Global properties of solutions to the Einstein-Boltzmann system with Bianchi I symmetry

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Overview
Matter is distributed in the universe, and the universe evolves in time.

- Einstein’s equations.

The universe is assumed to be homogeneous, and let us consider the Bianchi I symmetry, which is a generalization of the RW model, which is a homogeneous and isotropic universe.

Matter also evolves in time, and we use a kinetic equation to describe it. A lot of progress for the Einstein-Vlasov case, i.e. collisionless Boltzmann case, but not much for the Einstein-Boltzmann case.

- The Einstein-Boltzmann system with Bianchi I symmetry.

Result: if the universe is almost isotropic initially and initial data for the Boltzmann equation is sufficiently small, then we obtain global existence and asymptotic behavior of solutions.
Known: Vlasov + Bianchi I, [Nungesser, 10]
Known: Boltzmann + RW, [L, 13]
Result: Boltzmann + Bianchi I
Introduction
Boltzmann equation

\[ \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = Q(f, f) \]

▷ Matter = collection of particles.
▷ Distribution function, \( f = f(t, x, v) \), density of particles, \( f(t, x, v) \, dx \, dv \).
▷ Time \( t > 0 \), position \( x \in \mathbb{R}^3 \), velocity \( v \in \mathbb{R}^3 \).
▷ Particles collide.
▷ Two particles with velocities \( v \) and \( v_* \):

\[ (v, v_*) \leftrightarrow (v', v_*'). \]

▷ Energy and momentum conservations

\[ v' + v_*' = v + v_*, \quad |v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2. \]

▷ One parametrization

\[ v' = v - ((v - v_*) \cdot \omega) \omega, \quad v_*' = v_* + ((v - v_*) \cdot \omega) \omega, \quad \omega \in S^2. \]
Nonrelativistic case.

Another representation

\[
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in S^2.
\]
Boltzmann equation:

\[ \partial_t f + v \cdot \nabla_x f = Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, \sigma)(f(v') f(v_*) - f(v) f(v_*)) \, d\sigma \, dv_* . \]

Collision kernel \( B \) depends on physics.

**Special relativity**

- We want to consider fast moving particles.
- Space and time merge into the concept of spacetime,

\[ (t, x, y, z) = (x^0, x^1, x^2, x^3) = x^\alpha \in M. \]

- A manifold with the Minkowski metric \( \eta_{\alpha \beta} = \text{diag}(-1, 1, 1, 1) \).
- Four-dimensional vectors \( v^\alpha \in T_x M \) are measured by

\[ \eta_{\alpha \beta} v^\alpha v^\beta = v_\alpha v^\alpha = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2. \]
Speed of light $c = 1$. 

http://www.twow.net/ObjText/OtkCaLbStrB.htm
A worldline \( x^\alpha = x^\alpha(\tau) \) with the proper time \( \tau \).

http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/spacetime/
Four-velocity:
\[ v^\alpha = \frac{dx^\alpha}{d\tau}, \quad v_\alpha v^\alpha = -1. \]

Four-momentum \( p^\alpha = mv^\alpha \) (we assume \( m = 1 \), so \( p^\alpha = v^\alpha \)).

Mass shell condition.

Four-momentum \( p^\alpha \in P_x := T_x M \cap \{ p_\alpha p^\alpha = -1 \} \).

**Special relativistic Boltzmann equation**

- Distribution function, \( f(x^\alpha, p^\alpha) \).
- Spacetime variable \( x^\alpha \in M \) and four-momentum \( p^\alpha \in P_x \).
- Mass shell condition implies
  \[ p^0 = p^0(p) = \sqrt{1 + |p|^2}. \]
- Distribution function, \( f = f(t, x, p) \).
Two colliding particles with momenta $p^\alpha$ and $q^\alpha$:

$$(p^\alpha, q^\alpha) \leftrightarrow (p'^\alpha, q'^\alpha).$$

Energy-momentum conservations and the mass shell conditions:

$$p'^\alpha + q'^\alpha = p^\alpha + q^\alpha, \quad p'_\alpha p'^\alpha = -1, \quad q'_\alpha q'^\alpha = -1.$$

[Glassey-Strauss, 93], [Strain, 10], [Guo-Strain, 12], etc.

http://de.wikipedia.org/wiki/Hyperboloid
Einstein-Boltzmann with Bianchi I
Lorentzian metric

