Study of 2D O(N) Sigma Model by Renormalization Group Method

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It is a longstanding problem to prove quark confinement in the framework of 4D lattice gauge field theory. This is unsolved yet, and a similar 2D O(N) model is also a very hard problem. Here I show that a rigorous analysis of this model is possible, and establish our longstanding conjecture for large $N$. We start with the action

$$Z = \int \exp[-W_0(\phi, \psi)] \prod d\psi d\phi$$

$$W_0 = \frac{1}{2} \langle \phi, G_0^{-1} \phi \rangle + i \frac{\sqrt{N}}{\sqrt{N}} \langle (\phi^2 - N\beta, \psi) \rangle$$

where $G_0^{-1} = -\Delta + m_0^2$. The BST is the method to integrate $e^{-W_n}$ recursively (thus $e^{-W_n} \rightarrow e^{-W_{n+1}} \rightarrow \cdots$) by high-momentum part $\xi_n$ of $\phi_n$ and $\tilde{\psi}_n$ of $\psi_n$ by decomposing

$$\phi_n = A_{n+1} \phi_{n+1} + Q\xi_n, \quad \psi_n = A_{n+1} \psi_{n+1} + Q\tilde{\psi}_n$$

where

$$\phi_{n+1}(x) = (C\phi)_n(x) = \frac{1}{L^2} \sum_{\zeta \in \Delta_0} \phi_n(Lx + \zeta)$$

$$\psi_{n+1}(x) = (C\psi)_n(x) = L^2 (C\psi_n)(x) = \sum_{\zeta \in \Delta_0} \psi_n(Lx + \zeta)$$

and $\Delta_0$ is the square of size $L \times L$ centered at the origin, $2 \leq L$. This consists of averaging of spins over the blocks $\Delta_{Lx}$ (box centered at $Lx$) and scaling down ($Lx \rightarrow x$) of the coordinate $A_0 \subset Z^2 \rightarrow A_1 = L^{-1} A_0 \cap Z^2$. $Q$ is the matrix to form zero-average fluctuations $Q\xi_n$ and $Q\tilde{\psi}_n$ which we integrate out (Wilson’s idea). $A_n$ and $\tilde{A}$ are chosen so that the main gaussian terms of $W_n$ are diagonalized:

$$\langle \phi_n, G_n^{-1} \phi_n \rangle = \langle \phi_{n+1}, G_{n+1}^{-1} \phi_{n+1} \rangle + \langle \xi_n, Q^+ G_n^{-1} \xi_n \rangle$$

$$\langle \psi_n, H_n^{-1} \psi_n \rangle = \langle \psi_{n+1}, H_{n+1}^{-1} \psi_{n+1} \rangle + \langle \tilde{\psi}_n, Q^+ H_n^{-1} Q\tilde{\psi}_n \rangle$$

We prove that $W_n$ is given by the following form outside the domain-walls $D_w$ which have high-energy (thus with small probability) -subtle to define-:

$$W_n = \frac{1}{2} \langle \phi_n, G_n^{-1} \phi_n \rangle + i \frac{\sqrt{N}}{\sqrt{N}} \langle (\phi_n^2 - N\beta_n, \psi) \rangle$$

$$+ \frac{1}{2} \langle \psi_n, H_n^{-1} \psi_n \rangle + \gamma_n \langle \phi_n^2, E^+ G_n^{-1} E^+ \phi_n^2 \rangle$$

where $E^+$=projection to the block-wise zero average functions. Thus the last two terms are shown to be irrelevant since $E^+$ acts as a differentiation. Moreover we can show $\beta_n = \beta - cn(\rightarrow 0)$ and $\gamma_n = O((N\beta)^{-1})$. This establishes nonexistence of phase transition in the model.