Supersymmetric analysis of
discrete magnetic Schrödinger operators

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Abstract

In the last symposium (Jul. 2001), T. Shirai talked about the spectrum of the infinitely extended Sierpinski lattice [7, 3]. Their results are based on some relations between the spectra of an infinite regular graph and its line-graph. In this report, we extend their results to the cases of discrete magnetic Schrödinger operators on infinite regular graphs.

1 Definitions

A graph $G = (V(G), A(G))$ is a pair of the vertex set $V(G)$ and the oriented edge set $A(G)$. We say that two vertices $x, y$ are adjacent if there exists an edge $\alpha = xy$ which connects them. We denote $x \sim y$ if $x$ and $y$ are adjacent. Let $\alpha \in A(G)$,
which has direction from the origin \( x \in V(G) \) to the terminus \( y \in V(G) \). Then, we denote \( \alpha = xy, \ o(\alpha) = x \) and \( t(\alpha) = y \). We denote \( \overline{\alpha} \) the reverse edge of \( \alpha \), that is, \( \overline{\alpha} = yx \). We assume that \( \overline{\alpha} \in A(G) \) provided \( \alpha \in A(G) \).

Let

\[
A_x(G) = \{ \alpha \in A(G); o(\alpha) = x \},
\]

\[
\deg(x) = \# A_x(G).
\]

We call \( \deg(x) \) the degree of \( x \). If there exists a constant \( d \) such that \( \deg(x) = d \) for all \( x \in V(G) \), then the graph \( G \) is called \( d \)-regular. Regularity has important role in this report.

Throughout this report, we assume that (i) \( G \) is locally finite, that is, \( \deg(x) < \infty \) for all \( x \in V(G) \); (ii) \( G \) has no loop or multiple edge.

We define the discrete Laplacian \( \triangle G \) on a graph \( G \). We work on the Hilbert space

\[
\mathcal{l}^2(G) = \left\{ f : V(G) \to \mathbb{C}; \sum_{x \in V(G)} |f(x)|^2 < \infty \right\}.
\]

The discrete Laplacian \( \triangle G \) acts \( f \in \mathcal{l}^2(G) \) as follow:

\[
(\triangle G f)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} \left[ f(t(\alpha)) - f(x) \right]
= \frac{1}{\deg(x)} \left[ \sum_{\alpha \in A_x(G)} f(t(\alpha)) \right] - f(x)
= \frac{1}{\deg(x)} \left[ \sum_{y \sim x} f(y) \right] - f(x).
\]

We denote \( \text{Spec}(\triangle G) \) the spectrum of \( \triangle G \).

Remark 1. It is a well-known fact that \( \text{Spec}(\triangle G) \subset [0, 2] \). We remark that all of the operators in this report is bounded.

We use three graphs associated with a given graph \( G \). First one is the subdivision graph \( S(G) \). See Figure 2. We make \( S(G) \) by adding one vertex \( |\alpha| \) at the midpoint of each edges \( \alpha \in A(G) \). We note \( |\alpha| = |\alpha| \). Formally, we give

\[
V(S(G)) = V(G) \cup E(G),
A(S(G)) = \{ x\alpha, \alpha x; \ x \in V(G), \alpha \in A_x(G) \}.
\]

Here, we put

\[
E(G) = \{ |\alpha|; \ \alpha \in A(G) \}.
\]

We call \( E(G) \) the (unoriented) edge set of \( G \).
Figure 2: a graph $G$ and its subdivision graph $S(G)$

Second one is the line graph $L(G)$. See Figure 3. A vertex of $L(G)$ is an edge of $G$;

$$V(L(G)) = E(G).$$

The vertices $|\alpha|, |\beta| \in V(L(G))$ are adjacent on $L(G)$ if and only if $\alpha, \beta \in A(G)$ are adjacent on $G$;

$$A(L(G)) = \{\alpha\beta; \alpha, \beta \in A(G), \alpha \sim \beta\}.$$
We have a natural question on the relation among the spectra of the four laplacians, $\Delta_G, \Delta_{S(G)}, \Delta_{L(G)}, \Delta_{P(G)}$. Yu. Higuchi and T. Shirai gave the answer. In the next section, we review their results (See, [3, 7]).

2 Higuchi and Shirai’s results

In the last symposium (Jul. 2001), T. Shirai talked about the spectrum of the infinitely extended Sierpinski lattice $S_n$ [3, 7]. On the spectrum, the following theorem by Fukushima and Shima, and Teplyaev are known.

