Y;\mathbb{Q})$.
- 2019, NST, Filtered instanton cohomology $I^*_{[s,r]}$ and the filtered version $[\theta_Y^{[s,r]}] \in I^*_{[s,r]}$ of the obstruction class.

Definition

$r_s(Y) := \sup\{r \in \mathbb{R} \mid 0 = [\theta_Y^{[s,r]}] \in I^*_{[s,r]}\}$

Construction of $I^*_{[s,r]}$ and $[\theta_Y^{[s,r]}]$

Let Y be an oriented homology 3-sphere. Set $\mathcal{B}_Y := \Omega^1_Y \otimes \mathfrak{su}(2)/\mathrm{Map}^0(Y, SU(2))$, where $\mathrm{Map}^0(Y, SU(2))$ is the set of null-homotopic smooth maps and the action is given by $a * g := g^{-1}dg + g^{-1}ag$. The (perturbed) Chern–Simons functional

$$cs_h : \mathcal{B}_Y \to \mathbb{R}$$

is given by

$$cs([a]) := \frac{1}{8\pi^2}\int_Y \mathrm{Tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a) + h$$

for some perturbation $h \colon \mathcal{B}_Y \to \mathbb{R}$. The "critical point set" of cs_h is given by

$$R_h(Y) = \{[a] \in \mathcal{B}_Y \mid F(a) + *\mathrm{grad}\ h = 0\}.$$

Floer defined the Floer index

$$\mathrm{ind}_h \colon R_h(Y) \to \mathbb{Z}$$

under some good situation.

Gradient flow of cs

Fix a Riemann metric on Y. We equip an L^2-inner product on $\Omega^1_Y \otimes \mathfrak{su}(2)$ by

$$(a, b) := -\frac{1}{4\pi^2}\int_Y \mathrm{Tr}(a \wedge *b).$$

Then the formal gradient flow of cs w.r.t. the inner product is given by

$$\text{grad } (cs+h) \colon a \mapsto *_g(F(a)) + \text{grad } h.$$

A gradient flow $c : \mathbb{R} \to \Omega^1_Y \otimes \mathfrak{su}(2)$ of grad $(cs+h)$ corresponds to a solution to the ASD-equation

$$\frac{1}{2}(1 + *_{g+dt^2})(F(A) + \pi_h(A)) = 0,$$

where A is a connection on $Y \times \mathbb{R}$ given by $A|_{Y\times t} = c(t)$ such that $(dt\text{-component of } A) = 0$.

In the case of $Y = -\Sigma(2,3,5)$

The critical point set is

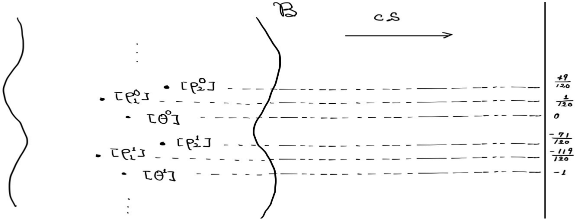
$$R(Y) \cong \{\rho_1^i, \rho_2^i, \theta^i\}_{i \in \mathbb{Z}}.$$

The critical values are

$$cs(\rho_1^i) = \frac{1}{120} + i,\ \ cs(\rho_2^i) = \frac{49}{120} + i \text{ and } cs(\theta^i) = i.$$

The Floer indicies are given by

$$\text{ind}(\rho_1^i) = 1 + 8i,\ \ \text{ind}(\rho_2^i) = 5 + 8i \text{ and } \text{ind}(\theta^i) = -3 + 8i.$$



Construction of I_*

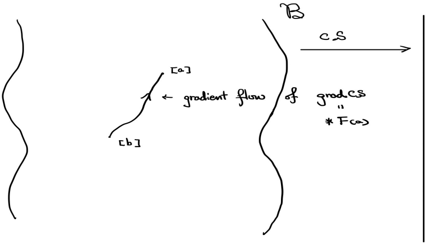
Suppose that $cs + h$ is Morse. ($\iff$ Hess $(cs+h)_a$: Ker$d_a^* \to$ Kerd_a^* is injective for any critical point a.) The instanton Floer chain is given by

$$CI_*(Y) := \mathbb{Z}\{[a] \in R_h(Y) \mid \mathrm{ind}_h([a]) = *\}.$$

The differential is defined by

$$\partial([a]) = \sum_{[b]\in R(Y),\ \mathrm{ind}([a])-\mathrm{ind}([b])=1} \#(M_h([a],[b])/\mathbb{R}),$$

where the space $M_h([a],[b])$ is the set of trajectories of $cs+h$ from $[a]$ to $[b]$.



