Block Spin Transformation of 2D O(N) sigma model, Toward solving a Millennium Problem

K.R.Ito

Inst. for Fundamental Sciences
Setsunan Univ.

2015 Feb 23, RIMS, Kyoto Univ.

From HomePage of Clay Institute

1. Construction of 4D YM Field Theory (Jaffe, Witten)
2. Solution of Navier-Stokes Equation (Feffermann)
3. Riemann Conjecture (Bombieri)

What kind of Analysis do we need in these problems ?

Difficulties in 4D LGT, 2D Sigma and NvS Eq

1. The system is non-linear. Difficult to find linear part (or Gaussian part)
2. There appear relevant terms (increasing coupling constats by BST)
3. Relevant term means non-gausian integral

History of 2D Spin Systems

2D O(N)Spin Model is simple, but hard to analyze.

1. 2D Ising spin, existence of spontaneous magnetization, R.Peierls (1936), L.Onsager (1944)
2. Kosterlitz-Thouless Transition in 2D XY model, J.Fröhlich and T.Spencer (1982)
3. non-existence of phase transition in the Heisenberg model with large N ($\sim$ quark confinement in YM_4) (this talk)

The Model

The 2D $O(N)$ Heisenberg model. Strong non-linearity:

$$\langle \cdots \rangle = \int (\cdots) \exp[\sum_{n.n.} \phi_x \phi_y] \prod_x \delta(\phi^2(x) - N\beta) d^N \phi_x$$

$$= \int (\cdots) \exp[\sum_{n.n.} \phi_x \phi_y + \frac{i}{\sqrt{N}} \langle : \phi^2 :, \psi \rangle] \prod_{x \in \Lambda} \frac{d\psi(x)}{2\pi} d^N \phi_x$$

where x, y are lattice points $x, y \in \Lambda \subset Z^2$ and

$$\langle : \phi^2 :, \psi \rangle = \sum_x (\phi^2(x) - N\beta) \psi(x)$$

(This technique turns out to be not so helpful.)

The Gibbs measure:

$$\langle f(\phi)\rangle = \int f(\phi)\exp[-W_0(\phi,\psi)]\prod_x d^N\phi(x)d\psi(x)$$

$$W_0 = \frac{1}{2}\langle\phi,(-\Delta+m_0^2)\phi\rangle + \frac{g_0}{2N}\langle :\phi^2:, :\phi^2:\rangle - \frac{i}{\sqrt{N}}\langle :\phi^2:,\psi\rangle$$

$$:\phi^2:(x) = \sum_{i=1}^N \phi_i^2(x) - NG(0), \quad \beta = G(0)$$

Here $(-\Delta)_{xy} = 4\delta_{xy} - \delta_{1,|x-y|}$ is the Lattice Laplacian on Z^2.
$G(0)=\beta$ means $m_0^2 \sim 32e^{-4\pi\beta}$:

$$G(x) = \frac{1}{-\Delta+m_0^2}(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{ipx}}{m_0^2+2\sum(1-\cos p_i)}\prod\frac{dp_i}{2\pi}$$

W_0 has two main Gaussian terms+(double well pot.):

$$\exp[-\frac{1}{2}\langle\phi,(-\Delta+m_0^2)\phi\rangle-\frac{g_0}{2N}\langle:\phi^2:,:\phi^2:\rangle]$$
$$\exp[-\mathrm{Tr}(G\psi)^2]=\exp[-\langle\psi,G_0^{\circ 2}\psi\rangle]$$

Here $g_0\geq 0$ can be chosen arbitrary. Here the second term is from the expansion of

$$\det{}^{-N/2}\left(1+\frac{2i}{\sqrt{N}}G\psi\right)\exp[-\sum_x i\sqrt{N}\beta\psi(x)]$$

RG is iterative integration over high p part of ϕ and ψ which keeps these Gaussian terms invariants.

