Global properties of solutions to the Einstein-Boltzmann system with Bianchi I symmetry

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Overview

Matter is distributed in the universe, and the universe evolves in time.

▷ Einstein's equations.

The universe is assumed to be homogeneous, and let us consider the Bianchi I symmetry, which is a generalization of the RW model, which is a homogeneous and isotropic universe.

Matter also evolves in time, and we use a kinetic equation to describe it. A lot of progress for the Einstein-Vlasov case, i.e. collisionless Boltzmann case, but not much for the Einstein-Boltzmann case.

▷ The Einstein-Boltzmann system with Bianchi I symmetry.

Result: if the universe is almost isotropic initially and initial data for the Boltzmann equation is sufficiently small, then we obtain global existence and asymptotic behavior of solutions.

Known : Vlasov $+$ Bianchi I, [Nungesser, 10]

Known : Boltzmann $+$ RW, [L, 13]

Result : Boltzmann $+$ Bianchi I

Introduction

Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = Q(f, f)$$

▷ Matter = collection of particles.
▷ Distribution function, $f = f(t, x, v)$, density of particles, $f(t, x, v)\, dx\, dv$.
▷ Time $t > 0$, position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$.
▷ Particles collide.
▷ Two particles with velocities v and v_*:

$$(v, v_*) \leftrightarrow (v', v_*').$$

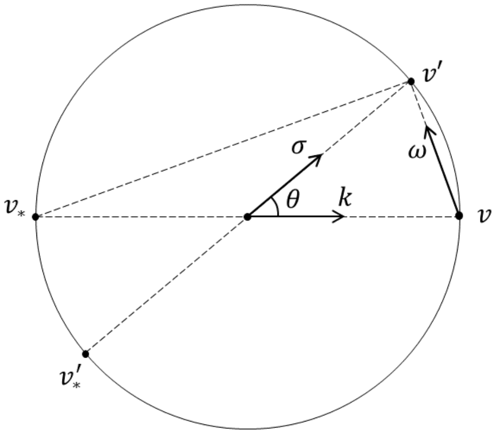
▷ Energy and momentum conservations

$$v' + v_*' = v + v_*, \quad |v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2.$$

▷ One parametrization

$$v' = v - ((v - v_*) \cdot \omega)\, \omega, \quad v_*' = v_* + ((v - v_*) \cdot \omega)\, \omega, \quad \omega \in \mathbb{S}^2.$$

▷ Nonrelativistic case.



▷ Another representation

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$

▷ Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma)(f(v')f(v'_*) - f(v)f(v_*))\, d\sigma\, dv_*.$$

▷ Collision kernel B depends on physics.

Special relativity

▷ We want to consider fast moving particles.

▷ Space and time merge into the concept of spacetime,

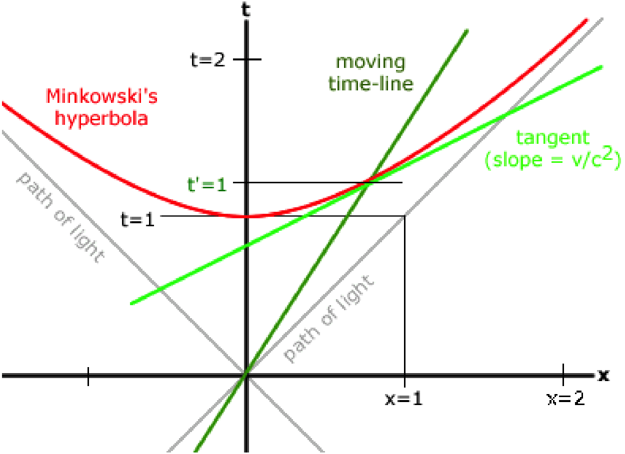
$$(t,x,y,z) = (x^0,x^1,x^2,x^3) = x^\alpha \in M.$$

▷ A manifold with the Minkowski metric $\eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$.

