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Existence of nonperturbative nonlocal field theory on noncommutative space and spiral source in renormalization group approach of matrix model

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1. Motivations

Random matrix theory (RMT, or MM):

statistical mechanics with dynamical variable: matrices with rank N

$$Z = \int dM_{ij}\, e^{-\beta \mathrm{tr} H(M)}, \quad \langle \mathcal{O}(\mathcal{M}) \rangle = \frac{1}{Z} \int dM_{ij}\, \mathcal{O}(M)\, e^{-\beta \mathrm{tr} H(M)}$$

- large-N limit: $\beta \to \beta_c,\ N \to \infty$ with $(\beta - \beta_c) N^{\star}$: fixed
  $\to$ critical phenomena (phase transition) (thermodynamic limit)

$\star$: universal, but dynamical $\leftarrow$ difficult to fix in general

$\to$ formalism to extract universal quantities $\to$ RG approach of MM

2. Review of large-N renormalization group

Brezin-Zinn-Justine '92

Begin with rank $(N+1)$ matrix: $M_{N+1} = \begin{pmatrix} M_N & v_i \\ {}^t v_i^* & \alpha \end{pmatrix}$

among $\int dM_{ij}$, integrate only one column and row:

$$e^{-S'_N(M_N,g)} = \lambda_N(g) \int dv dv^* d\alpha \, e^{-S_{N+1}(M_N,v,v^*,\alpha,g)}$$

→ result: regarded as rank N matrix model and read change of g

e.g.:

$$S_{N+1}(M_{N+1}) = (N+1)\mathrm{tr}_{N+1}\left(\frac{1}{2}M_{N+1}^2 + \frac{g}{4}M_{N+1}^4\right)$$

$$= (N+1)\left[\mathrm{tr}_{N+1}\left(\frac{1}{2}M_N^2 + \frac{g}{4}M_N^4\right) + v^*v\right] + (N+1)g\left[v^*M_Nv + \frac{1}{2}(v^*v)^2\right] + f(\alpha)$$

$$\to \quad S'_N(M_N) = (N+1)\left[\mathrm{tr}_N\left(\frac{1}{2}M_N^2 + \frac{g}{4}M_N^4\right)\right] + g\,\mathrm{tr}_N M_N^2 + \mathcal{O}(g^2)$$

$$\rightarrow \quad S'(M'_N) = N\mathrm{tr}_N\left(\frac{1}{2}M'^2_N + \frac{g'}{4}M'^4_N\right), \quad M_N = \rho M'_N, \quad \rho^2 = \left(\frac{\frac{N}{2}}{\frac{N+1}{2} + g}\right)$$

$g' = g - \frac{1}{N}(g + 4g^2)$: change of coupling under $N \rightarrow N - 1$

$$\rightarrow \quad \beta(g) \equiv \frac{\partial g}{\partial\left(\frac{1}{N}\right)} = -g - 4g^2 \quad \rightarrow \quad g^* = -1/4\ (-1/12), \quad \gamma_1 = 2\ (5/2)$$

<u>Exact result (2D quantum gravity)</u>:

David '89, Distler-Kawai '89

$$Z_{h=0} = \Delta^{\gamma_1} f(\Delta N^{2/\gamma_1}), \quad \Delta = g_c - g, \quad \gamma_1 = 5/2,$$

string susceptibility

- Advantages: simple one column & row calc. leads to large-N limit

  $N \to \infty \Leftrightarrow \infty$ times one column & row RG $\Leftrightarrow$ fixed pt. (universal)

  universality of $N \to \infty$ cf. thermodynamic limit, continuum limit
- Drawbacks:
  - unclear notion of high/low energy modes ($\to$ locality of RG!)

    Wilson-Kogut '74
  - space-time interpretation of matrices in string theory

$\to$ assign the notion of 'energy' to each matrix element naturally, and then develop new large-N RG based on it

$\to$ expect nice correspondence to RG in usual field theory, in particular, locality (in the space of matrices!)

