Hida distribution construction of $P(\phi)_d$ ($d \geq 4$) indefinite metric quantum field models without BPHZ renormalization

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Abstract

By removing the divergent functions (i.e., the divergent Fynman graphs) a system of "Schwinger functions $!I$ which corresponds to a $(\Phi^4)_4$ Euclidean quantum field theory is constructed. The system of "Schwinger functions”, which are defined through the Hida distributions, satisfies the property of OS 2) (Euclidean covariance), OS 4) (Symmetry) and OS 5) (Cluster property). It does not satisfy OS 3) (Reflection positivity). Then, for the system of $!I$Schwinger functions $!I$ a possibility that it admits an analytic continuation to a system of $!I$Wightman functions $!I$satisfying the modified Wightman axioms is discussed.

1 Explanation of well known results on $d = 2$ by means of probabilistic words

Below, let $d = 4$ or $d = 2$ with an adequate understanding. Let $\hat{W}$ be the random variable such that $\hat{W}(\omega) \in S'(\mathbb{R}^d \to \mathbb{R})$, $P - a.e. \omega \in \Omega$, and for each $\varphi \in S(\mathbb{R}^d \to \mathbb{R})$, $< \hat{W}, \varphi >_{S',S}$ is a real valued Gaussian random variable (white noise on $\mathbb{R}^d$) satisfying

$$E \left[ < \hat{W}, \varphi >_{S',S} \right] = 0, \quad (1)$$

$$E \left[ < \hat{W}, \varphi_1 >_{S',S} \cdot < \hat{W}, \varphi_2 >_{S',S} \right] = \int_{\mathbb{R}^d} \varphi_1(x) \varphi_2(x) dx, \quad \forall \varphi_1, \varphi_2 \in S(\mathbb{R}^d \to \mathbb{R}). \quad (2)$$

Let $J_{d=2}$ be the integral kernel of the pseudo differential operator on $S(\mathbb{R}^d)$ such that $(-\Delta + 1)^{-\frac{1}{2}}$ with $\Delta = \Delta_{d=2}$ the Laplace operator on $\mathbb{R}^2$. For $\varphi, f_j \in S(\mathbb{R}^2 \to \mathbb{R})$, $j = 1, \ldots, n$, let

$$\phi(f_j) = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} f_j(x) J_{d=2}^{\frac{1}{2}}(x - y) dx \right) \hat{W}(y) dy, \quad (3)$$

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and
\[ <: \phi_{d=2}^4 :, \varphi > \]
\[ = \int_{\mathbb{R}^4} \varphi(x) \prod_{i=1}^{4} J_{\phi_{d=2}}^2(x - y_i) dx \]
\[ \cdot \dot{W}(y_1) \cdots \dot{W}(y_4) dy_1 \cdots dy_4. \quad (4) \]
Then,
\[ e^{-\lambda <: \phi_{d=2}^4 :, 1_\Lambda >} \in \cap_{p \geq 1} L^p(\Omega; P), \quad (6) \]
and the Schwinger function for \( d = 2 \)
\[ S_n(f_1, \ldots, f_n) \equiv \frac{1}{Z(\lambda; \Lambda)} \mathbb{E} \left[ \phi(f_1) \cdots \phi(f_n) e^{-\lambda <: \phi_{d=2}^4 :, 1_\Lambda >} \right], \quad (7) \]
is well defined.

2 Formulation for \( d = 4 \)

By changing the 2-dimensional space time Gaussian white noise process by 4-dimensional space time ones in the above discussion, and if we apply the same considerations to \((\Phi^4)_4\) quantum field model, then all the terms in \((\phi(f_1) \cdots \phi(f_n)) \ (<: \phi_{d=4}^4 :, 1_\Lambda >)^k\) will not have the right to be random variables.