- Black hole

http://plato.stanford.edu/entries/spacetime-singularities/
The Einstein-Boltzmann system with Bianchi I symmetry

A cosmological model

Space-time diagram: normal distance & time

https://telescoper.wordpress.com/2015/01/05/faster-than-the-speed-of-light/

Ho Lee (Kyung Hee University)
Expanding universe

http://www.physicsoftheuniverse.com/topics_bigbang_expanding.html
Vlasov equation with Bianchi symmetry

- A metric $^4g = g_{\alpha\beta} dx^\alpha dx^\beta$ is given,

\[
\frac{\partial f}{\partial t} - \Gamma^\alpha_{\beta\gamma} \frac{p^\beta p^\gamma}{p^0} \frac{\partial f}{\partial p^\alpha} = 0,
\]

(cf. geodesic equations: $\dot{x}^\alpha = p^\alpha$ and $\dot{p}^\alpha = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma$).

- Mass shell condition: $p_\alpha p^\alpha = -1$.

- A basis $\{e_a\}$ is given such that $[e_\alpha, e_\beta] = \eta^\gamma_{\alpha\beta} e_\gamma$ and $\nabla_{e_\beta} e_\alpha = \Gamma^\gamma_{\alpha\beta} e_\gamma$,

\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\xi} \left( e_\beta (g_{\xi\gamma}) + e_\gamma (g_{\beta\xi}) - e_\xi (g_{\gamma\beta}) + \eta^\delta_{\gamma\beta} g_{\xi\delta} + \eta^\delta_{\xi\gamma} g_{\beta\delta} - \eta^\delta_{\beta\xi} g_{\gamma\delta} \right),
\]

which is called Koszul’s formula.

- A coordinate basis $\{\partial_\alpha\}$, i.e. $[\partial_\alpha, \partial_\beta] = 0$,

\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\xi} \left( \partial_\beta (g_{\xi\gamma}) + \partial_\gamma (g_{\beta\xi}) - \partial_\xi (g_{\gamma\beta}) \right),
\]

which is the usual Christoffel symbols.
The Einstein-Boltzmann system with Bianchi I symmetry

The metric is assumed to be $^4g = -dt^2 + g$ with $g = g_{ab}(t)dx^a dx^b$.

An $n$-dimensional manifold $M$ is given.

Isometry group $G_r$ of dimension $r$.

Transformation generated by a vector field $V$ with $L_V g = 0$,

$$r \leq \frac{1}{2}n(n + 1).$$

Isotropy group of dimension $d = r - n \leq \frac{1}{2}n(n - 1)$,
(cf. translations and rotations in $\mathbb{R}^3$).

Bianchi spacetime: $d = 0$.

Killing vector fields with basis $\{e_a\}$

$$[e_a, e_b] = C^c_{ab}e_c,$$

where $C^c_{ab}$ are called the structure constants.
The Vlasov equation with Bianchi symmetry:

\[ \frac{\partial f}{\partial t} - g^{ad} \left( (p^0)^{-1} C^e_{\, dc} p^c p_e + \dot{g}_{bd} p^b \right) \frac{\partial f}{\partial p^a} = 0. \]

In covariant momenta \( p_a = g_{ab} p^b \),

\[ \frac{\partial f}{\partial t} - (p^0)^{-1} C^e_{\, ac} p^c p_e \frac{\partial f}{\partial p_a} = 0. \]

Energy-momentum tensor

\[ T_{\alpha\beta} = \int_{\mathbb{R}^3} f(t, p) \frac{p^\alpha p^\beta}{-p_0} | \det g |^{\frac{1}{2}} \, dp. \]

We only need

\[ \rho := T^{00} = (\det g)^{-\frac{1}{2}} \int_{\mathbb{R}^3} f(t, p_*) (1 + g^{cd} p_c p_d)^{\frac{1}{2}} \, dp_*, \]

\[ S_{ab} := T_{ab} = (\det g)^{-\frac{1}{2}} \int_{\mathbb{R}^3} f(t, p_*) p_a p_b (1 + g^{cd} p_c p_d)^{-\frac{1}{2}} \, dp_*, \]

where \( p = (p^1, p^2, p^3) \) and \( p_* = (p_1, p_2, p_3) \).
Einstein-Vlasov system with Bianchi symmetry