**Theorem 2 (Fukushima and Shima (1992), Teplyaev (1998)).**

$$\text{Spec}(-\Delta_{S_n}) = \bigcup_{k=0}^{\infty} \left[ g^{-k}\left(\frac{n+1}{2n}\right) \cup g^{-k}\left(\frac{n+3}{2n}\right) \right] \cup \left\{ \frac{n+1}{n} \right\}$$

Here, $g(x) = -2nx^2 + (n+3)x$. 

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They proved this theorem using approximation by finite lattices. Higuchi and Shirai gave new proof based on some relations between the spectra of an infinite regular graph and its line-graph, without any approximation. They proved Theorem 2 as a conclusion of the following four lemmas.

Lemma 3 (Shirai [6]). Let $G$ be a $d$-regular graph with $d \geq 3$. We have
\[
\text{Spec}(-\Delta_{S(G)}) = \psi^{-1} (\text{Spec}(-\Delta_G)) \cup \{1\}
\]
Here, $\psi(x) = 2(2x - x^2)$.

Lemma 4 (Shirai [6]). Let $G$ be a $d$-regular graph with $d \geq 3$. We have
\[
\text{Spec}(-\Delta_{L(G)}) = \frac{2}{2d - 2} \text{Spec}(-\Delta_G) \cup \left\{\frac{d + 2}{d}\right\}.
\]
Shirai proved Lemma 3 and Lemma 4 using the weak Weyl criterion on essential spectrum; He constructed a weak sequence for $-\Delta_{S(G)}$ from a eigenvector of $-\Delta_G$ and vice versa.

Lemma 5 (Shirai [6]). Let $G$ be $d$-regular with $d \geq 3$. We have
\[
\text{Spec}(-\Delta_{P(G)}) = \phi^{-1} (\text{Spec}(-\Delta_G)) \cup \{1\} \cup \left\{\frac{n + 3}{n + 1}\right\},
\]
Here, $\phi(x) = -dx^2 + (d + 2)x$.

This Lemma 5 can be obtained from Lemmas 3 and 4 and the fact Eq. (1).

Lemma 6 (Higuchi and Shirai [3]). Let $S_n$ be $n$-dim. Sierpinski lattice. Then, there exists a $(n + 1)$-regular graph $G_n$ such that
\[
P(G_n) = G_n \quad \text{and} \quad S_n = L(G_n).
\]
Outline of HS’s proof of Theorem 2. By Lemmas 5 and 6, we have the equation of the set $\text{Spec}(-\Delta_{G_n})$,
\[
\text{Spec}(-\Delta_{G_n}) = \phi^{-1} (\text{Spec}(-\Delta_{G_n})) \cup \{1\} \cup \left\{\frac{n + 3}{n + 1}\right\},
\]
Since the map from $\text{Spec}(-\Delta_{G_n})$ to RHS is a contraction map, there exists a unique solution of this equation and we can derive $\text{Spec}(-\Delta_{G_n})$ exactly. Since $S_n = L(G_n)$, Lemmas 4 implies the desired result. For more detail, see Refs. [3, 7, 6].

Our goals in this report are (i) we give another simplest proof of Lemma 3 and Lemma 4 using supersymmetry; (ii) we extend these to magnetic Schrödinger case. We devote the next section to (i) and do the last section to (ii).

Remark 7. Shirai proved that 1 is always the infinitely degenerate eigenvalue of $-\Delta_{S(G)}$ and $(d + 2)/d$ is always the infinitely degenerate eigenvalue of $-\Delta_{L(G)}$. Though we are interested in these eigenvalues, we omit the discussions on these (Remark 18).
3 Supersymmetry

In this section, we give our new proof on the relations among the spectra of $G$, $S(G)$ and $L(G)$. We summarize the facts on supersymmetry. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $A$ be a densely defined closed linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

**Theorem 8** (Deift [1]). *We have*

$$\text{Spec } (AA^*) \setminus \{0\} = \text{Spec } (A^*A) \setminus \{0\}$$

*with taking account of multiplicity.*

**Corollary 9** (I. Shigekawa [5]). *Let*

$$D = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2$$

*and*

$$H = D^2 = \begin{pmatrix} A^*A & 0 \\ 0 & A^*A \end{pmatrix}.$$  

*Then, we have that*

$$\text{Spec } (H) \setminus \{0\} = \text{Spec } (A^*A) \setminus \{0\} = \text{Spec } (AA^*) \setminus \{0\}$$