Denote
\[ x \equiv (t, \overrightarrow{x}) \in \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4, \quad \xi \equiv (\tau, \overrightarrow{\xi}) \in \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4. \]

Let
\[ F[\varphi](\xi) = \int_{\mathbb{R}^4} e^{-2\pi \sqrt{-1} \varsigma \cdot x} \varphi(x) dx, \quad F^{-1}[\varphi](x) = \int_{\mathbb{R}^4} e^{2\pi \sqrt{-1} \xi \cdot \varphi(\xi)} d\xi \]

For each \( \epsilon > 0 \), let \( j_{\epsilon}^2(\xi), j_\epsilon(\xi), j_{-\epsilon}(\xi), \) and \( j(\xi) \), resp., be the symbol of the pseudo differential operators, resp., such that
\[ j_{\epsilon}^2(\xi) \equiv (|\xi|^2 + 1 + \epsilon(|\xi|^2 + 1)^2)^{-\frac{1}{2}}, \quad j_\epsilon(\xi) \equiv (|\xi|^2 + 1)^{-\frac{1}{2}}, \quad j_{-\epsilon}(\xi) \equiv (|\xi|^2 + 1)^{-\frac{1}{2}}, \quad j(\xi) \equiv (|\xi|^2 + 1). \]

and define
\[ (J_{\epsilon}^2 \varphi)(x) = F^{-1}(j_{\epsilon}^2 \hat{\varphi})(x), \quad (J_{\epsilon} \varphi)(x) = F^{-1}(j_{\epsilon} \hat{\varphi})(x), \]
\[ (J_{-\epsilon} \varphi)(x) = F^{-1}(j_{-\epsilon} \hat{\varphi})(x), \quad (J \varphi)(x) = F^{-1}(j \hat{\varphi})(x). \]
Symbolically
\[ J_{\epsilon}^2 = (-\Delta_{d=4} + 1 + \epsilon(-\Delta_{d=4} + 1)^2)^{-\frac{1}{2}}, \quad J_{\epsilon} = (-\Delta_{d=4} + 1)^{-\frac{1}{2}}, \]
\[ J_{-\epsilon} = (-\Delta_{d=4} + 1 + \epsilon(-\Delta_{d=4} + 1)^2)^{-1}, \quad J = (-\Delta_{d=4} + 1)^{-1}, \]

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where
\[ \Delta_{d=4} \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \text{with} \quad x = (t, x, y, z). \]

Let
\[ < \phi, \varphi >_{S'} = \int_{\mathbb{R}^4} \left( \int_{\mathbb{R}^4} \varphi(x) J_\frac{1}{2}(x - y) dx \right) \dot{W}(y) dy, \quad \text{for} \quad \varphi \in \mathcal{S}'(\mathbb{R}^4 \rightarrow \mathbb{R}). \]  
(8)

Also, define an \(\mathcal{S}'(\mathbb{R}^4 \rightarrow \mathbb{R})\)-valued random variable \(\phi_\epsilon(\omega)\) such that
\[ < \phi_\epsilon(\omega), \varphi >_{S'} = \int_{\mathbb{R}^4} \left( \int_{\mathbb{R}^4} \varphi(x) J_\frac{1}{2}(x - y) dx \right) \dot{W}(y) dy, \quad \text{for} \quad \varphi \in \mathcal{S}'(\mathbb{R}^4 \rightarrow \mathbb{R}), \]  
(9)

and for \(p \in \mathbb{N}\), define
\[ < \phi_\epsilon^p, \varphi > = \int_{(\mathbb{R}^4)^p} \left\{ \int_{\mathbb{R}^4} \varphi(x) \prod_{i=1}^p J_\frac{1}{2}(x - y_i) dx \right\} : \prod_{j=1}^p \dot{W}(y_j) : \prod_{j=1}^p dy_j, \]  
(10)

We then see that (cf. [FelMagRivSe], [CRiv])
\[(<: \phi_\epsilon^k, 1_A >)^k = \sum_{G' \in A'(k; \epsilon, \Lambda)} G' + \sum_{G \in A(k; \epsilon, \Lambda)} G, \]  
(11)

where \(A'(k; \epsilon, \Lambda)\) is a set of random variables such that for \(G'(k; \epsilon, \Lambda) \in A'(k; \epsilon, \Lambda)\) there exists an \(n \in \mathbb{N} \cup \{0\}\) (in fact \(0 \leq n \leq 4k\)) and
\[ \lim_{\epsilon \downarrow 0, \Lambda \uparrow \mathbb{R}^4} E \left[ (\phi(f_1) \cdots \phi(f_n) G'(k; \epsilon, \Lambda)) \right] \text{ diverges,} \quad \forall f_j \in \mathcal{S}(\mathbb{R}^4), \ j = 1, \ldots, n \]  
and \(A(k; \epsilon, \Lambda)\) is a set of random variables such that for \(G(k; \epsilon, \Lambda) \in A(k; \epsilon, \Lambda)\)
\[ \lim_{\epsilon \downarrow 0, \Lambda \uparrow \mathbb{R}^4} E \left[ (\phi(f_1) \cdots \phi(f_k) G(k; \epsilon, \Lambda)) \right] \text{ converges,} \quad \forall f_j \in \mathcal{S}(\mathbb{R}^4), \ j = 1, \ldots, n, \ \forall n \in \mathbb{N} \cup \{0\}, \]  
(12)