▷ Four-dimensional vectors $v^\alpha \in T_x M$ are measured by

$$\eta_{\alpha\beta}v^\alpha v^\beta = v_\alpha v^\alpha = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2.$$

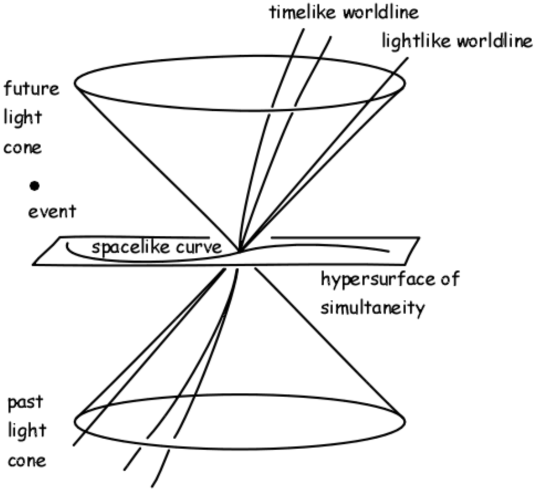
▷ Speed of light $c = 1$.



CONSTRUCTION OF MINKOWSKI'S SPACETIME DIAGRAM

http://www.twow.net/ObjText/OtkCaLbStrB.htm

▷ A worldline $x^\alpha = x^\alpha(\tau)$ with the proper time τ.



http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/spacetime/

▷ Four-velocity:

$$v^\alpha = \frac{dx^\alpha}{d\tau}, \quad v_\alpha v^\alpha = -1.$$

▷ Four-momentum $p^\alpha = mv^\alpha$ (we assume $m = 1$, so $p^\alpha = v^\alpha$).
▷ Mass shell condition.
▷ Four-momentum $p^\alpha \in P_x := T_x M \cap \{p_\alpha p^\alpha = -1\}$.

Special relativistic Boltzmann equation

▷ Distribution function, $f(x^\alpha, p^\alpha)$.
▷ Spacetime variable $x^\alpha \in M$ and four-momentum $p^\alpha \in P_x$.
▷ Mass shell condition implies

$$p^0 = p^0(p) = \sqrt{1 + |p|^2}.$$

▷ Distribution function, $f = f(t, x, p)$.

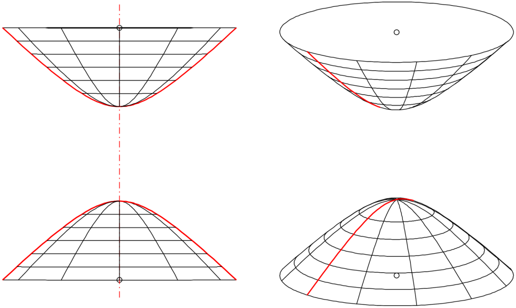
▷ Two colliding particles with momenta p^α and q^α:

$$(p^\alpha, q^\alpha) \leftrightarrow (p'^\alpha, q'^\alpha).$$

▷ Energy-momentum conservations and the mass shell conditions:

$$p'^\alpha + q'^\alpha = p^\alpha + q^\alpha, \quad p'_\alpha p'^\alpha = -1, \quad q'_\alpha q'^\alpha = -1.$$

▷ [Glassey-Strauss, 93], [Strain, 10], [Guo-Strain, 12], etc.