3. Review of fuzzy sphere

spin L SU(2) rep. (angular momentum operator) $\hat{J}_1, \hat{J}_2, \hat{J}_3$:

$$[\hat{J}_i\,,\,\hat{J}_j] = i\epsilon_{ijk}\hat{J}_k, \qquad N = (2L+1)\text{-dim. matrices}$$

$\hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 = L(L+1)$: eq. of S^2 → noncommutative (fuzzy sphere)

functions on S^2 → $N \times N$ matrices

$$Y_{\ell m} = \sum_{i_1,\cdots,i_\ell} c^{(\ell m)}_{i_1\cdots i_\ell} x^{i_1}\cdots x^{i_\ell} \quad \to \quad T_{\ell m} = \sum_{i_1,\cdots,i_\ell} c^{(\ell m)}_{i_1\cdots i_\ell} \hat{J}^{i_1}\cdots \hat{J}^{i_\ell}$$

$$\phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m} \quad \to \quad \hat{\phi} = \sum_{\ell=0}^{N-1}\sum_{m=-\ell}^{\ell} c_{\ell m} T_{\ell m} \quad : \quad N \times N \text{ matrix} \; \left(\sum_{\ell=0}^{N-1}\sum_{m=-\ell}^{\ell} 1 = N^2\right)$$

sp. of functions on S^2 with $\ell \leq 2L$ ≃ sp. of $N \times N$ Hermitian matrices

not closed — closed! (alg. or ring)

(as vector. sp.)

- Laplacian & integration:

$$\int d\Omega\, Y_{\ell m} Y^*_{\ell' m'} = \frac{1}{N}\mathrm{tr}(T_{\ell m} T^\dagger_{\ell' m'}) = \delta_{\ell\ell'}\delta_{mm'},$$

$$\Delta Y_{\ell m} = \ell(\ell+1) Y_{\ell m} \quad \rightarrow \quad [\hat{J}_i, [\hat{J}_i, T_{\ell m}]] = \ell(\ell+1) T_{\ell m}$$

- Regularization of field theory on S^2:

field theory on S^2: $\phi(\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\phi_{\ell m} Y_{\ell m}(\theta,\varphi)$

matrix model: $\phi = \sum_{\ell=0}^{N-1}\sum_{m=-\ell}^{\ell}\phi_{\ell m} T_{\ell m}$

equivalence of action (change of basis):

$$S(\{\phi_{\ell m}\}) = \int \rho^2 d\Omega \left(\frac{1}{2\rho^2}\phi(\theta,\varphi)\Delta\phi(\theta,\varphi) + \frac{m^2}{2}\phi(\theta,\varphi)^2 + \frac{g}{4}\phi(\theta,\varphi)^4\right)$$

$$= \frac{\rho^2}{N}\mathrm{tr}_N\left(\frac{1}{2\rho^2}\phi[\hat{J}_i, [\hat{J}_i, \phi]] + \frac{m^2}{2}\phi^2 + \frac{g}{4}\phi^4\right)$$

with rotational symmetry! $$U(R) T_{\ell m} U(R)^{-1} = \sum_{m'=-\ell}^{\ell} T_{\ell m'} R^{\ell}_{mm'}(R)$$

- Realization of the notion of angular momentum or energy in each matrix elements: $\phi_{\ell m}$
- $N-1=2L$: UV cutoff → $N\to\infty$ to recover $C^{\infty}(S^2)$

$$\hat{x}_i=\alpha\hat{J}_i \;\to\; \hat{x}_1^2+\hat{x}_2^2+\hat{x}_3^2=\rho^2,\quad \rho^2=\frac{N^2\alpha^2}{4},\qquad [\hat{x}_i,\hat{x}_j]=i\alpha\epsilon_{ijk}\hat{x}_k$$

noncommutativity

$N^2\pi\alpha^2\simeq 4\pi\rho^2$: S^2 divided by N^2 cells (cf. lattice $a\sim\alpha$)

we take $N\to\infty$ with noncommutativity α fixed

→ field theory on fuzzy sphere

- Moyal plane: $[x,y]=i\theta$: ∞-dim. → inadequate for large-N RG

4. Large-N RG on fuzzy sphere

Formulation:

Start by matrix model with $N \times N$ Hermitian matrix

$\phi = \sum_{\ell=0}^{2L} \sum_{m=-\ell}^{\ell} \phi_{\ell m} T_{\ell m}$ $(2L = N - 1)$:

$$S_N = \frac{\rho_N^2}{N} \mathrm{tr}_N \left(\frac{1}{2\rho_N^2} \phi [\hat{J}_i, [\hat{J}_i, \phi]] + \frac{m_N^2}{2} \phi^2 + \frac{g_N}{4} \phi^4 \right)$$

we integrate only cutoff modes $\phi_{2L\, m}$ $(m = -2L, \cdots, 2L)$

→effective action:

$$e^{-S_{N-1}} = \int \prod_{m=-2L}^{2L} d\phi_{2L\, m}\, e^{-S_N}, \quad \left(\sum_{m=-2L}^{2L} 1 = 4L + 1 = 2N - 1 = N^2 - (N-1)^2 \right)$$

and rewrite S_{N-1} as $(N-1)^2$ matrix model → read $(m_{N-1}^2,\ g_{N-1})$

- Integrate “highest energy” modes → expect “local” & nice RG

  ※compared BZ, only different basis we favor → important for locality

- Not only large-N RG as well as RG of field theory on fuzzy sphere

  RG of noncommutativity, nonlocality

  → understand nonlocal nature, scale dependence of QFT

- However, it seems (at least to me) that S_{N-1} can be again rewritten in terms of standard operation among matrices with rank $N-1$:

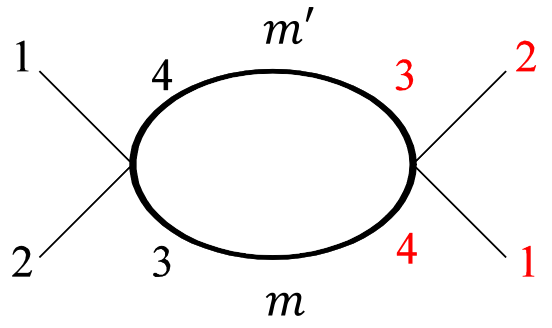
$$e^{-S_{N-1}(\{\phi_{\ell' m}\})} = \int \prod_{m=-2L}^{2L} d\phi_{2L\,m}\, e^{-S_N(\{\phi_{\ell m}\})}$$

nontrivial function of $\phi_{\ell' m}$

- We compute RHS in perturbation theory with respect to g_N

5. Properties of large-N RG

1. multi trace operators are generated in general



$$\Delta_{2L}^2 \mathrm{tr}_N \left(\phi^2 T_{2L\,m} T_{2L\,m'}\right) \mathrm{tr}_N(\phi^2 T_{2L\,m'} T_{2L\,m})$$

$$\langle \phi_{2L\,m}\phi_{2L\,m'}\rangle = \Delta_{2L}(-1)^m \delta_{mm'}, \quad \Delta_{2L} = \frac{1}{N(N-1)+\rho_N^2 m_N^2}$$

However, above eq.= $\dfrac{\Delta_{2L}^2}{(2N-1)N}\left(\mathrm{tr}_N \phi^4 - \dfrac{1}{2N}\mathrm{tr}_N\left(\phi^2\left[\hat{J}_i,[\hat{J}_i,\phi^2]\right]\right)+\cdots\right)$

cf. field theory case: $g^2\int dx \int dy\, \phi(x)^2\phi(y)^2\Delta(x-y)^2$: bilocal field

$\Delta(x-y)$: highly massive mode propagator

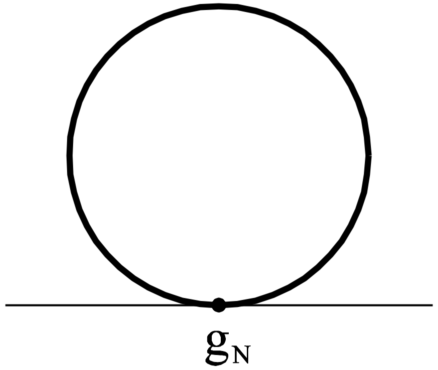
→ short distance, rapidly damp as $x-y$: large

