Lifts of holonomy representations and the volume of a link complement

Hiroshi Goda

(Tokyo University of Agriculture and Technology)

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$\S 1$. Introduction

Mostow rigidity $\longrightarrow$

a complete hyperbolic 3-mfd M is geometrically determined by its fundamental group $\pi_1(M)$.

$\longrightarrow$ the hyperbolic volume vol(M) should theoretically be computable from a presentation of $\pi_1(M)$.

$$\begin{array}{ccc} \pi_1(M) & \longrightarrow & \mathrm{vol}(M) \\ \downarrow & & \downarrow \mathcal{A}_n \\ H_1(M) & \longrightarrow & \text{Alexander polynomial} \end{array}$$

$\S$ 2. Background

Reidemeister torsion twisted by the ajoint representation

associated with an $\mathrm{SL}(2, \mathbb{C})$-rep. [Porti]

$\downarrow$

A relation between the non-acyclic R-torsion and a zero of the

acyclic R-torsion (Yamaguchi [Y, '08]) (knot case)

$\downarrow$

Twisted Alexander invariants and non-abelian R-torsion (Dubois &

Yamaguchi [DY]) (link case)

A volume formula of a closed hyperbolic 3-manifold

using the Ray-Singer analytic torsion (by Müller ['12])

$\downarrow$[Müller '93]

A volume formula of a cusped hyperbolic 3-manifold

using R-torsion (by Menal-Ferrer & Porti [MFP, '14])

$\downarrow$ [Y]

A volume formula of a hyperbolic knot complement using the

twisted Alexander invariant [G]

$\downarrow$ [DY]

Link case

§3. The Alexander polynomial

K: a knot in S^3, $E(K) := S^3 - \overset{\circ}{N}(K)$

$\pi_1(E(K)) := G(K) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle$: Wirtinger pre.

$$\alpha : G(K) \longrightarrow H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle \text{ : epimorphism}$$

$$\mu(\text{meridian}) \longmapsto t$$

That is, $\alpha(x_1) = \alpha(x_2) = \cdots = \alpha(x_k) = t.$

α induces the ring homomorphism between group rings over $\mathbb{Z}$:

$$\widetilde{\alpha} : \mathbb{Z}G(K) \to \mathbb{Z}[t^{\pm 1}]$$

$F_k = \langle x_1, x_2, \cdots, x_k \rangle$: the free group of rank k

$\phi : F_k \to G(K)$: epi. $\overset{\text{extend by linearity}}{\longrightarrow}$ $\widetilde{\phi} : \mathbb{Z}F_k \to \mathbb{Z}G$

$\Phi : \widetilde{\alpha} \circ \widetilde{\phi} : \mathbb{Z}F_k \to \mathbb{Z}[t^{\pm 1}]$: ring homo.

$$M := \Phi\Big(\frac{\partial r_i}{\partial x_j}\Big) \ \ (\in M_{k-1,k}(\mathbb{Z}[t^{\pm 1}])) : \ \text{the Alexander matrix,}$$

where $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}, \ \ \dfrac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij}x_i^{-1}, \ \ \dfrac{\partial (uv)}{\partial x_j} = \dfrac{\partial u}{\partial x_j} + u\dfrac{\partial v}{\partial x_j}$

(Fox's free differential calculus)

Example. $\dfrac{\partial}{\partial x}(xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1})$

$$= \frac{\partial x}{\partial x} + x\frac{\partial}{\partial x}(y^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1})$$

$$= 1 + x\Big(\frac{\partial y^{-1}}{\partial x} + y^{-1}\frac{\partial}{\partial x}(x^{-1}yxy^{-1}xyx^{-1}y^{-1})\Big)$$

$$= 1 + xy^{-1}\Big(\frac{\partial x^{-1}}{\partial x} + x^{-1}\frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1})\Big)$$

$$= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}\frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1}) = \cdots =$$

$$1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}$$

$$\stackrel{\widetilde{\alpha}}{\longmapsto} 1 - t^{-1} + 1 + 1 - t = -\frac{1}{t} + 3 - t$$

M_ℓ: the sub matrix of M deleting ℓ-column

Definition. The Alexander polynomial : $\Delta_K(t) = \det M_\ell$.