▷ Einstein’s equations in covariant form

\[ G_{\alpha\beta} = 8\pi T_{\alpha\beta}. \]

▷ Einstein’s equations in 3+1 form

\[ \partial_t g_{ab} = -2k_{ab}, \]
\[ \partial_t k_{ab} = R_{ab} + (g^{cd}k_{cd})k_{ab} - 2(g^{cd}k_{bd})k_{ac} - 8\pi S_{ab} + 4\pi g_{ab}(S - \rho) \]

with constraint equations

\[ R - k^{ij}k_{ij} + k^2 = 16\pi \rho, \]
\[ \nabla^i k_{ij} = 8\pi T_{0j}. \]

▷ [Rendall, 94], [Hayoung Lee, 04], rotationally symmetry, reflection symmetry, [Nungesser, 10, 12], [Nungesser-Andersson-Bose-Coley, 14], [L, 13].

▷ The present work is a joint work with Nungesser.
Einstein-Vlasov with Bianchi I symmetry

▷ Bianchi I symmetry: the structure constants $C^c_{ab} = 0$.

▷ The Vlasov equation reduces to

$$\frac{\partial f}{\partial t} + 2k^a_b p^b \frac{\partial f}{\partial p^a} = 0 \quad \text{or} \quad \frac{\partial f}{\partial t} = 0.$$ 

▷ The Einstein equations reduce to

$$\partial_t g_{ab} = -2k_{ab},$$

$$\partial_t k_{ab} = (g^{cd} k_{cd}) k_{ab} - 2(g^{cd} k_{bd}) k_{ac} - 8\pi S_{ab} + 4\pi g_{ab}(S - \rho).$$

▷ The matter terms

$$\rho = (\det g)^{-\frac{1}{2}} \int_{\mathbb{R}^3} f(t, p_*) (1 + g^{cd} p_c p_d)^{\frac{1}{2}} dp_*,$$

$$S_{ab} = (\det g)^{-\frac{1}{2}} \int_{\mathbb{R}^3} f(t, p_*) p_a p_b (1 + g^{cd} p_c p_d)^{-\frac{1}{2}} dp_*.$$

▷ Solutions tend to the Einstein-de Sitter model, i.e. $-dt^2 + t^\frac{4}{3} (dx^2 + dy^2 + dz^2)$. 
**Einstein-Boltzmann with Bianchi I symmetry**

- The Boltzmann equation will be
  \[ \frac{\partial f}{\partial t} + 2k_b^a p^b \frac{\partial f}{\partial p^a} = Q(f, f) \quad \text{or} \quad \frac{\partial f}{\partial t} = Q(f, f). \]

- Representation of a momentum \( p \in T_x M \):
  \[ p = p^a E_a = \hat{p}^a e_a, \]
  where \( \{ E_a \} \) is the given basis and \( \{ e_a \} \) an orthonormal basis such that
  \( g(E_a, E_b) = g_{ab} \) and \( g(e_a, e_b) = \eta_{ab} \).

- The Boltzmann equation in an orthonormal frame
  \[ \frac{\partial \hat{f}}{\partial t} + \hat{k}_b^a \hat{p}^b \frac{\partial \hat{f}}{\partial \hat{p}^a} = Q(\hat{f}, \hat{f}), \]
  where \( e_a = e_a^b E_b, p^a = e_b^a \hat{p}^b \) and \( \hat{k}_{ab} = e_c^a e_d^b k_{cd} \).
Orthonormal frame

Roughly speaking..

http://astro.physics.sc.edu/selfpacedunits/Unit57.html
The Boltzmann equation: in Strain’s framework [Strain, 10],

\[ \frac{\partial f}{\partial t} = (\det g)^{-\frac{1}{2}} \int \int v_M \sigma(h, \theta) \left( f(p'_*) f(q'_*) - f(p_*) f(q_*) \right) d\omega dq_* , \]

and parametrization of post-collision momenta

\[
\begin{pmatrix} p'^0 \\ p'_i \end{pmatrix} = \begin{pmatrix} \frac{p^0 + q^0}{2} + \frac{h}{2} \frac{n_i e^i_j \omega^j}{\sqrt{s}} \\ \frac{p_i + q_i}{2} + \frac{h}{2} \left( g_{ij} e^j_k \omega^k + \left( \frac{n^0}{\sqrt{s}} - 1 \right) \frac{n_j e^j_k \omega^k n_i}{g^{ab} n_a n_b} \right) \end{pmatrix},
\]