Construction of I_*

If we give a topology on $M_h([a],[b])$, we use the identification

$$M_h([a],[b]) \cong \{A \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{k,loc}} \mid (*)\}/\mathcal{G},$$

where the conditions $(*)$ are given by

- $A - p^*a \in L^2_k(Y \times (-\infty,-1])$, $A - p^*b \in L^2_k(Y \times [1,\infty))$ and
- $(1 + *_{g+dt^2})(F(A) + \pi_h(A)) = 0$ (ASD equation),

where the map p is the projection $Y \times \mathbb{R} \to Y$. The gauge group $\mathcal{G}$ is

$$\left\{ g \in \text{Map}\ (Y \times \mathbb{R}, SU(2))_{L^2_{k,\mathbf{loc}}} \;\middle|\; \begin{array}{l} g^*p^*a \in L^2_k(Y \times (-\infty,-1]), \\ g^*p^*b \in L^2_k(Y \times [1,\infty)) \end{array} \right\}.$$

(One can check that the group $\mathcal{G}$ acts on the space $\{A \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{k,\mathbf{loc}}} \mid (*)\}$.)

Construction of I_*

Theorem (Floer)

There exists a nice class of perturbations $h : \mathcal{B} \to \mathbb{R}$ of cs satisfying the following conditions:

- The map ∂ is well-defined, i.e. , $M_h([a],[b])$ has a structure of a manifold of dimension $\text{ind}([a]) - \text{ind}([b])$ such that $\mathbb{R}$ action on $M_h([a],[b])$ is proper and free if $\text{ind}([a]) - \text{ind}([b]) > 0$ and $M_h([a],[b])/\mathbb{R}$ is compact if $\text{ind}([a]) - \text{ind}([b]) = 1$. Moreover, there is a method to give orientations on $M_h([a],[b])$.
- $\partial^2 = 0$ holds.
- The chain homotopy type of (CI_*, ∂) does not depend on h.

The instanton (co) homology is given by $I_*(Y) := H_*(CI_*, \partial)$.

Example

$$I_*(-\Sigma(2,3,5)) \cong \begin{cases} \mathbb{Z} \text{ if } * = 1, 5 \mod 8 \\ 0 \text{ otherwise} \end{cases}$$

The obstruction class $[\theta]$

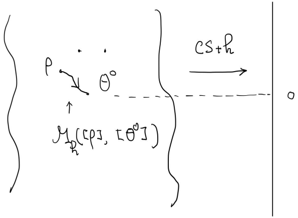
Definition

The homomorphism $\theta : CI_1 \to \mathbb{Z}$ is given by $[a] \mapsto \#M_h([a], [\theta^0])$.

One can see that $\partial^* \theta = 0$. Therefore, the map θ determines a class $[\theta] \in I^1(Y)$. Although, the definition of the map θ depends on the choice of h, the cohomology class does not depend on the choice of h.

Example

If Y$=-\Sigma(2,3,5)$, $\theta : CI_1 \to \mathbb{Z}$ satisfies $\theta(\rho_1^0) = \pm 1$. In this case, $[\theta]$ generates $I^1(Y)$.



Construction of $I^*_{[s,r]}$

Set $\lambda_Y := \frac{1}{4}\min\{|a-b| \mid a, b \in cs|_{R(Y)}\}$.

Definition

For $s \in \mathbb{R}_{\leq 0} \amalg \{-\infty\}$ and $r \in \mathbb{R}_{\geq 0} \setminus cs|_{R(Y)}$, we define

$$CI^{[s,r]}_*(Y) := \mathbb{Z}\left\{ [a] \in R_h(Y) \;\middle|\; \begin{array}{l} \mathrm{ind}([a]) = *, \\ s - \lambda_Y < (cs + h)([a]) < r \end{array} \right\}.$$

The differential $\partial^{[s,r]}$ is given by the restriction of ∂. The filtered instanton cohomology is given by

$$I^*_{[s,r]}(Y) := H_*(\mathrm{Hom}\ (CI^{[s,r]}_*(Y), \mathbb{Z}), (\partial^{[s,r]})^*)).$$

Theorem (Fintushel–Stern, '92)

If we take a small perturbation h to define $I^*_{[s,r]}(Y)$, the chain homotopy type of $(\mathrm{Hom}\ (CI^{[s,r]}_*(Y), \mathbb{Z}), (\partial^{[s,r]})^*)$ does not depend on the choice of h.