Example. $K = 4_1$ (figure 8 knot),

$$G(K) = \langle x,\, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle$$

4_1

$$\Delta_K(t) = \det\left(-\frac{1}{t} + 3 - t\right) = -\frac{1}{t} + 3 - t$$

§4. The twisted Alexander invariant

$\rho : G(K) \to \mathrm{SL}(n, \mathbb{C})$: rep.

$\overset{\text{extend by linearity}}{\longrightarrow} \quad \widetilde{\rho} : \mathbb{Z}G(K) \to M(n, \mathbb{C})$

$\widetilde{\rho} \otimes \widetilde{\alpha} : \mathbb{Z}G(K) \to M(n, \mathbb{C}[t^{\pm 1}])$: tensor prod., ring homo.

$\Phi : (\widetilde{\rho} \otimes \widetilde{\alpha}) \circ \widetilde{\phi} : \mathbb{Z}F_k \to M(n, \mathbb{C}[t^{\pm 1}])$: ring homo.

$$M := \Phi\Big(\frac{\partial r_i}{\partial x_j}\Big) \ \ (\in M_{n(k-1),nk}(\mathbb{C}[t^{\pm 1}])) :$$

The Alexander matrix associated with ρ

M_ℓ: the sub matrix of M deleting 'ℓ'-column

Definition. The twisted Alexander invariant :

$$\Delta_{K,\rho}(t) = \frac{\det M_\ell}{\det \Phi(x_\ell - 1)}$$

Example. $K = 4_1$ (figure 8 knot), $\exists \rho : G(K) \to \mathrm{SL}(2, \mathbb{C})$ s.t.

$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: X, \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} =: Y$, where $u^2+u+1 = 0$.

$$\frac{\partial r}{\partial x} = 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}$$

$$\Phi\left(\frac{\partial r}{\partial x}\right) = I - \frac{1}{t}XY^{-1}X^{-1} + XY^{-1}X^{-1}Y + XY^{-1}X^{-1}YXY^{-1} - tXY^{-1}X^{-1}YXY^{-1}XYX^{-1}$$

$$\Phi(y-1) = tY - I$$

$$\Delta_{K,\rho}(t) = \frac{\det \Phi\left(\frac{\partial r}{\partial x}\right)}{\det \Phi(y-1)} = \frac{1/t^2(t-1)^2(t^2-4t+1)}{(t-1)^2} \doteq t^2 - 4t + 1$$

K: a hyperbolic knot in the 3-sphere.

$$\rho_n^{\pm} : G(K) \xrightarrow{\text{Hol.}} \mathrm{PSL}(2,\mathbb{C}) \xrightarrow{\pm} \mathrm{SL}(2,\mathbb{C}) \xrightarrow[\sigma_n]{\text{irr.}} \mathrm{SL}(n,\mathbb{C})$$

$\Delta_{K,\rho_n^{\pm}}(t)$: the twisted Alexander invariant

Set $\mathcal{A}^{\pm}_{K,2k}(t) := \dfrac{\Delta_{K,\rho^{\pm}_{2k}}(t)}{\Delta_{K,\rho^{\pm}_{2}}(t)}$ and $\mathcal{A}_{K,2k+1}(t) := \dfrac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}$

$(\Delta_{K,\rho^{+}_{2k+1}}(t) = \Delta_{K,\rho^{-}_{2k+1}}(t))$

Theorem [G].

$$\lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{A}^{\pm}_{K,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}$$

$\$\,5$. Lifts of the holonomy representation

M: an oriented, complete, hyperbolic 3-manifolds of finite volume with $\partial\overline{M} \cong T_1^2 \cup \cdots \cup T_b^2$.

$\mathrm{Hol}_M : \pi_1(M,p) \to \mathrm{Isom}^+\mathbb{H}^3 \cong \mathrm{PSL}(2,\mathbb{C})(= \mathrm{SL}(2,\mathbb{C})/\{\pm 1\})$

η : a lift of Hol_M to $\mathrm{SL}(2,\mathbb{C})$. Thus we have a map:

$$\mathrm{Hol}_{(M,\eta)} : \pi_1(M,p) \to \mathrm{SL}(2,\mathbb{C}).$$

Definition. (1) η is positive on T_i^2 if for <u>all</u> $\gamma \in \pi_1(T_i^2, p)$, we have

$$\text{trace } \mathrm{Hol}_{(M,\eta)}(\gamma) = +2.$$

(2) A lift η is acyclic if η is non-positive on each T_i^2.