Decompose $\langle \phi, (-\Delta + m_0^2)\phi \rangle$ into many Gaussians with covariances $\Gamma_n = Q^+ G_n^{-1} Q$:

$$\langle \phi, (-\Delta + m_0^2)\phi \rangle = \langle \phi, G_0^{-1}\phi \rangle$$
$$= \langle \phi_1, G_1^{-1}\phi_1 \rangle + \langle z_0, \underbrace{Q^+ G_0^{-1} Q}_{\Gamma_0^{-1}} z_0 \rangle$$

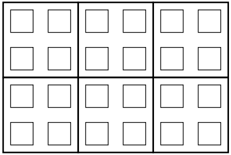
and continue, where

$$\phi_n(x) = (C\phi_n)(x) = \frac{1}{L^2} \sum_{\zeta \in \Delta_0} \phi_{n-1}(Lx + \zeta)$$
$$G_n(x, y) = \frac{1}{L^4} \sum_{\zeta, \xi \in \Delta_0} G_{n-1}(Lx + \zeta, Ly + \xi)$$

and

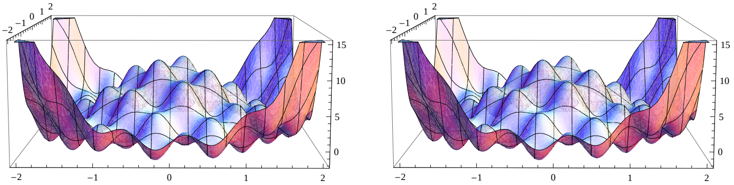
$$\begin{aligned} \phi_n(x) &= A_{n+1}\phi_{n+1} + Qz_n, \\ Qz_n &= Q\Gamma_n^{1/2}\xi_n = \text{block average zero fluctuations} \\ &\quad \text{where} \xi_n = \mathrm{N}(1,0) \end{aligned}$$

Put $\Lambda_n = \{L^{-n}\Lambda \cap Z^2\}$ and see boxes (of side length $L, L^2, L^3, \cdots$) where ξ_k lives on the lattice of width L^k



Fluctuations z_n constrained by Double-Wells

$\exp[-\frac{g}{N}((\phi_{n+1}+z_n)^2 - N\beta_n)^2]$, yields strong constraint on z_n :



Fluctuations $\xi_n(x)$ perpenficular to $\phi_{n+1}(x)$ have $N-1$ degrees of freedom of gaussian fields. They propagate along the bottom of the bottle

BST=Perturbation around the Gaussians:

C leaves the fundamental Gaussian measures invariant. They are left inv by C:

$$G_n(x,y) \quad = \quad (CG_{n-1}C^{+})(x,y) \sim G_0(x,y)$$

$$\begin{aligned} &\exp[-W_{n+1}(\{\phi_{n+1}\})] \\ &\equiv \int \exp[-W_n(\{\phi_n\})] \prod_{x\in\Lambda_{n+1}} \delta(\phi_{n+1}(x) - (C\phi_n)(x)) \\ &\times \prod_{\zeta\in\Lambda_n} d\phi_n(\zeta) \end{aligned}$$

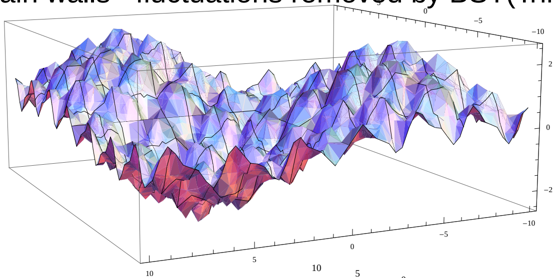
We expect W_n keeps its main terms invariant

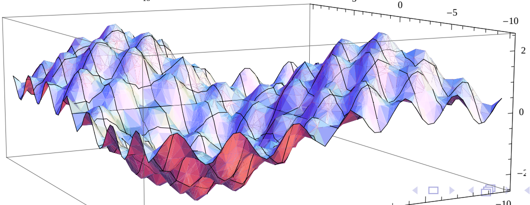
$$
\begin{aligned}
W_n(\phi_n, \psi_n) &= \frac{1}{2}\langle \phi_n, G_n^{-1}\phi_n\rangle + \frac{g_n}{2N}\langle \phi_n^2 :_{G_n}, \phi_n^2 :_{G_n}\rangle \\
&\quad + \text{correction} \\
G_n(0) &= \beta_n \sim \beta_0 - \text{const.} n
\end{aligned}
$$

Problem: W_n keeps its form with small irrelevant corections ?