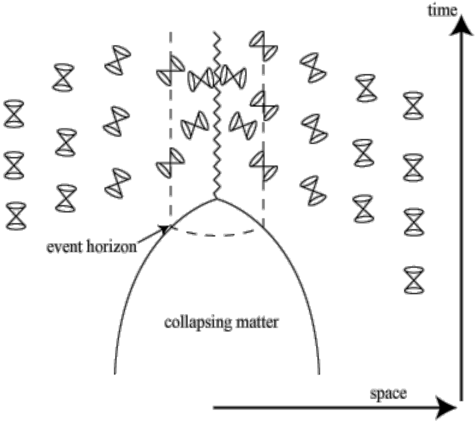


http://de.wikipedia.org/wiki/Hyperboloid

Einstein-Boltzmann with Bianchi I

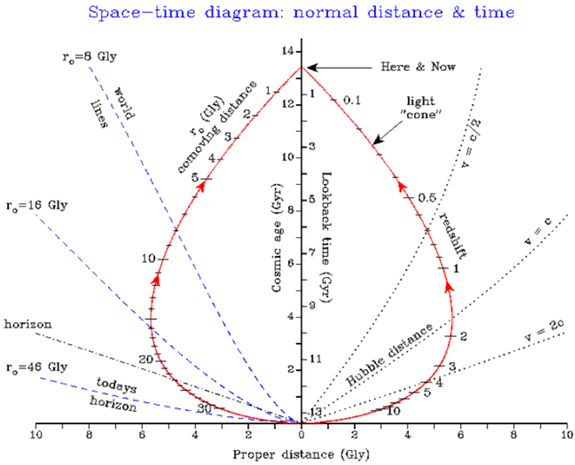
Lorentzian metric

▷ Black hole



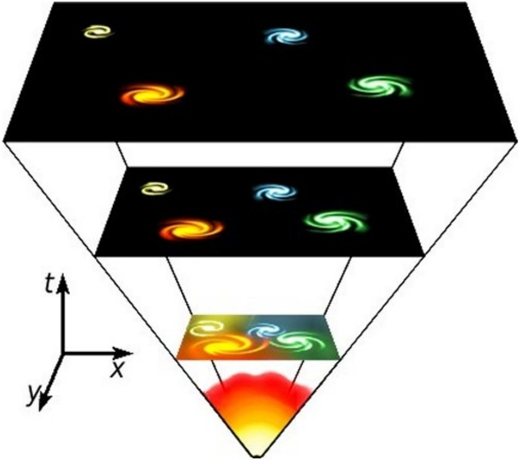
http://plato.stanford.edu/entries/spacetime-singularities/

▷ A cosmological model



https://telescoper.wordpress.com/2015/01/05/faster-than-the-speed-of-light/

▷ Expanding universe



http://www.physicsoftheuniverse.com/topics_bigbang_expanding.html

Vlasov equation with Bianchi symmetry

▷ A metric ${}^4g = g_{\alpha\beta}dx^\alpha dx^\beta$ is given,

$$\frac{\partial f}{\partial t} - \Gamma^a{}_{\beta\gamma}\frac{p^\beta p^\gamma}{p^0}\frac{\partial f}{\partial p^a} = 0,$$

(cf. geodesic equations: $\dot{x}^\alpha = p^\alpha$ and $\dot{p}^\alpha = -\Gamma^\alpha{}_{\beta\gamma}p^\beta p^\gamma$).

▷ Mass shell condition: $p_\alpha p^\alpha = -1$.

▷ A basis $\{e_a\}$ is given such that $[e_\alpha, e_\beta] = \eta^\gamma{}_{\alpha\beta}e_\gamma$ and $\nabla_{e_\beta}e_\alpha = \Gamma^\gamma{}_{\alpha\beta}e_\gamma$,

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\xi}\Big(e_\beta(g_{\xi\gamma}) + e_\gamma(g_{\beta\xi}) - e_\xi(g_{\gamma\beta}) + \eta^\delta{}_{\gamma\beta}g_{\xi\delta} + \eta^\delta{}_{\xi\gamma}g_{\beta\delta} - \eta^\delta{}_{\beta\xi}g_{\gamma\delta}\Big),$$

which is called Koszul's formula.

▷ A coordinate basis $\{\partial_\alpha\}$, i.e. $[\partial_\alpha, \partial_\beta] = 0$,

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\xi}\Big(\partial_\beta(g_{\xi\gamma}) + \partial_\gamma(g_{\beta\xi}) - \partial_\xi(g_{\gamma\beta})\Big),$$

which is the usual Chirstoffel symbols.