Proposition [MFP]. M_γ : the mfd obtained by a Dehn filling along γ. A lift η of Hol_M to $\mathrm{SL}(2,\mathbb{C})$ extends to a lift Hol_{M_γ} to $\mathrm{SL}(2,\mathbb{C})$ $\iff$ trace $\mathrm{Hol}_{(M,\eta)}(\gamma) = -2$.

Proposition [MFP]. Assume that, for each boundary component T^2_ℓ,

$$(*)\begin{cases} \text{the map } \iota_* : H_1(T^2_\ell; \mathbb{Z}/2\mathbb{Z}) \to H_1(M; \mathbb{Z}/2\mathbb{Z}) \\ \text{induced by the inclusion } \iota \text{ has non trivial kernel,} \end{cases}$$

then all lifts of Hol_M are non-positive on each T^2_ℓ, i.e., all lifts are acyclic.

Example. a knot complement. ($\because$ $\exists$ a Seifert surface), in general any 3-manifold with $\partial M = T^2$ (i.e., $b = 1$) [eg. Hempel 6.8].

$6. Results of Dubois & Yamaguchi

M: an oriented, complete, hyperbolic 3-manifolds of finite volume with $\partial\overline{M} \cong T_1^2 \cup \cdots \cup T_b^2$.

They investigated a relation between the R-tosion and the twisted Alexander invariants under the following condition:

$$(**) \begin{cases} \text{the hom. } i_* : H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \text{ is onto} \\ \text{and its restriction } (i|_{T_\ell^2})_* \text{ has rank one for all } \ell. \end{cases}$$

$$(*) \Longleftarrow (**) \Longleftarrow (***)$$

$$(***) \begin{cases} M \text{ is the complement of a algebraically split link } L \\ \text{(homologically trivial link) in a homology 3-sphere.} \end{cases}$$

i.e., $L = K_1 \cup \cdots \cup K_b$ such that $\ell k(K_i, K_j) = 0$ for all $i,\ j$.

Example. (algebraically split link) Every knot.

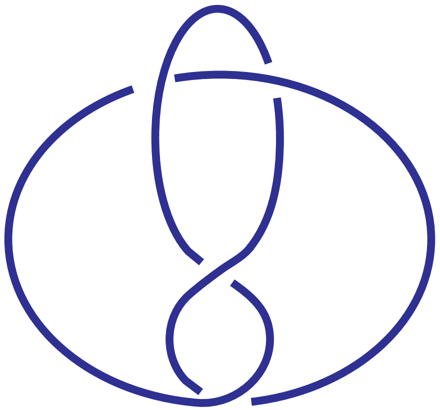
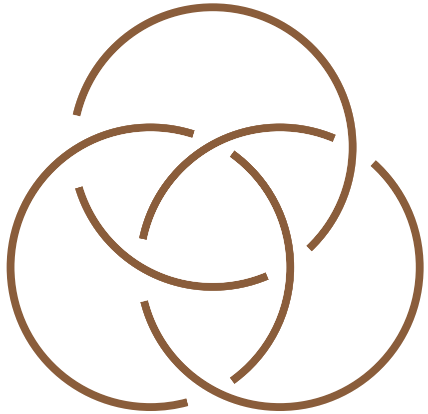


Figure 8 (4_1)

Whitehead link



Borromean link

Suppose $\exists \varphi : \pi_1(M) \to \mathbb{Z}^n$ epi. & $\exists (a_1^{(\ell)}, \ldots, a_n^{(\ell)}) \in \mathbb{Z}^n_{>0}$

such that

$$\varphi(\pi_1(T_\ell^2)) = \langle t_1^{a_1^{(\ell)}} \cdots t_n^{a_n^{(\ell)}} \rangle \quad \text{for all } T_\ell^2.$$

Theorem [DY]. Suppose $(**)$. Then, we have:

$$\lim_{t_1,\ldots,t_n \to 1} \frac{\Delta_{M,\varphi\otimes Ad\circ\rho_2}(t_1,\ldots,t_n)}{\prod_{\ell=1}^b (t_1^{a_1^{(\ell)}} \cdots t_n^{a_n^{(\ell)}} - 1)} = (-1)^b \cdot \mathbb{T}(M; Ad\circ\rho_2; \{\lambda_\ell\}),$$

where λ_ℓ is a longitudinal basis on T_ℓ^2.