By using (12) we can define a \(\text{Hida distribution:}\)
\[ G = \lim_{\epsilon \downarrow 0, \Lambda \uparrow \mathbb{R}^4} G(k; \epsilon, \Lambda). \]  
(13)

**Definition 1** [Tensor Product]

Let \(F\) and \(G\) be the random variables in \(\cap_{p \geq 1} L^p(\Omega; P)\) defined by the multiple stochastic integral with respect to \(\dot{W}\) such that
\[ F = \int_{(\mathbb{R}^4)^n} f(x_1, \ldots, x_n) : \prod_{k=1}^n \dot{W}(x_k) : dx_1 \cdots dx_n, \]  
(14)
\[ G = \int_{(\mathbb{R}^4)^m} g(x_1, \ldots, x_m) : \prod_{k=1}^{m} \dot{W}(x_k) : dx_1 \cdots dx_n. \]

The tensor product \( F \otimes G \) of \( F \) and \( G \) is defined by

\[ F \otimes G = \int_{(\mathbb{R}^4)^{n+m}} f(x_1, \ldots, x_n)g(x_{n+1}, \ldots, x_{n+m}) : \prod_{k=1}^{n+m} \dot{W}(x_k) : dx_1 \cdots dx_{n+m}. \]

As a usual notation by [GrotStreit], the notion expressed by \( F \otimes G \) above is equivalent with the Wick product \( F \diamond G \).

**Definition 2** [simplified definition of Hida-distributions]

For \( f^{(n)}(x_1, \ldots, x_n) \in \mathcal{S}((\mathbb{R}^4)^n), \ n \in \mathbb{N} \) (\( f^{(0)} \) is understood as a constant), a random variable \( \varphi \) defined by

\[ \varphi = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^4)^n} f^{(n)}(x_1, \ldots, x_n) : \prod_{k=1}^{n} \dot{W}(x_k) : dx_1 \cdots dx_n, \]

is said to be in \((\mathcal{S})_r\) for \( r \in \mathbb{N} \cup \{0\} \) if

\[ \| \varphi \|_{L^2(\Omega, P), r} \equiv \sum_{n=0}^{\infty} n! \left\| \prod_{k=1}^{n} (-\Delta_{x_k} + |x_k|^2 + 1^r) f^{(n)}(x_1, \ldots, x_n) \right\|_{L^2((\mathbb{R}^4)^n)}^2 < \infty, \]

where \( \Delta_{x_k} \equiv \frac{\partial^2}{\partial t_k^2} + \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} + \frac{\partial^2}{\partial z_k^2} \) for \( x_k = (t_k, x_k, y_k, z_k) \in \mathbb{R}^4 \).

\[ \| \varphi \|_{L^2(\Omega, P), r}^2 \text{ is equivalent with } E \left[ \sum_{n=0}^{\infty} \int_{(\mathbb{R}^4)^n} \prod_{k=1}^{n} (-\Delta_{x_k} + |x_k|^2 + 1^r) f^{(n)}(x_1, \ldots, x_n) : \prod_{k=1}^{n} \dot{W}(x_k) : dx_1 \cdots dx_n \right]^2. \]

We say that a sequence of random variables \( \{G_{\epsilon}\}_{\epsilon > 0} \) defines a Hida distribution \( G \in (\mathcal{S})_{-r} \) if there exists a constant \( K < \infty \) and

\[ |E[G_{\epsilon} \cdot \varphi]| \leq K \| \varphi \|_{L^2(\Omega, P), r}, \quad \forall \varphi \in (\mathcal{S})_r, \]

and the limit exists for \( \forall \varphi \in (\mathcal{S})_r \):

\[ \lim_{\epsilon \downarrow 0} E[G_{\epsilon} \varphi] \]

The following continuous linear functional \( G \) on \((\mathcal{S})_r\) is called as a Hida distribution in \((\mathcal{S})_{-r} \):

\[ < G, \varphi > = \lim_{\epsilon \downarrow 0} E[G_{\epsilon} \varphi] \]
Remark 1 \( i \) If the defining sequence \( \{G_e\}_{e>0} \) of a Hida distribution \( G \) is composed by elements of \( n \)-times multiple stochastic integrals, then

\[
E \left[ G_e \cdot \left( \int_{(\mathbb{R}^4)^m} f(x_1, \ldots, x_m) : \prod_{k=1}^m W(x_k) : dx_1 \cdots dx_m \right) \right] = 0, \quad (21)
\]

for any \( f \in \mathcal{S}(\mathbb{R}^m) \) with \( m \neq n \).