where \( h^2 = (p^\alpha - q^\alpha)(p^\alpha - q^\alpha) \), \( s = -n_\alpha n^\alpha \) and \( n^\alpha = p^\alpha + q^\alpha \).
The Boltzmann equation in the framework of [Glassey-Strauss, 93]:

\[ \frac{\partial f}{\partial t} = (\det g)^{-\frac{1}{2}} \int \int \frac{v_M \sqrt{s(n^0)^2} \sigma(h, \theta)}{((n^0)^2 - (n_a e^a_b \xi^b)^2)^{3/2}} \left( f(p^*_0) f(q^*_i) - f(p^*_i) f(q^*_*) \right) d\xi dq_*, \]

and parametrization of post-collision momenta

\[
\begin{pmatrix}
  p'^0_0 \\
p'^0_i
\end{pmatrix} = \begin{pmatrix}
  \frac{p^0 + q^0}{2} + \frac{h}{2} \frac{n_i \xi^i_j \omega^j}{\sqrt{(n^0)^2 - (n_i e^i_j \xi^j)^2}} \\
  \frac{p_i + q_i}{2} + \frac{h}{2} \frac{n^0 g_{ij} e^j_k \xi^k}{\sqrt{(n^0)^2 - (n_i e^i_j \xi^j)^2}}
\end{pmatrix}.
\]

Differentiability for the relativistic Boltzmann equation [Guo-Strain, 12].

We have the Einstein-Boltzmann system with Bianchi I symmetry.
Results
Einstein’s equations for given matter terms

- Einstein’s equations

\[ \partial_t g_{ab} = -2k_{ab}, \]
\[ \partial_t k_{ab} = (g^{cd}k_{cd})k_{ab} - 2(g^{cd}k_{bd})k_{ac} - 8\pi S_{ab} + 4\pi g_{ab}(S - \rho). \]

- Assume that \( f(t, p) = \hat{f}(t, \hat{p}) \leq \varepsilon \exp(t^{-\frac{5}{4}}|\hat{p}|^2) \) and \( C^1 \).
- Local existence by [Rendall, 94].
- Global-in-time existence by [Rendall, 94].
- Asymptotic behavior by [Nungesser, 10] such that

\[ g_{ab}(t) = t^{\frac{4}{3}} \tilde{g}_{ab}(t) \quad \text{and} \quad \tilde{g}_{ab}(t) = G_{ab} + O(\varepsilon t^{-1}), \]

assuming smallness and using bootstrap argument.
Einstein’s equations for given matter terms

- Decompose

\[ k_{ab} = \sigma_{ab} - H g_{ab}, \quad H = -\frac{1}{3} k, \quad k = g^{ab} k_{ab}, \]

where \( H \) is called the Hubble variable and \( k \) the mean curvature.

- Assume that \( \sigma_{ab} \) the trace free part is small in the sense that

\[ F := \frac{1}{4H^2} \sigma_{ab} \sigma_{ab}. \]

- In the Robertson-Walker case, i.e. \( g_{ab} = R^2(t) \eta_{ab} \), we have \( \sigma_{ab} = 0 \).

- Without smallness we have

\[ \frac{1}{3t} \leq H(t) \leq \frac{2}{3t}. \]

- Assuming smallness we have

\[ \frac{2}{3t(1 + \varepsilon t^{-1})} \leq H(t) \leq \frac{2}{3t}. \]

- In the Robertson-Walker case, \( H(t) = \frac{2}{3} t^{-1} \).
Bootstrap argument: \( F(t) \leq \varepsilon(1 + t)^{-\frac{3}{2}} \implies F(t) \leq \varepsilon(1 + t)^{-2+\varepsilon} \).

Equation for \( F \):

\[
\dot{F} = -3H \left( 1 - \frac{2}{3} F - \frac{8\pi S}{9H^2} - \frac{4\pi S_{ab}\sigma^{ab}}{3H^3 F} \right) F \sim -2t^{-1} F.
\]

We eventually obtain \( F \sim \varepsilon t^{-2} \).