- ▷ The metric is assumed to be ${}^4g = -dt^2 + g$ with $g = g_{ab}(t)dx^a dx^b$.
- ▷ An n-dimensional manifold M is given.
- ▷ Isometry group G_r of dimension r.
- ▷ Transformation generated by a vector field V with $L_V g = 0$,

$$r \leq \frac{1}{2}n(n+1).$$

- ▷ Isotropy group of dimension $d = r - n \leq \frac{1}{2}n(n-1)$,
  (cf. translations and rotations in $\mathbb{R}^3$).

- ▷ Bianchi spacetime: $d = 0$.
- ▷ Killing vector fields with basis $\{e_a\}$

$$[e_a, e_b] = C^c{}_{ab} e_c,$$

where $C^c{}_{ab}$ are called the structure constants.

▷ The Vlasov equation with Bianchi symmetry:

$$\frac{\partial f}{\partial t} - g^{ad}\Big((p^0)^{-1}C^e{}_{dc}p^c p_e + \dot{g}_{bd}p^b\Big)\frac{\partial f}{\partial p^a} = 0.$$

▷ In covariant momenta $p_a = g_{ab}p^b$,

$$\frac{\partial f}{\partial t} - (p^0)^{-1}C^e{}_{ac}p^c p_e \frac{\partial f}{\partial p_a} = 0.$$

▷ Energy-momentum tensor

$$T_{\alpha\beta} = \int_{\mathbb{R}^3} f(t,p)\frac{p_\alpha p_\beta}{-p_0}|\det{}^4g|^{\frac{1}{2}}\,dp.$$

▷ We only need

$$\rho := T^{00} = (\det g)^{-\frac{1}{2}}\int_{\mathbb{R}^3} f(t,p_*)(1+g^{cd}p_c p_d)^{\frac{1}{2}}\,dp_*,$$

$$S_{ab} := T_{ab} = (\det g)^{-\frac{1}{2}}\int_{\mathbb{R}^3} f(t,p_*)p_a p_b(1+g^{cd}p_c p_d)^{-\frac{1}{2}}\,dp_*,$$

where $p = (p^1, p^2, p^3)$ and $p_* = (p_1, p_2, p_3)$.

Einstein-Vlasov system with Bianchi symmetry

▷ Einstein's equations in covariant form

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}.$$

▷ Einstein's equations in 3+1 form

$$\partial_t g_{ab} = -2k_{ab},$$
$$\partial_t k_{ab} = R_{ab} + (g^{cd}k_{cd})k_{ab} - 2(g^{cd}k_{bd})k_{ac} - 8\pi S_{ab} + 4\pi g_{ab}(S-\rho)$$

with constraint equations

$$R - k^{ij}k_{ij} + k^2 = 16\pi\rho,$$
$$\nabla^i k_{ij} = 8\pi T_{0j}.$$

▷ [Rendall, 94], [Hayoung Lee, 04], rotationally symmetry, reflection symmetry, [Nungesser, 10, 12], [Nungesser-Andersson-Bose-Coley, 14], [L, 13].

▷ The present work is a joint work with Nungesser.

Einstein-Vlasov with Bianchi I symmetry

▷ Bianchi I symmetry: the structure constants $C^c{}_{ab} = 0$.

▷ The Vlasov equation reduces to

$$\frac{\partial f}{\partial t} + 2k^a_b p^b \frac{\partial f}{\partial p^a} = 0 \quad \text{or} \quad \frac{\partial f}{\partial t} = 0.$$

▷ The Einstein equations reduce to

$$\begin{aligned}
\partial_t g_{ab} &= -2k_{ab}, \\
\partial_t k_{ab} &= (g^{cd}k_{cd})k_{ab} - 2(g^{cd}k_{bd})k_{ac} - 8\pi S_{ab} + 4\pi g_{ab}(S - \rho).
\end{aligned}$$

▷ The matter terms

$$\begin{aligned}
\rho &= (\det g)^{-\frac{1}{2}} \int_{\mathbb{R}^3} f(t, p_*)(1 + g^{cd}p_c p_d)^{\frac{1}{2}} \, dp_*, \\
S_{ab} &= (\det g)^{-\frac{1}{2}} \int_{\mathbb{R}^3} f(t, p_*) p_a p_b (1 + g^{cd}p_c p_d)^{-\frac{1}{2}} \, dp_*.
\end{aligned}$$

▷ Solutions tend to the Einstein-de Sitter model, i.e. $-dt^2 + t^{\frac{4}{3}}(dx^2 + dy^2 + dz^2)$.