$7. A volume formula of a link complement

Proposition.[MFP'12] Let M be a hyperbolic mfd with b cusps. Then,

$$\dim_{\mathbb{C}} H_0(M;\rho_n) = 0,$$

$$\dim_{\mathbb{C}} H_1(M;\rho_n) = a,$$

$$\dim_{\mathbb{C}} H_2(M;\rho_n) = a,$$

where $a = b$ if n is odd, and $a = \sharp$ of cusps for which the lift of the holonomy is positive if n is even,

i.e., η is acyclic $\Rightarrow \dim_{\mathbb{C}} H_i(M;\rho_n) = 0$ when n is even.

Proposition.[MFP'14] Let $G_\ell (< \pi_1(M))$ be some fixed realization of $\pi_1(T^2_\ell)$ as a subgroup of $\pi_1(M)$. For each T^2_ℓ choose a non-trivial cycle $\theta_\ell \in H_1(T^2_\ell;\mathbb{Z})$, and non-trivial vector $w_\ell \in V_n$ fixed by $\rho_n(G_\ell)$.

1. A basis of $H_1(M;\rho_n)$ is given by $i_{\ell *}([w_\ell \otimes \theta_\ell]),\ (\ell = 1,\dots,b)$.
2. A basis of $H_2(M;\rho_n)$ is given by $i_{\ell *}([w_\ell \otimes T^2_\ell]),\ (\ell = 1,\dots,b)$.

Set

$$\mathcal{T}_{2k+1}(M,\eta) := \frac{\mathbb{T}(M;\rho_{2k+1};\{\theta_\ell\})}{\mathbb{T}(M;\rho_3;\{\theta_\ell\})},$$
$$\mathcal{T}_{2k}(M,\eta) := \frac{\mathbb{T}(M;\rho_{2k};\{\theta_\ell\})}{\mathbb{T}(M;\rho_2;\{\theta_\ell\})}.$$

Theorem [MFP'14]. (1) For any η,

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M,\eta)|}{(2k+1)^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

(2) If η is acyclic, then

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k}(M,\eta)|}{(2k)^2} = \frac{\mathrm{Vol}(M)}{4\pi}$$

Applying the idea of the proof of Dubois $\&$ Yamaguchi's Theorem,

we have:

Theorem[G]. Let L be an algebraically split link. Then,

$$\lim_{k\to\infty} \frac{\log|\mathcal{A}_{L,2k+1}(1,\ldots,1)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{A}_{L,2k}(1,\ldots,1)|}{(2k)^2} = \frac{\mathrm{Vol}(L)}{4\pi}$$

§8. The irreducible representation σ_n of SL(2,$\mathbb{C}$)

V_n : the vector space of 2-variables homogeneous polynomials on $\mathbb{C}$ with degree $n-1$, i.e.,

$$\begin{aligned} V_n &= \{a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1} \mid a_1, \ldots, a_n \in \mathbb{C}\} \\ &= \mathrm{span}_{\mathbb{C}} \langle x^{n-1}, x^{n-2}y, x^{n-3}y^2, \ldots, xy^{n-2}, y^{n-1} \rangle \end{aligned}$$

The action of $A \in$ SL(2,$\mathbb{C}$) is expressed as:

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p\Big(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\Big) \quad \text{for } p \begin{pmatrix} x \\ y \end{pmatrix} \in V_n$$

(V_n, σ_n) : the rep. given by this action of $\mathrm{SL}(2,\mathbb{C})$ where σ_n means the homomorphism from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{GL}(V_n)$.

(The fact is, $\sigma_n(A) \in$ **SL**$(n, \mathbb{C})$.)

Theorem. Every irreducible n-dim. representation of $\mathrm{SL}(2, \mathbb{C})$ is unique and equivalent to (V_n, σ_n).