*and*

$$\text{Spec } (D) \setminus \{0\} = (\sqrt{\text{Spec } (H)} \cup -\sqrt{\text{Spec } (H)}) \setminus \{0\}.$$  

In physics literatures, $D$ is called a supercharge and $H$ is called a SUSY-Hamiltonian. We remark that we can ignore the condition, which $A$ must be densely defined and closed, since all of our operators is bounded (Remark 1).
3.1 the spectra of bipartite graph

We start to prove a well-known fact on spectrum of graph using supersymmetry. Let $G$ be bipartite, that is,

$$V(G) = V_1 \cup V_2,$$
$$V_1 \cap V_2 = \emptyset,$$
$$x \not\sim y \text{ for all } x, y \in V_i \ (i = 1, 2).$$

![Bipartite Graph and Non-Bipartite Graph](image)

Figure 7: bipartite graph and non-bipartite graph

Lemma 10. If $G$ is bipartite, then $\text{Spec}(-\Delta_G)$ is symmetric w.r.t. 1.

Proof. We have $l^2(G) = l^2(V_1) \oplus l^2(V_2)$. Let $\phi_{12}$ be an operator from $l^2(V_1)$ to $l^2(V_2)$ defined by

$$(\phi_{12}f)(y) = \frac{1}{\deg(y)} \sum_{x \sim y} f(x).$$

Then, we have

$$\Delta_G + 1 = \begin{pmatrix} 0 & \phi_{12} \\ \phi_{12}^* & 0 \end{pmatrix}.$$

Thus, $\text{Spec}(-\Delta_G - 1)$ is symmetric w.r.t. 0.

Similarly, we define the operator $\phi_{21}$ from $l^2(V_2)$ to $l^2(V_1)$ by

$$(\phi_{21}g)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} g(y).$$

Then we have $\phi_{21} = \phi_{12}^*$. These operators $\phi_{12}$ and $\phi_{21}$ are used in our new proofs in the following.
3.2 the spectra of subdivision graph

We consider the relation between $G$ and $S(G)$.

**Lemma 11 (SUSY version of Lemma 3).** For arbitrary graph $G$, we have

$$\text{Spec}(-\Delta_{S(G)}) = \psi^{-1}(\text{Spec}(-\Delta_G)) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

![Figure 8: the subdivision graph as a bipartite graph](image)

**Proof.** $S(G)$ is bipartite (See, Fig. 8). In fact, we can take $V_1 = V(G)$ and $V_2 = E(G)$;

$$V(S(G)) = V(G) \cup E(G) = V_1 \cup V_2.$$

Thus, we have

$$\Delta_{S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix}.$$

Moreover, we can see

$$(\Delta_{S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(\Delta_G + 2) & 0 \\ 0 & \phi_{12}\phi_{21} \end{pmatrix}.$$ 

Indeed, we can write $\phi_{12}$ and $\phi_{21}$ as follows:

$$(\phi_{12}f)(|\alpha|) = \frac{1}{2}[f(t(\alpha)) + f(o(\alpha))],$$

$$(\phi_{21}g)(x) = \frac{1}{\text{deg}(x)} \sum_{\alpha \in A_x(G)} g(|\alpha|).$$
Therefore,

\[
(\phi_{21}\phi_{12}f)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} [\phi_{12}f](|\alpha|)
\]

\[
= \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} \frac{1}{2}[f(t(\alpha)) + f(o(\alpha))]
\]

\[
= \frac{1}{2} \left( \frac{1}{\deg(x)} \left[ \sum_{\alpha \in A_x(G)} f(t(\alpha)) \right] + f(x) \right)
\]

\[
= \frac{1}{2} (\triangle_G + 2)f(x)
\]

Thus, we obtain

\[
\text{Spec} (\triangle_{S(G)} + 1) \setminus \{0\} = \pm \sqrt{\text{Spec} \left( \frac{1}{2}(\triangle_G + 2) \right) \setminus \{0\}}.
\]

Thus, the spectral mapping theorem implies Lemma 1.

Via supersymmetry, we can not see that 1 is an infinitely degenerate eigenvalue of \(-\triangle_{S(G)}\). We need another discussion, but omit it here (Remark 18).

We remark that we do not need the regularity condition as in Lemma 3.