The obstruction class $[\theta^{[s,r]}]$

Definition

For $s \in \mathbb{R}_{\leq 0} \amalg \{-\infty\}$ and $r \in \mathbb{R}_{\geq 0} \setminus cs|_{R(Y)}$, we have the homomorphism $\theta^{[s,r]} : CI_1^{[s,r]} \to \mathbb{Z}$ given by $[a] \mapsto \# M_h([a], [\theta^0])$.

One can see that $(\partial^{[s,r]})^* \theta = 0$. Therefore, the map $\theta^{[s,r]}$ determines a class $[\theta^{[s,r]}] \in I^1_{[s,r]}(Y)$. Moreover, for a small perturbation h, the class $[\theta^{[s,r]}] \in I^1_{[s,r]}(Y)$ is well-defined.

Example

Suppose that Y$=-\Sigma(2,3,5)$.

- If $0 < r < \frac{1}{120}$, then the map $\theta^{[s,r]} : CI_1^{[s,r]} \to \mathbb{Z}$ is zero map since $CI_1^{[s,r]} = 0$.
- If $\frac{1}{120} < r$, then the map $\theta^{[s,r]} : CI_1^{[s,r]} \to \mathbb{Z}$ gives an isomorphism.

Definition of r_s

Definition

For a given homology 3-sphere Y,

$$r_s(Y) := \sup\{r \mid 0 = [\theta^{[s,r]} \otimes \mathrm{Id}_{\mathbb{Q}}] \in I^1_{[s,r]}(Y;\mathbb{Q})\}$$

Example

Suppose that $Y = -\Sigma(2,3,5)$.

- If $0 < r < \frac{1}{120}$, then $0 = [\theta^{[s,r]}] \in I^1_{[s,r]}$.
- If $\frac{1}{120} < r$, then $0 \neq [\theta^{[s,r]}] \in I^1_{[s,r]}$.

Therefore, $r_s(-\Sigma(2,3,5)) = \frac{1}{120}$.

Negative definite inequality of $\{r_s\}$

For a negative definite cobordism W with $\partial W = Y_0 \amalg Y_1$ and $H_1(W, \mathbb{R}) = 0$, $s \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ and $r \in \mathbb{R}_{\geq 0} \setminus (cs(R(Y_0)) \amalg cs(R(Y_1)))$, we have the cobordism map

$$CW : I^*_{[s,r]}(Y_1; \mathbb{Q}) \to I^*_{[s,r]}(Y_0; \mathbb{Q})$$

with $CW(\theta^{[s,r]}_{Y_1}) = c(W)\theta^{[s,r]}_{Y_0}$, where $c(W)$ is non-zero rational number. This map is defined by counting the solution to ASD-moduli space for W. This gives an inequality

$$r_s(Y_0) \leq r_s(Y_1).$$

Moreover, If $r_s(Y_1) < \infty$ and $r_s(Y_0) = r_s(Y_1)$, one can construct an irreducible $SU(2)$-representation of $\pi_1(W)$. Therefore, if $\pi_1(W) = 1$ and $r_s(Y_1) < \infty$, we have

$$r_s(Y_0) < r_s(Y_1).$$

Cobordism inequality of $\{r_s\}$

To prove $r_0(Y_0\#Y_1) \geq \min\{r_0(Y_0), r_0(Y_1)\}$, we need to show if $[\theta^{[0,r]}_{Y_i}] = 0$ for $i = 0$ and 1 then $[\theta^{[0,r]}_{Y_0\#Y_1}] = 0$. Let W be a cobordism with $\partial W = Y_0\#Y_1 \amalg (-Y_0) \amalg (-Y_1)$ obtained by adding a 3-handle on $Y_0\#Y_1$. There are four kinds of maps on the instanton chain complex induced by W;

- $p_0CW : CI^{[0,r]}_*(Y_0\#Y_1) \to CI^{[0,r]}_*(Y_0) \otimes CI^{[0,r]}_*(Y_1)$,
- $p_1CW : CI^{[0,r]}_*(Y_0\#Y_1) \to CI^{[0,r]}_*(Y_1)$,
- $p_2CW : CI^{[0,r]}_*(Y_0\#Y_1) \to CI^{[0,r]}_*(Y_0)$ and
- $p_3CW : CI^{[0,r]}_*(Y_0\#Y_1) \to \mathbb{Q}$.

Moreover, these maps satisfy nice equations related to $[\theta^{[0,r]}_{Y_0}]$, $[\theta^{[0,r]}_{Y_1}]$ and $[\theta^{[0,r]}_{Y_0\#Y_1}]$. Using such equations and the assumption $[\theta^{[0,r]}_{Y_i}] = 0$, one can see $[\theta^{[0,r]}_{Y_0\#Y_1}] = 0$.

Thank you!