\( ii \) Since,

\[
\left\| \left( \prod_{k=1}^n (-\Delta x_k + |x_k|^2 + 1)^r \right) f \right\|_{L^2((\mathbb{R}^4)^n)}
\]

\[
= \left\{ \int_{(\mathbb{R}^4)^n} \left( \prod_{k=1}^n (|x_k|^2 + 1)^{-2} (|x_k|^2 + 1)^2 (-\Delta x_k + |x_k|^2 + 1)^r f \right)^2 \, dx_1 \cdots dx_n \right\}^\frac{1}{2}
\]

\[
\leq \left( \sup_{x_1, \ldots, x_n} \left( \prod_{k=1}^n (|x_k|^2 + 1)^{-2} (-\Delta x_k + |x_k|^2 + 1)^r f \right) \right) \cdot \left\| \prod_{k=1}^n (|x_k|^2 + 1)^{-2} \right\|_{L^2((\mathbb{R}^4)^n)}
\]

\[
\leq K \cdot p_{m,k}(f), \quad \forall f(x_1, \ldots, x_n) \in \mathcal{S}((\mathbb{R}^4)^n), \quad (22)
\]

where \( p_{m,k}() \) is a semi-norm of \( \mathcal{S}((\mathbb{R}^4)^n) \) such that

\[
p_{m,k}(f) = \sum_{|\alpha| \leq m} \sup_\mathfrak{x} (1 + |\mathfrak{x}|^2)^k |D^\alpha f(\mathfrak{x})|, \quad (23)
\]

with \( \mathfrak{x} = (x_1, \ldots, x_n) \in (\mathbb{R}^4)^n, \quad x_k = (t_k, x_k, y_k, z_k) \in \mathbb{R}^4, \)

\[\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_k = (\alpha_{k1}, \alpha_{k2}, \alpha_{k3}, \alpha_{k4}), \quad |\alpha| = \sum_{k=1}^4 \sum_{i=1}^4 \alpha_{ki}, \]

\[D^\alpha = \prod_{k=1}^n \left( \frac{\partial}{\partial t_k} \right)^{\alpha_{k1}} \left( \frac{\partial}{\partial x_k} \right)^{\alpha_{k2}} \left( \frac{\partial}{\partial y_k} \right)^{\alpha_{k3}} \left( \frac{\partial}{\partial z_k} \right)^{\alpha_{k4}}. \]

By this, if a Hida distribution \( G \) is defined through a sequence \( \{G_e\}_{e>0} \) of \( n \)-times multiple stochastic integrals, then it can be identified with an element of \( \mathcal{S}'((\mathbb{R}^4)^n) \):

\[\exists K < \infty \text{ and } \exists m, k \in \mathbb{N} \text{ that depend only on } r \text{ and } \]

\[| < G, \varphi > | \leq K \cdot p_{m,k}(f) \]

\[\mathcal{S}'((\mathbb{R}^4)^n) \ni G : \mathcal{S}((\mathbb{R}^4)^n) \ni f \longmapsto < G, \varphi >, \quad (24)\]

for

\[
\varphi = \int_{(\mathbb{R}^4)^n} f(x_1, \ldots, x_n) : \prod_{k=1}^n W(x_k) : dx_1 \cdots dx_n.
\]

\[\blacksquare\]
Well defined terms for $d = 4$ and the strategy of the consideration

Through the discussions in the previous sections, we can define

$$e^{-\lambda <\phi^4; 1_\Lambda>} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(k!)} \left(<\phi^4; 1_\Lambda>\right)^k.$$  \hspace{1cm} (25)