Equation for \( \bar{g}_{ab} \):

\[
\dot{\bar{g}}_{ab} = 2 \left( H - \frac{2}{3} t^{-1} \right) \bar{g}_{ab} - 2t^{-\frac{4}{3}} \sigma_{ab},
\]

and note that \( (H - \frac{2}{3} t^{-1}) \) is integrable.

We eventually obtain \( |\bar{g}_{ab}| \leq C \) and

\[
g_{ab}(t) = t^{\frac{4}{3}} \left( G_{ab} + O(\varepsilon t^{-1}) \right),
\]

together with \( F(t) \leq CF(t_0)t^{-2} \) and \( H(t) = \frac{2}{3} t^{-1} (1 + O(\varepsilon t^{-1})) \).
The Boltzmann equation in a given spacetime

The Boltzmann equation

\[ \frac{\partial f}{\partial t} = (\det g)^{-\frac{1}{2}} \int \int v_M \sigma(h, \theta) \left( f(p') f(q') - f(p) f(q) \right) d\omega dq, \]

Consider first the Robertson-Walker case, i.e. 

\[ -dt^2 + R^2(dx^2 + dy^2 + dz^2). \]

Take weight function \( e^{\|p*\|^2} \) and multiply this to the equation

\[ \frac{\partial(e^{\|p*\|^2} f(t, p*))}{\partial t} = R^{-3} \int \int \ldots \]

\[ \ldots (e^{\|p*\|^2} f(p*) e^{\|q*\|^2} f(q*) - e^{\|p*\|^2} f(p*) e^{\|q*\|^2} f(q*)) e^{-\|q*\|^2} d\omega dq, \]

if we have an identity \( |p|^2 + |q|^2 = |p*|^2 + |q*|^2. \)

In the end,

\[ \frac{d}{dt} \|f(t)\| \leq CR^{-3} \|f(t)\|^2 \quad \text{and} \quad \|f(t)\| \leq \|f(0)\| + C\|f(t)\|^2, \]

if \( R^{-3} \) is integrable.
The post-collision momentum:

\[ p'_i = \frac{p_i + q_i}{2} + \frac{h}{2} \frac{n^0 g_{ij} e^j_k \xi^k}{\sqrt{(n^0)^2 - (n_i e^i_j \xi^j)^2}} = \frac{p_i + q_i}{2} + \frac{Rh}{2} \frac{n^0 \xi_i}{\sqrt{(n^0)^2 - R^{-2}(n \cdot \xi)^2}}, \]

\[ Rh = |p_* - q_*| \sqrt{1 - \frac{|p_* + q_*|^2 \cos^2 \theta_0}{R^2(p^0 + q^0)^2}}. \]

If \( \lim_{t \to \infty} R(t) = \infty \), we have

\[ p'_* \to \frac{p_* + q_*}{2} + \frac{|p_* - q_*|}{2} \xi \quad \text{and} \quad q'_* \to \frac{p_* + q_*}{2} - \frac{|p_* - q_*|}{2} \xi, \]

which is the parametrization of the nonrelativistic case. In other words, at late times the post-collision momenta with lower indices behave like in the nonrelativistic case. Hence, we will eventually have \( |p'_*|^2 + |q'_*|^2 = |p_*|^2 + |q_*|^2 \).

We obtain a small solution such that

\[ f(t, p_*) \leq \varepsilon \exp(-|p_*|^2) \quad \text{or} \quad \hat{f}(t, \hat{p}) \leq \varepsilon \exp(-R^2|\hat{p}|^2). \]
In the Bianchi I case we may choose \( \exp(\bar{g}^{ab} p_a p_b) \) to get

\[
\hat{f}(t, \hat{p}) \leq \varepsilon \exp(-t^{4/3} |\hat{p}|^2) \left( = \varepsilon \exp(-\bar{g}^{ab} p_a p_b) \right).
\]

For a small \( \varepsilon \) such that \( \frac{d}{dt} [t^{-\varepsilon} \bar{g}^{ab}(t)] \leq 0 \), we have

\[
\hat{f}(t, \hat{p}) \leq \varepsilon \exp(-t^{4/3 - \varepsilon} |\hat{p}|^2) \left( = \varepsilon \exp(-t^{-\varepsilon} \bar{g}^{ab} p_a p_b) \right).
\]

Differentiability of solutions: [Guo-Strain, 12].
Thank you very much.