Einstein-Boltzmann with Bianchi I symmetry

▷ The Boltzmann equation will be

$$\frac{\partial f}{\partial t} + 2k^a_b p^b \frac{\partial f}{\partial p^a} = Q(f,f) \quad \text{or} \quad \frac{\partial f}{\partial t} = Q(f,f).$$

▷ Representation of a momentum $p \in T_x M$:

$$p = p^a \mathbf{E}_a = \hat{p}^a \mathbf{e}_a,$$

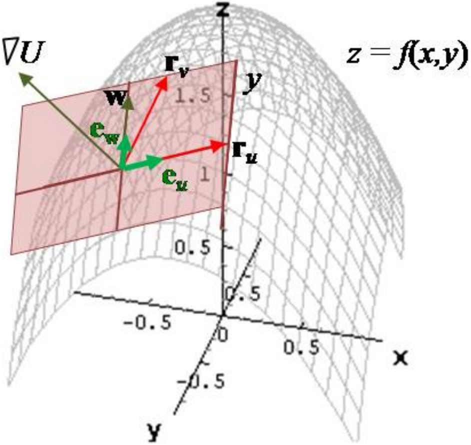
where $\{\mathbf{E}_a\}$ is the given basis and $\{\mathbf{e}_a\}$ an orthonormal basis such that $g(\mathbf{E}_a, \mathbf{E}_b) = g_{ab}$ and $g(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$.

▷ The Boltzmann equation in an orthonormal frame

$$\frac{\partial \hat{f}}{\partial t} + \hat{k}^a_b \hat{p}^b \frac{\partial \hat{f}}{\partial \hat{p}^a} = Q(\hat{f}, \hat{f}),$$

where $\mathbf{e}_a = e^b_a \mathbf{E}_b$, $p^a = e^a_b \hat{p}^b$ and $\hat{k}_{ab} = e^c_a e^d_b k_{cd}$.

▷ Orthonormal frame



http://math.etsu.edu/multicalc/prealpha/Chap3/Chap3-6/part3.htm

Roughly speaking..

http://astro.physics.sc.edu/selfpacedunits/Unit57.html

▷ The Boltzmann equation: in Strain's framework [Strain, 10],

$$\frac{\partial f}{\partial t} = (\det g)^{-\frac{1}{2}} \iint v_M \sigma(h,\theta)\Big(f(p'_*)f(q'_*) - f(p_*)f(q_*)\Big) d\omega dq_*,$$

and parametrization of post-collision momenta

$$\begin{pmatrix} p'^0 \\ p'_i \end{pmatrix} = \left(\begin{array}{c} \dfrac{p^0+q^0}{2} + \dfrac{h}{2}\dfrac{n_i e^i_j \omega^j}{\sqrt{s}} \\ \dfrac{p_i+q_i}{2} + \dfrac{h}{2}\left(g_{ij} e^j_k \omega^k + \left(\dfrac{n^0}{\sqrt{s}} - 1\right) \dfrac{n_j e^j_k \omega^k n_i}{g^{ab} n_a n_b} \right) \end{array} \right),$$

where $h^2 = (p_\alpha - q_\alpha)(p^\alpha - q^\alpha)$, $s = -n_\alpha n^\alpha$ and $n^\alpha = p^\alpha + q^\alpha$.