The equality holds for $P - a.e. \omega \in \Omega$, because both sides of the equality are real valued random-variables, regardless they are integrable or not. The number of terms (graphs) of $<\phi^4; 1_\Lambda>$, is estimated by $(k!)^2$:

$$<\phi^4; 1_\Lambda>^k = \sum_{G \in \tilde{A}(k; \epsilon, \Lambda)} G,$$  \hspace{1cm} (26)

where, each $G \in \tilde{A}(k; \epsilon, \Lambda)$ is a tensor product of multiple stochastic integrals, and in the sense of Fynman graph it is a graph with $k$-verticies, also the cardinarity of the set $\tilde{A}(k; \epsilon, \Lambda)$ is the order $(k!)^2$. By the notation of (11), $\tilde{A}(k; \epsilon, \Lambda)$ can be expressed by the direct sum

$$\tilde{A}(k; \epsilon, \Lambda) = A'(k; \epsilon, \Lambda) \oplus A(k; \epsilon, \Lambda).$$  \hspace{1cm} (27)

For $r \geq 0$, let

$$\Lambda_r = \{(t, x, y, z) \in \mathbb{R}^4 \mid \sqrt{t^2 + x^2 + y^2 + z^2} \leq r\}.$$  \hspace{1cm} (28)

For each $k \in \mathbb{N} \cup \{0\}$, define $A_r(\epsilon, k)$, a subset of $A(k; \epsilon, \Lambda_r) \subset \tilde{A}(k; \epsilon, \Lambda)$ in (28), as follows:

$$A_r(\epsilon, k) = \text{the set with the elements } G \in \tilde{A}(k; \epsilon, \Lambda_r) \text{ (cf. (40))}, \text{ each of which is a tensor product of stochastic integrals, and each component of the tensor product is identified with a connected Fynman graph who is in type O or type I}.$$  \hspace{1cm} (29)

Where, type O resp., and type I is defined as follows:

**Type O** is the set of graphs, each element of which satisfies the following:

- (O-i) each vertex of the graph has at least 1 free (open) leg,
- (O-ii) there exists at least 1 vertex which has more than 2 free legs,
- (O-iii) there exists a connected path passing through every vertex of the graph exactly once,
- (O-iv) in the graph there is no sub graph such that two vertices connect two or three legs each other.

**Type I** is the set of graphs, each element of which satisfies the following:

- (I) between each vertex and the other vertex in the graph there exists exactly only one (directed) path that connects these two vertices, and each vertex of the graph has at least 1 free (open) leg.
We would like to show that the limit
\[ S_n(\phi(f_1), \ldots, \phi(f_n)) \equiv \lim_{r \to \infty} \left\{ \lim_{\epsilon \to 0} E \left[ \phi(f_1) \cdots \phi(f_n) \left( \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left( \sum_{G \in A_r(\epsilon, k)} G \right) \right) \right] \right\}, \tag{30} \]
for \( f_j \in S(\mathbb{R}^4 \to \mathbb{R}) \), \( j = 1, \ldots, n \), \( n \in \mathbb{N} \cup \{0\} \), defines a system of "Schwinger functions" \( \{S_n\}_{n \in \mathbb{N} \cup \{0\}} \) in a "modified sense". Because of (0.21), (O-i), (O-ii) and (I), since the sum is precisely a finite sum, and the expectation is clearly well-defined:
\[ E \left[ \phi(f_1) \cdots \phi(f_n) \left( \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left( \sum_{G \in A_r(\epsilon, k)} G \right) \right) \right] = E \left[ \phi(f_1) \cdots \phi(f_n) \left( \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \left( \sum_{G \in A_r(\epsilon, k)} G \right) \right) \right], \tag{31} \]
and the limit of (30) can be taken within the framework of finite sum of Hida distributions (cf. Definition 2).