Example. Set $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \big(\in \mathbf{SL}(2, \mathbb{C})\big)$. $X^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

$$X^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ y \end{pmatrix}$$

$$p\big(X^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\big) = p \begin{pmatrix} x - y \\ y \end{pmatrix}$$

$$(x-y)^2 = 1x^2 - 2xy + 1y^2,\ (x-y)y = 1xy - y^2,\ y^2 = 1y^2$$

$$\sigma_3(X) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^T$$

$$
\begin{aligned}
(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3, \\
(x-y)^2y &= \qquad\quad x^2y - 2xy^2 + y^3, \\
(x-y)y^2 &= \qquad\qquad\qquad xy^2 - y^3, \\
y^3 &= \qquad\qquad\qquad\qquad\quad y^3
\end{aligned}
$$

$$
\sigma_4(X) = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T
$$

In the same way, we have:

$$\sigma_n(X) = \begin{pmatrix} 1 & -{}_{n-1}\mathbf{C}_1 & {}_{n-1}\mathbf{C}_2 & \cdots & (-1)^{n-2}{}_{n-1}\mathbf{C}_{n-2} & (-1)^{n-1} \\ 0 & 1 & -{}_{n-2}\mathbf{C}_1 & \cdots & (-1)^{n-3}{}_{n-2}\mathbf{C}_{n-3} & (-1)^{n-2} \\ 0 & 0 & 1 & \cdots & (-1)^{n-4}{}_{n-3}\mathbf{C}_{n-4} & (-1)^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}^T$$

Example. $K = 4_1$, $G(K) = \langle x,\, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle$.

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = X,\ \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} = Y$$

: holonomy rep. where $(u^2 + u + 1 = 0)$.

$$\sigma_n(Y) = \begin{pmatrix} 1 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & 0 \\ u & 1 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots & & 0 \\ u^2 & 2u & 1 & 0 & \cdots\cdots\cdots\cdots\cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & \\ u^{n-2} & {}_{n-2}\mathbf{C}_1 u^{n-3} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & 1 & 0 \\ u^{n-1} & {}_{n-1}\mathbf{C}_1 u^{n-2} & {}_{n-1}\mathbf{C}_2 u^{n-3} & \cdots \quad \cdots & {}_{n-1}\mathbf{C}_{n-2} u & 1 \end{pmatrix}^T$$

§9. Experimentation using a computer $K = 4_1$, $\mathrm{Vol}(K) = 2.0298832\cdots$

$$\mathrm{Vol}(K) = 4\pi \lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2}$$

$$\Delta_{K,\rho_2}(t) = \frac{1/t^2(t-1)^2(t^2-4t+1)}{(t-1)^2} \doteq t^2 - 4t + 1$$

$$\Delta_{K,\rho_4}(t) = \frac{1}{t^4}(t^2-4t+1)^2 \doteq (t^2-4t+1)^2$$

$$\mathcal{A}_{K,4}(t) = \frac{\Delta_{K,\rho_4}(t)}{\Delta_{K,\rho_2}(t)} = \frac{(t^2-4t+1)^2}{t^2-4t+1} = t^2-4t+1$$

$$4\pi\frac{\log|\mathcal{A}_{K,4}(1)|}{4^2} = \frac{\pi\log 2}{4} \approx 0.54440\cdots\cdots$$

$$\mathbf{Vol}(K) = 4\pi \lim_{k\to\infty} \frac{\log |\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2}$$

$$\Delta_{K,\rho_3}(t) = -1/t^3(t-1)(t^2-5t+1) \doteq (t-1)(t^2-5t+1)$$

$$\Delta_{K,\rho_5}(t) = -\frac{1}{t^5}(t-1)(t^4-9t^3+44t^2-9t+1)$$

$$\doteq (t-1)(t^4-9t^3+44t^2-9t+1)$$

$$\mathcal{A}_{K,5}(t) = \frac{\Delta_{K,\rho_5}(t)}{\Delta_{K,\rho_3}(t)} = \frac{t^4-9t^3+44t^2-9t+1}{t^2-5t+1}$$

$$4\pi\frac{\log |\mathcal{A}_{K,5}(1)|}{5^2} = \frac{4\pi \log \frac{28}{3}}{5^2} \approx 1.12273\cdots\cdots$$