▷ The Boltzmann equation in the framework of [Glassey-Strauss, 93]:

$$\frac{\partial f}{\partial t} = (\det g)^{-\frac{1}{2}} \iint \frac{v_M \sqrt{s}(n^0)^2 \sigma(h,\theta)}{((n^0)^2 - (n_a e^a_b \xi^b)^2)^{3/2}} \Big(f(p'_*) f(q'_*) - f(p_*) f(q_*) \Big) d\xi dq_*,$$

and parametrization of post-collision momenta

$$\begin{pmatrix} p'^0 \\ p'_i \end{pmatrix} = \left(\begin{array}{c} \dfrac{p^0 + q^0}{2} + \dfrac{h}{2} \dfrac{n_i \xi^i_j \omega^j}{\sqrt{(n^0)^2 - (n_i e^i_j \xi^j)^2}} \\ \dfrac{p_i + q_i}{2} + \dfrac{h}{2} \dfrac{n^0 g_{ij} e^j_k \xi^k}{\sqrt{(n^0)^2 - (n_i e^i_j \xi^j)^2}} \end{array} \right).$$

▷ Differentiabiity for the relativistic Boltzmann equation [Guo-Strain, 12].

▷ We have the Einstein-Boltzmann system with Bianchi I symmetry.

Results

Einstein's equations for given matter terms

▷ Einstein's equations

$$\partial_t g_{ab} = -2k_{ab},$$
$$\partial_t k_{ab} = (g^{cd}k_{cd})k_{ab} - 2(g^{cd}k_{bd})k_{ac} - 8\pi S_{ab} + 4\pi g_{ab}(S-\rho).$$

▷ Assume that $f(t,p) = \hat{f}(t,\hat{p}) \leq \varepsilon \exp(t^{-\frac{5}{4}}|\hat{p}|^2)$ and C^1.
▷ Local existence by [Rendall, 94].
▷ Global-in-time existence by [Rendall, 94].
▷ Asymptotic behavior by [Nungesser, 10] such that

$$g_{ab}(t) = t^{\frac{4}{3}}\bar{g}_{ab}(t) \quad \text{and} \quad \bar{g}_{ab}(t) = G_{ab} + O(\varepsilon t^{-1}),$$

assuming smallness and using bootstrap argument.

▷ Decompose

$$k_{ab} = \sigma_{ab} - H g_{ab}, \quad H = -\frac{1}{3}k, \quad k = g^{ab}k_{ab},$$

where H is called the Hubble variable and k the mean curvature.

▷ Assume that σ_{ab} the trace free part is small in the sense that

$$F := \frac{1}{4H^2}\sigma_{ab}\sigma^{ab}.$$

▷ In the Robertson-Walker case, i.e. $g_{ab} = R^2(t)\eta_{ab}$, we have $\sigma_{ab} = 0$.

▷ Without smallness we have

$$\frac{1}{3t} \leq H(t) \leq \frac{2}{3t}.$$

▷ Assuming smallness we have

$$\frac{2}{3t(1+\varepsilon t^{-1})} \leq H(t) \leq \frac{2}{3t}.$$

▷ In the Robertson-Walker case, $H(t) = \frac{2}{3}t^{-1}$.

▷ Bootstrap argument: $F(t) \leq \varepsilon(1+t)^{-\frac{3}{2}} \Longrightarrow F(t) \leq \varepsilon(1+t)^{-2+\varepsilon}$.
▷ Equation for F:

$$\dot{F} = -3H\left(1 - \frac{2}{3}F - \frac{8\pi S}{9H^2} - \frac{4\pi S_{ab}\sigma^{ab}}{3H^3F}\right)F \sim -2t^{-1}F.$$

▷ We eventually obtain $F \sim \varepsilon t^{-2}$.
▷ Equation for $\bar{g}_{ab}$:

$$\dot{\bar{g}}_{ab} = 2\left(H - \frac{2}{3}t^{-1}\right)\bar{g}_{ab} - 2t^{-\frac{4}{3}}\sigma_{ab},$$

and note that $\left(H - \frac{2}{3}t^{-1}\right)$ is integrable.
▷ We eventually obtain $|\bar{g}_{ab}| \leq C$ and

$$g_{ab}(t) = t^{\frac{4}{3}}\Big(G_{ab} + O(\varepsilon t^{-1})\Big),$$

together with $F(t) \leq CF(t_0)t^{-2}$ and $H(t) = \frac{2}{3}t^{-1}(1 + O(\varepsilon t^{-1}))$.