In fact the limit exists and we have Theorem's 4.1 and 4.2 below. To give the statements we prepare some notions and notations:
Denote the number of the elements of \( A_r(\epsilon, k) \) by \( N(A(k)) \), and give an index to each \( G \in A_r(\epsilon, k) \) to indicate it as \( G_j(k; \epsilon, r), \ j = 1, \ldots, N(A(k)) \). Then,
\[ A_r(\epsilon, k) = \left\{ G_j(k; \epsilon, r) \right\}_{j=1, \ldots, N(A(k))}, \quad k \in \mathbb{N}. \tag{32} \]
Recall that for \( f_j \in S(\mathbb{R}^4), \ j = 1, 2, \ldots, \)
\[ \phi(f_1)\phi(f_2) = : \phi(f_1)\phi(f_2) : + E[\phi(f_1)\phi(f_2)] \]
\[ \phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4) = : \phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4) : + ( : \phi(f_1)\phi(f_2) : E[\phi(f_3)\phi(f_4)] + ( : \phi(f_1)\phi(f_3) : E[\phi(f_2)\phi(f_4)] + ( : \phi(f_2)\phi(f_3) : E[\phi(f_1)\phi(f_4)] + ( : \phi(f_2)\phi(f_4) : E[\phi(f_1)\phi(f_3)] + ( : \phi(f_3)\phi(f_4) : E[\phi(f_1)\phi(f_2)] + \sum \text{distinguished } i_i \text{'s} \]
and
\[ : \phi(f_1) \cdots \phi(f_n) : = \int_{(\mathbb{R}^4)^n} \prod_{j=1}^{n} (J^2 f_j)(x_j) : \prod_{j=1}^{n} W(x_j) : dx_1 \cdots dx_n. \tag{33} \]
By using the above, through a simple evaluation we have the following:
Lemma 3.1. For each \( r \geq 0, \epsilon > 0 \) and \( k \geq 1 \), let \( A_{r}(\epsilon, k) \) be the set of Fyman graphs (i.e., set of multiple stochastic integrals) defined by (29). Then, by using the expression (32), the following hold:

\[
N(A(k)) \simeq (k!)^{2^{r}},
\]

(34)

and by denoting the number of free legs of \( G_{j}(k; \epsilon, r) \in A_{r}(\epsilon, k) \) by \( N_{f}(G_{j}(k; \epsilon, r)) \),

\[
k + 2 \leq N_{f}(G_{j}(k; \epsilon, r)) \leq 4k, \quad j = 1, \ldots, N(A(k)),
\]

(35)

also (cf. (33))

\[
E \left[ \{ \phi(f_{1}) \cdots \phi(f_{n}) : G_{j}(k; \epsilon, r) \} \right] = 0, \quad \text{if } n \neq N_{f}(G_{j}(k; \epsilon, r)),
\]

(36)

\[
E \left[ \{ \phi(f_{1}) \cdots \phi(f_{n}) \} G_{j}(k; \epsilon, r) \right] = 0, \quad \text{if } n - 2 < k,
\]

(37)

moreover, there exists \( M < \infty \) and

\[
\left| E \left[ \{ \phi(f_{1}) \cdots \phi(f_{n}) : G_{j}(k; \epsilon, r) \} \right] \right| \leq M^{k}(n!) \prod_{i=1}^{n} \| f_{i} \|_{L^{1}(\mathbb{R}^4)},
\]

(38)

for \( \forall f_{i} \in \mathcal{S}(\mathbb{R}^{4} \to \mathbb{R}),\ i = 1, \ldots, n, \quad \forall n \in \mathbb{N}; \quad \forall G_{j}(k; \epsilon, r) \in A_{r}(\epsilon, k); \quad \forall k \in \mathbb{N}, \quad \forall \epsilon > 0, \quad \text{and}\ \forall r > 0 \quad (\text{if } n \neq N_{f}(G_{j}(k; \epsilon, r)), \text{then "right hand side of (33)"} = 0).

\[
\square
\]

From Lemma 4.1, we immediately have

**Theorem 3.1**. For each \( k \in \mathbb{N} \) and \( j = 1, \ldots, N(A(k)) \), there exists a Hida distribution \( G_{j}(k) \), which is a Fyman graph, such that

\[
\lim_{r \to \infty} \lim_{\epsilon \downarrow 0} G_{j}(k; \epsilon, r) = G_{j}(k) \quad \text{with} \quad N_{f}(G_{j}(k)) = N_{f}(G_{j}(k; \epsilon, r))
\]

(39)

and the set \( A_{r}(\epsilon, k) \) converges to a set of Hida distributions \( A(k) \),

\[
\lim_{r \to \infty} \lim_{\epsilon \downarrow 0} A_{r}(\epsilon, k) = A(k) = \{ G_{j}(k) \}_{j=1, \ldots, N(A(k))}.
\]

(40)

Denote

\[
\left\langle \{ \phi(f_{1}) \cdots \phi(f_{n}) : \} , G_{j}(k) \right\rangle \equiv \lim_{r \to \infty} \lim_{\epsilon \downarrow 0} E \left[ \{ \phi(f_{1}) \cdots \phi(f_{n}) : \} G_{j}(k; \epsilon, r) \right],
\]