n(**even**)	$\frac{4\pi \log \lvert \mathcal{A}^{+}_{K,n}(1) \rvert}{n^2}$	$\frac{4\pi \log \lvert \mathcal{A}^{-}_{K,n}(1) \rvert}{n^2}$	n(**odd**)	$\frac{4\pi \log \lvert \mathcal{A}_{K,n}(1) \rvert}{n^2}$
4	$0.54439\cdots$	$1.40724\cdots$	5	$1.12273\cdots$
8	$1.66441\cdots$	$1.84668\cdots$	9	$1.76436\cdots$
12	$1.86678\cdots$	$1.94781\cdots$	13	$1.90158\cdots$
16	$1.93822\cdots$	$1.98381\cdots$	17	$1.95494\cdots$
20	$1.97121\cdots$	$2.00039\cdots$	21	$1.98076\cdots$
24	$1.98914\cdots$	$2.00940\cdots$	25	$1.99522\cdots$
28	$1.99994\cdots$	$2.01483\cdots$	29	$2.00412\cdots$
32	$2.00696\cdots$	$2.01836\cdots$	33	$2.00999\cdots$

§10. On Evaluation by Root of unity (in progress)

Theorem [DY'12]. M_q: q-fold cyclic covering of M, and $s = t^q$

$$\Rightarrow \quad \Delta_{M_q}^{\hat{\rho}\otimes\hat{\alpha}}(s) = \prod_{k=0}^{q-1} \Delta_M^{\rho\otimes\alpha}(e^{2\pi k\sqrt{-1}/q}t)$$

Corollary.

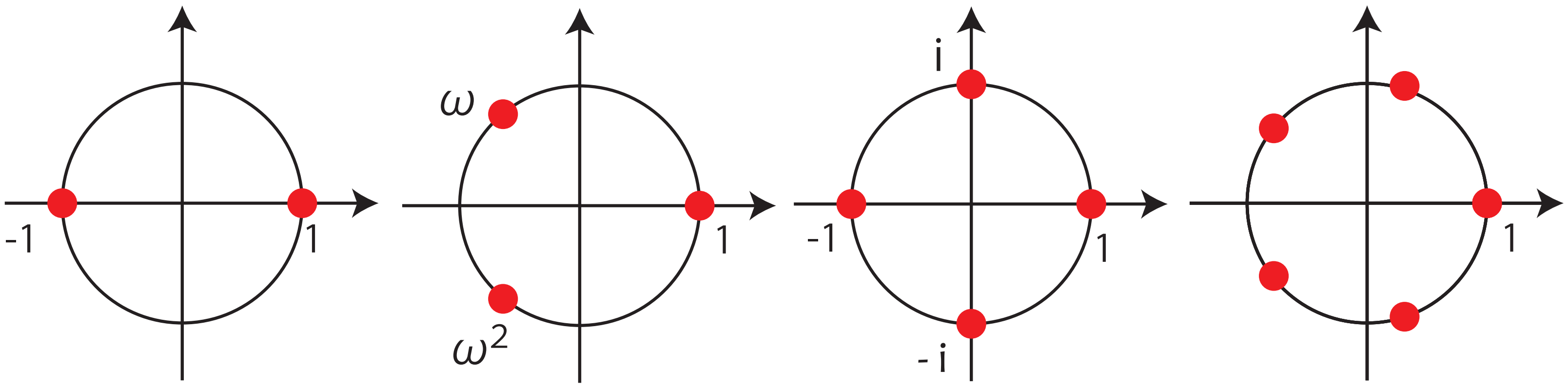
$$\lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(-1)|}{(2k)^2} = \frac{\mathrm{Vol(K)}}{4\pi}.$$

$(\because)$ Suppose $t = 1,\ q = 2$, then $s = 1$.

$$\Delta_{M_2}^{\hat{\rho}\otimes\hat{\alpha}}(1) = \Delta_M^{\rho\otimes\alpha}(1) \cdot \Delta_M^{\rho\otimes\alpha}(-1)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$2\mathrm{Vol}(M) \qquad \mathrm{Vol}(M) \qquad \mathrm{Vol}(M)$$



3-rd root of unity $\Rightarrow$ OK

4-th root of unity $\Rightarrow$ OK

5-th root of unity $\Rightarrow$???

Thank you very much for your attention !