The Boltzmann equation in a given spacetime

▷ The Boltzmann equation

$$\frac{\partial f}{\partial t} = (\det g)^{-\frac{1}{2}} \iint v_M \sigma(h,\theta)\Big(f(p'_*)f(q'_*) - f(p_*)f(q_*)\Big) d\omega dq_*,$$

▷ Consider first the Robertson-Walker case, i.e. $-dt^2 + R^2(dx^2 + dy^2 + dz^2)$.

▷ Take weight function $e^{|p_*|^2}$ and multiply this to the equation

$$\frac{\partial(e^{|p_*|^2} f(t,p_*))}{\partial t} = R^{-3} \iint \cdots$$

$$\cdots \Big(e^{|p'_*|^2} f(p'_*) e^{|q'_*|^2} f(q'_*) - e^{|p_*|^2} f(p_*) e^{|q_*|^2} f(q_*)\Big) e^{-|q_*|^2} d\omega dq_*,$$

if we have an identity $|p'_*|^2 + |q'_*|^2 = |p_*|^2 + |q_*|^2$.

▷ In the end,

$$\frac{d}{dt}\|f(t)\| \leq CR^{-3}\|f(t)\|^2 \quad \text{and} \quad \|f(t)\| \leq \|f(0)\| + C\|f(t)\|^2,$$

if R^{-3} is integrable.

▹ The post-collision momentum:

$$p_i' = \frac{p_i + q_i}{2} + \frac{h}{2}\frac{n^0 g_{ij} e^j_k \xi^k}{\sqrt{(n^0)^2 - (n_i e^i_j \xi^j)^2}} = \frac{p_i + q_i}{2} + \frac{Rh}{2}\frac{n^0 \xi_i}{\sqrt{(n^0)^2 - R^{-2}(n\cdot\xi)^2}},$$

$$Rh = |p_* - q_*|\sqrt{1 - \frac{|p_* + q_*|^2 \cos^2\theta_0}{R^2(p^0+q^0)^2}}.$$

▹ If $\lim_{t\to\infty} R(t) = \infty$, we have

$$p_*' \to \frac{p_* + q_*}{2} + \frac{|p_* - q_*|}{2}\xi \quad \text{and} \quad q_*' \to \frac{p_* + q_*}{2} - \frac{|p_* - q_*|}{2}\xi,$$

which is the parametrization of the nonrelativistic case. In other words, at late times the post-collision momenta with lower indices behave like in the nonrelativistic case. Hence, we will eventually have $|p_*'|^2 + |q_*'|^2 = |p_*|^2 + |q_*|^2$.

▹ We obtain a small solution such that

$$f(t,p_*) \le \varepsilon \exp(-|p_*|^2) \quad \text{or} \quad \hat{f}(t,\hat{p}) \le \varepsilon \exp(-R^2|\hat{p}|^2).$$

▷ In the Bianchi I case we may choose $\exp(\bar{g}^{ab}p_a p_b)$ to get

$$\hat{f}(t,\hat{p}) \leq \varepsilon \exp(-t^{\frac{4}{3}}|\hat{p}|^2)\Big(= \varepsilon \exp(-\bar{g}^{ab}p_a p_b)\Big).$$

▷ For a small ε such that $\frac{d}{dt}\left[t^{-\varepsilon}\bar{g}^{ab}(t)\right] \leq 0$, we have

$$\hat{f}(t,\hat{p}) \leq \varepsilon \exp(-t^{\frac{4}{3}-\varepsilon}|\hat{p}|^2)\Big(= \varepsilon \exp(-t^{-\varepsilon}\bar{g}^{ab}p_a p_b)\Big).$$

▷ Differentiability of solutions: [Guo-Strain, 12].

Thank you very much

Thank you very much.