### 3.3 the spectra of line graph

We consider the relation between \(G\) and \(L(G)\).

**Lemma 12 (SUSY version of Lemma 4).** Let \(G\) be \(d\)-regular with \(d \geq 3\). Then, we have

\[
\text{Spec} (-\triangle_{L(G)}) = \frac{2}{2d - 2} \text{Spec}(-\triangle_G) \cup \left\{ \frac{d + 2}{d} \right\}.
\]

**Proof.** We use same \(\phi_{21}\) and \(\phi_{12}\) as in Lemma 1 and we have

\[
\triangle_{S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix} \quad \text{on } l^2(V_1) \oplus l^2(V_2).
\]

Here, \(V_1 = V(G), V_2 = E(G)\). We can identify \(E(G)\) and \(V(L(G))\). (See, Fig. 9.) If \(G\) is \(d\)-regular, then \(L(G)\) is \(2d - 2\)-regular. Therefore, \(l^2(L(G))\) and \(l^2(V_2)\) is unitary equivalent through the unitary operator \(U\) defined by

\[
U : l^2(L(G)) \rightarrow l^2(V_2), \quad Uf = \sqrt{d - 1} f.
\]
\[ V_1 = V(G) \]

\[ V_2 = E(G) \]

\[ \text{Figure 9: Identification between } E(G) \text{ and } V(L(G)) \]

Using this \( U \), we obtain

\[
(\triangle_{S(G)} + 1)^2 = \begin{pmatrix}
\frac{1}{2}(\triangle_G + 2) & 0 \\
0 & \begin{pmatrix} \frac{d-1}{d} & \frac{d}{d-1} \end{pmatrix} U^* \end{pmatrix}
\]

by direct computations. Thus,

\[
\text{Spec} \left( \frac{1}{2}(\triangle_G + 2) \right) \setminus \{0\} = \text{Spec} \left( \frac{d-1}{d} (\triangle_{L(G)} + \frac{d}{d-1}) \right) \setminus \{0\}.
\]

Via supersymmetry, we can not see that \((d+2)/d\) is an infinitely degenerate eigenvalue of \(-\triangle_{L(G)}\). We need another discussion, but omit it here (Remark 18).

\section{discrete magnetic Schrödinger operator}

For simplicity, we assume that the transition probability on \( G \) is isotropic. We can remove this restriction.

We introduce the space of 1-forms (vector potentials) on graph \( G \).

\[ C^1(G) = \{ \theta : A(G) \to \mathbb{R}; \; \theta(\alpha) = -\theta(\alpha) \}. \]

We define the discrete magnetic Schrödinger operator \( H_{\theta,G} \) with a 1-form \( \theta \) by

\[
H_{\theta,G} f(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} [e^{i\theta(\alpha)} f(t(\alpha)) - f(x)].
\]

\[
= \frac{1}{\deg(x)} \left[ \sum_{\alpha \in A_x(G)} e^{i\theta(\alpha)} f(t(\alpha)) \right] - f(x).
\]
Our problem is whether we can extend Lemma 11 and Lemma 12 for $H_{\theta,G}$.

**Remark 13.** In ordinary, $H_{\theta,G}$ is defined with the opposite sign. Then $H_{\theta,G}$ is non-negative. But, in this report, we want to compare it to the discrete Laplacian, so we choose this sign.

For later use, we introduce a quantity related to 1-form. Let $C$ be an oriented cycle on $G$, i.e.,

$$C = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \subset A(G)$$

such that $t(\alpha_i) = o(\alpha_{i+1})$ ($\alpha_n = \alpha_0$). For this cycle $C$ and $\theta$, we set

$$\Psi(\theta, C) = \sum_{\alpha \in C} \theta(\alpha).$$

We call this $\Psi(\theta, C)$ the magnetic flux through the cycle $C$.