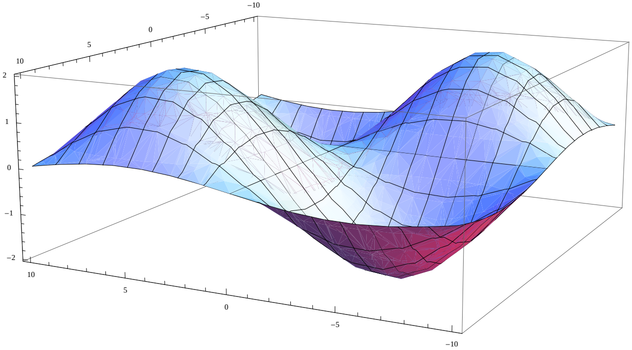
$$\begin{aligned} \text{large or violent } \phi &= \text{Long Domain Walls + Short Domain walls} \\ D(\phi_n) &= D_w(\phi_{n+1}) \cup R(z_n) \end{aligned}$$

Short Domain walls =fluctuations removed by BST(Trimming)





Block Spin=Trimming short waves



Fluctuations $\xi_n(x)$ perpenficular to $\phi_n(x)$ have $N-1$ degrees of freedom of gaussian fields.

RG=Contraction Map on Banach Space $\mathcal{H}$

Namely we consider of Flow of Space $\mathcal{K}_n$ of Spin Configurations

$$\mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots \supset \mathcal{K}_n$$

$\mathcal{K}_n$ =smoothly propagating spin waves on the surfaces of balls

1. no domain walls
   $|\phi_n(x)\phi_n(y) - N\beta_n| < N^{1/2+\varepsilon} \exp[(c/10)|x-y|]$
   $\forall x, y \in K$
2. $|\phi_n(x)^2 - N\beta_n| < N^{1/2+\varepsilon}$
3. $|\nabla\phi_n(x)| < N^{1/2+\varepsilon}$

Block spin $\phi_n(x)$ =Low Mom. spin +High Mom. spin
=Next order BS+ Zero Ave. Fluct.

$$= \sum_y A_{n+1}(x,y)\phi_{n+1}(y) + \sum_y (Q)_{xy}\zeta_y$$

Approximately, $A \in \mathrm{Mat}(L^2, 1)$, $Q \in \mathrm{Mat}(L^2, L^2 - 1)$:

$$A(x,y) \sim \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, Q(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q^+ acts as a differential operator.

$$Q = U|Q|, \quad UU^+ = E^\perp = \text{proj to } \mathcal{N}(C)$$

Serious difficulty is

$$\phi_n(x) = A_n \phi_{n+1} + Q\xi(x) \sim \phi_{n+1}([x/L]) + Q\xi(x)$$

Namely $\phi_n(x), |x| < L/2$ contain L^2 of $\phi_{n+1}([x/L])$. Thus

$$\sum_{x\in\Lambda_n} \phi_n^2(x) \sim L^2 \sum_{x\in\Lambda_{n+1}} \phi_{n+1}^2(x)$$

$$\sum_{x\in\Lambda_n} (:\phi_n^2:_{G_n}(x))^2 \sim L^2 \sum_{x\in\Lambda_{n+1}} (:\phi_{n+1}^2(x):_{G_{n+1}})^2$$

ϕ^4 term increases exponentially in n, i.e. relevant term.

BUT THIS DOES NOT HAPPEN.

Theorem on the RG flow

The main part of W_n is represented by 3 terms and three parameters β_n, g_n, γ_n and m_n^2, :

$$
\begin{aligned}
W_n(\phi_n, \psi_n) &= \frac{1}{2}\langle \phi_n, G_n^{-1}\phi_n + \frac{g_n}{2N}\langle : \phi_n^2 :_{G_n}, : \phi_n^2 :_{G_n} \rangle \\
&\quad + \frac{1}{2}\gamma_n < \phi_n^2, E^{\perp} G_n^{-1} E^{\perp} \phi_n^2 >
\end{aligned}
$$

where

1. $G_n^{-1} = -\Delta + m_n^2$, $m_n^2 = L^{2n} m_0^2$
2. $\gamma_n = (N\beta_n)^{-1}$.
3. $g_n = g^* = O(1) > 0$ (fixed point)
4. $E^{\perp} =$ projection to $\mathcal{N}(C) = \{f; Cf = 0\}$

- the first two terms = marginal (main term)
- the last term is irrelevant. it fades e away.
- $(:\phi_n^2:)^2$ is relevant but g_n converges to a constant in the scaling region

All this means is that the system simple and is close to ϕ_4^4 model (triviality model). The flow is described by three parameters

$$m_n^2 = L^{2n} m_0^2 \sim \exp[-4\pi\beta + 2n \log L] \to O(1),$$
$$\beta_n = \beta - \text{const}.n \to O(1)$$
$$\gamma_n = O((\beta_n N)^{-1})$$
$$g_n = O(1)$$

Main Conclusion

This means that system goes to the single-well potential, and then absence of phase transitions follows.