(41)

then there exists \( M < \infty \) and

\[
\left| \left\langle \{ \phi(f_{1}) \cdots \phi(f_{n}) : \} , G_{j}(k) \right\rangle \right| \leq M^{k}(n!) \prod_{i=1}^{n} \| (-\Delta_{d=4} + 1)^{2} f_{i} \|_{L^{2}(\mathbb{R}^4)},
\]

(42)
\[
\left| \langle (\phi(f_1) \cdots \phi(f_n) \cdot), G_j(k) \rangle \right| \leq M^k (n!) \prod_{i=1}^{n} p_{4,k}(f_i),
\]
(43)

for \( \forall f_i \in \mathcal{S}(\mathbb{R}^4 \to \mathbb{R}), \ i = 1, \ldots, n, \ \forall n \in \mathbb{N}; \ \forall G_j(k) \in A(k); \ \forall k \in \mathbb{N} \)
(if \( n \neq N_f(G_j(k)) \), then ”right hand side of (42) and (43)” = 0), where \( p_{m,k}(f) \) is the semi-norm defined by (23).

\[ \square \]

By the above theorem, we can set the following Definition.

**Definition 3** Let \( A(0) = \{1\} \), namely, \( G_1(0) = 1 \) and \( N(A(0)) = 1 \). For each \( \lambda \in \mathbb{Z} \) let \( \{S^\lambda_n\}_{n \in \mathbb{N} \cup \{0\}} \) be a system of Schwartz distributions such that

\[
S^\lambda_0 = S^\lambda_1 = 0,
\]
(44)

\[
< S^\lambda_n, f_1 \otimes \cdots \otimes f_n > = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left( \sum_{j=1}^{N(A(k))} \langle (\phi(f_1) \cdots \phi(f_n)), G_j(k) \rangle \right)
\]

\[
= \sum_{k=0}^{n-2} \frac{(-\lambda)^k}{k!} \left( \sum_{j=1}^{N(A(k))} \langle (\phi(f_1) \cdots \phi(f_n)), G_j(k) \rangle \right)
\]

for \( f_i \in \mathcal{S}(\mathbb{R}^4), \ i = 1, \ldots, n, \ n \in \mathbb{N} \).

\[ \square \]

For the statements of the next theorem, we recall the OS (Osterwalder-Schrader) axioms (cf., e.g., [Si1]):

It is said that a set of functions \( \{S_n\} \) is a system of Schwinger functions with Osterwalder-Schrader axioms OS’1-5, if it satisfies

OS 1’) (Temperedness + Analytic continuity)

\[
S^p_n \in \mathcal{S}'(\mathbb{R}^{dn}), \quad S^p_n(f^*) = S^p_n(\theta f^*),
\]

and \( S^p_n \) is a Laplace transform of some \( M_n \in \mathcal{S}'((\mathbb{R}_+^d)^{n-1}) \), precisely

\[
S^p_n(x_1, \ldots, x_n, \bar{x}_n) = \int_{(\mathbb{R}_+^d)^{n-1}} M(\tau_1, \xi_1, \ldots, \tau_{n-1}, \xi_{n-1})
\]

\[
\times \exp \left\{ \sum_{j=1}^{n-1} \left( \sqrt{-1} \xi_j \cdot (\bar{x}_{j+1} - \bar{x}_j) - \tau_j (x_{j+1}^0 - x_j^0) \right) \right\} d\xi_1 d\tau_1 \cdots d\xi_{n-1} d\tau_{n-1};
\]

(46)

OS 2) (Euclidean covariance);

OS 3) (Reflection positivity);

For the statements of the next theorem, we recall the OS (Osterwalder-Schrader) axioms (cf., e.g., [Si1]):
OS 4) (Symmetry);
OS 5) (Cluster property).

Theorem 3.2 Let \( \{S_\lambda^\lambda\}_{n \in \mathbb{N} \cup \{0\}} \) be the system of Schwartz distributions defined by Definition 3, then it is a system of "modified Schwinger functions" in the sense that it satisfies OS 2), OS 4) and OS 5).

Remark 2.
We have an affirmative rigorous result on the analyticity OS 1') for \( \{S_\lambda^\lambda\}_{n \in \mathbb{N} \cup \{0\}} \). It will be announced in forthcoming papers.

References


References