### 4.1 the spectra of subdivision graph

**Lemma 14 (magnetic case of Lemma 11).** Let $G$ be an arbitrary graph. Assume that $\theta \in C^1(G)$ and $\theta_S \in C^1(S(G))$ satisfy that

$$\theta(\alpha) = \theta_S(o(\alpha)|\alpha|) + \theta_S(|\alpha|t(\alpha)) \quad \text{for all } \alpha \in A(G).$$

Then,

$$\text{Spec} \left( -H_{\theta_S,S(G)} \right) = \psi^{-1}(\text{Spec} \left( -H_{\theta,G} \right)) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

**Proof.** Let

$$(\phi_{12} f)(|\alpha|) = \frac{1}{2} \sum_{\beta \in \{\alpha, \pi\}} e^{i\theta_S(|\alpha|t(\beta))} f(t(\beta)),$$

$$(\phi_{21} g)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} e^{i\theta_S(|\alpha|)} g(|\alpha|).$$

Then, by direct computations, we obtain that

$$H_{\theta_S,S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix},$$

$$(H_{\theta_S,S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(H_{\theta,G} + 2) & 0 \\ 0 & \phi_{12}\phi_{21} \end{pmatrix}.$$ 

This implies the desired result. \qed
Remark 15. The assumption of this Lemma 14 is natural. These $\theta$ and $\theta_S$ has same magnetic flux for same cycle. Let
\[ C = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}, \]
\[ C_S = \{\alpha_{01}, \alpha_{02}, \alpha_{10}, \alpha_{11}, \ldots, \alpha_n, 0, \alpha_n, 1\}. \]
See Figure 10. Then, we have
\[
\Psi(\theta_S, C_S) = \sum_{\alpha \in C_S} \theta_S(\alpha) = \sum_{i=0}^{n} (\theta_S(\alpha_{i,0}) + \theta_S(\alpha_{i,1}))
\]
\[
= \sum_{i=0}^{n} \theta(\alpha_i) = \sum_{\alpha \in C} \theta(\alpha) = \Psi(\theta, C).
\]

Of course, via supersymmetry, we can not see that 1 is an infinitely degenerate eigenvalue of $-H_{\theta, S(G)}$. We need another discussion, but omit it here. (cf. Remark 18).

4.2 the spectra of line graph

Lemma 16 (magnetic case of Lemma 12 (Lemma 4)). Let $G$ be $d$-regular with $d \geq 3$. Assume that $\theta \in C^1(G)$, $\theta_S \in C^1(S(G))$, $\theta_L \in C^1(L(G))$ satisfy that
\[
\theta(\alpha) = \theta_S(o(\alpha)|\alpha|) + \theta_S(|\alpha|t(\alpha)) \text{ for all } \alpha \in A(G),
\]
\[
\theta_L(\alpha\beta) = \theta_S(|\alpha|x) + \theta_S(x|\beta|) \text{ for all } \alpha\beta \in A(L(G)).
\]
Then
\[
\text{Spec} (-H_{\theta, L(G)}) = \frac{2}{2d-2} \text{Spec} (-H_{\theta, G}) \cup \left\{ \frac{d+2}{d} \right\}.
\]

Proof. We use same identification between $E(G)$ and $V(L(G))$, $l^2(V_2)$ and $l^2(L(G))$ using $U$. Then, using same $\phi_{12}$ and $\phi_{21}$ in the proof of Lemma 14, we have
\[
(-H_{\theta, S(G)} + 1)^2 = \begin{pmatrix}
\frac{1}{2}(-H_{\theta, G} + 2) & 0 \\
0 & U \begin{pmatrix}
\frac{d-1}{d} & 0 \\
0 & \frac{d}{d-1}
\end{pmatrix} U^*
\end{pmatrix}.
\]
Thus, we can obtain the desired result. \hfill \square

Figure 11: Same cycle on \( G \) and \( L(G) \)

Remark 17. The assumption of this Lemma 16 is natural. These \( \theta \) and \( \theta_S \) has same magnetic flux for same cycle. Let

\[
C = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}, \\
C_L = \{\alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_n\alpha_0\}.
\]

Then, it holds that \( \Psi(\theta_L, C_L) = \Psi(\theta, C) \). The pair of \( C = \{\alpha_0, \alpha_1, \alpha_2\} \) and \( C_L = \{\alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_0\} \) in Figure 11 is an example. But, \( L(G) \) maybe has some cycles, which has no corresponding cycles on \( G \). The cycle \( \{\alpha_2\alpha_0, \alpha_0\beta, \beta\alpha_2\} \) in Figure 11 is an example. These cycles have zero magnetic flux.

Remark 18. As in Remark 7, though we omit the discussions on the eigenvalue 1 of \( -H_{\theta,G} \) and the eigenvalue \( (d + 2)/d \) of \( -H_{\theta,L(G)} \), these are corresponding to \( \ker \phi_{12} \) and \( \ker \phi_{21} \). In other words, these eigenvalues are zero-modes in SUSY context. So, we must investigate these states in detail [4].

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References