Sketch of the Proof

Main Ideas and Theorems:

Set $\phi_n = A_{n+1}\phi_{n+1} + z_n$, $z_n = Q\xi_n$ so that

$$\begin{aligned}
< \phi_n, G_n^{-1}\phi_n > &= < \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} > + < \xi_n, \Gamma_n^{-1}\xi_n >, \\
\Gamma_n^{-1} &= Q^+ G_n^{-1} \sim Q^+(-\Lambda)Q > O(1) \\
:\phi_n^2(x):_{G_n} &= :\phi_{n+1}^2(x):_{G_{n+1}} + q(x) \\
q(x) &= 2\phi_{n+1}(x) z_n(x) + :z(x)_n^2:_{\Gamma_n}
\end{aligned}$$

Calculate

$$\begin{aligned} P(\varphi_{n+1}, p) &= \int \exp\left[\frac{i}{\sqrt{N}}\langle (p-q), \lambda\rangle\right] d\mu(\xi) \prod d\lambda_x \\ d\mu(\xi) &= \exp[-\langle \xi, \Gamma_n^{-1}\xi\rangle] \prod d\xi_x \end{aligned}$$

the distribution function of $2\phi_{n+1}(x) z_n(x) + :z(x)_n^2:_{\Gamma_n}$ with respect to $d\mu(\xi)$

Thorem 1:

$$\begin{aligned} P(p,\varphi) &= \exp[-\frac{1}{4N}\langle p, \frac{1}{M}p\rangle] \\ M &= \Gamma_n^{\circ 2} + 2(\phi_n\phi_n)\Gamma_n \end{aligned}$$

This is approximately Gaussian which depends on the domain wall $\phi_n(x)\phi_n(y) - N\beta_n$

Definition of Domain Wall

Domain walls are paved set such that

$$|\phi_n(x)\phi_n(y) - N\beta_n| > N^{1/2+\varepsilon}\exp[(c/10)|x-y|]$$

$$\forall x \in D_w, \exists y \in D_w$$

$1/2$ is the central limit theorem for $\sum : \xi_i^2 :$. Outside of D_w,

$$|\phi_n(x)\phi_n(y) - N\beta_n| < N^{1/2+\varepsilon}\exp[(c/10)|x-y|]$$

$$\forall x \in D_w^c, \forall y \in D_w^c$$

Thus $\phi_n(x)\phi_n(y) = NG_n(x,y)$ on $(D_w)^c$

Theorem 2

Domain Wall region D_w has high energy:

$$\int \exp[-\frac{1}{2} < \varphi_n, G_0^{-1}\varphi_n >_{D_w}] d\mu(\xi) < \exp[-N^{2\varepsilon}|D_w|]$$

Outside of D_w, we can can replace $\varphi\varphi$ by NG_n, and we have a Gaussian integral over p.

We integrate over ξ under the influence of long spin wave by p variables. Using $:\varphi_n^2:^2 = (:\varphi_{n+1}^2: + p)^2$, we replace ξ^4 by p^2:
Theorem 3

$$
\begin{aligned}
&\int \exp[-\frac{g_n}{2N}\langle :\varphi_n^2:, :\varphi_n^2:\rangle + (.....)] d\mu(\xi) \\
&= \int \exp[-\frac{g_n}{2N}\langle :\varphi_{n+1}^2: + p, :\varphi_{n+1}^2: + p\rangle] P(p,\varphi) \prod dp
\end{aligned}
$$

$$
P(p) = \exp[-\langle p, M^{-1} p\rangle / 4N]
$$

This can be done by steepesr descent+pertuebation. Then $g_n \to g^*$ (convergence).

Final Step:

Though W_n contains relevant ϕ^4 term, its strength g_n coverges to a constant $g^* = O(1)$, and the position of its bottom β_n cnverges to 0

Thank you very much for your attension and patience!