→ derivative exp. is good → local field with derivative expansion

above: <u>derivative exp. in the space of matrices ← locality of our RG</u>

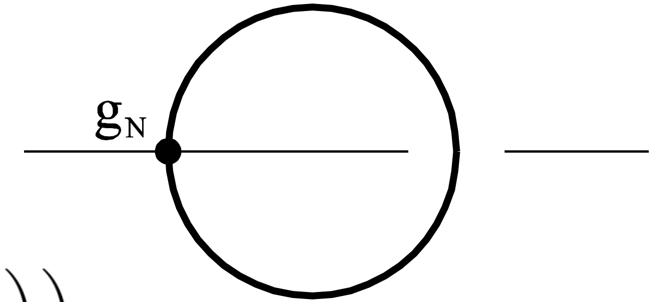
2. nonplanar diagram generates nonlocal interactions

e.g.: mass correction



planar: $\sum_{m,m'} \langle \phi_{2L\,m}\phi_{2L\,m'} \rangle \, \mathrm{tr}_N \left(\phi^2 T_{2L\,m} T_{2L\,m'} \right)$

$$= (2N-1)\Delta_{2L} \sum_{\ell,m} \phi^*_{\ell\,m}\phi_{\ell\,m} = N(2N-1)\Delta_{2L}\mathrm{tr}_N(\phi^2)$$



nonplanar:

$$\sum_{m,m'} \langle \phi_{2L\,m}\phi_{2L\,m'} \rangle \, \mathrm{tr}_N \left(\phi T_{2L\,m} \phi T_{2L\,m'} \right)$$

$$= \frac{1}{2}N(2N-1)\Delta_{2L} \sum_{\ell,m} (-1)^\ell \phi^*_{\ell m}\phi_{\ell m} \frac{1}{N}\left(1 - \frac{\ell(\ell+1)}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right)$$

"antipode transformation" $\phi = \sum_{\ell,m} \phi_{\ell m} T_{\ell m} \mapsto \phi^A \equiv \sum_{\ell,m} (-1)^\ell \phi_{\ell m} T_{\ell m}$

$$\frac{1}{2}N(2N-1)\Delta_{2L}\left[\mathrm{tr}_N\left(\phi^A \phi\right) + \frac{1}{N}\mathrm{tr}_N\left([\hat{J}_i, \phi^A][\hat{J}_i, \phi]\right) + \cdots \right]$$

antipode transf.: $\phi = \sum_{\ell,m} \phi_{\ell m} T_{\ell m} \longmapsto \phi^A \equiv \sum_{\ell,m} (-1)^\ell \phi_{\ell m} T_{\ell m}$

counterpart: $Y_{\ell m}(\theta, \varphi) \longmapsto Y_{\ell m}(\pi - \theta, \varphi + \pi) = (-1)^\ell Y_{\ell m}(\theta, \varphi)$

matrix model "knows" most natural discrete transf. on S^2

Note:

equality!

$$\frac{1}{N} \mathrm{tr}_N \left(\phi \phi^A\right) = \int d\Omega \, \phi(\theta, \varphi) \phi(\pi - \theta, \varphi + \pi)$$

nonlocal interaction

In the spirit of RG, it is natural to introduce terms with ϕ^A from the beginning

We are led to start by action with antipodal interaction

$$S_N = S_N^{(\mathrm{kin.})} + S_N^{(\mathrm{pot.})}$$

$$S_N^{(\mathrm{kin.})} = \frac{1}{N}\mathrm{tr}_N\left(\frac{1}{2}\phi[\hat{J}_i,[\hat{J}_i,\phi]] + \frac{\rho_N^2 m_N^2}{2}\phi^2 - \frac{\zeta_N}{2}\phi^A[\hat{J}_i,[\hat{J}_i,\phi]] + \frac{\rho_N^2 \tilde{m}_N^2}{2}\phi\phi^A\right)$$

$$S_N^{(\mathrm{pot.})} = \frac{\rho_N^2}{4N}\mathrm{tr}_N\left(\kappa_N^{(0)}\phi^4 + \kappa_N^{(1)}\phi^3\phi^A + \kappa_N^{(2\alpha)}\phi^2(\phi^A)^2 + \kappa_N^{(2\beta)}(\phi\phi^A)^2\right)$$

by pert. theory in $\kappa_N^{(a)}$ up to 1st nontrivial order

$\longrightarrow$ RG of 6 parameters!

Note again that $\displaystyle \frac{1}{N}\kappa_N^{(0)}\mathrm{tr}_N\phi^4 = \kappa_N^{(0)}\int d\Omega\,\phi(\theta,\varphi)^4$

while $\displaystyle \frac{1}{N}\kappa_N^{(1)}\mathrm{tr}_N\left(\phi^3\phi^A\right) = \kappa_N^{(1)}\int d\Omega\,\phi(\theta,\varphi)^3\phi(\pi-\theta,\varphi+\pi)$

nonlocal int.

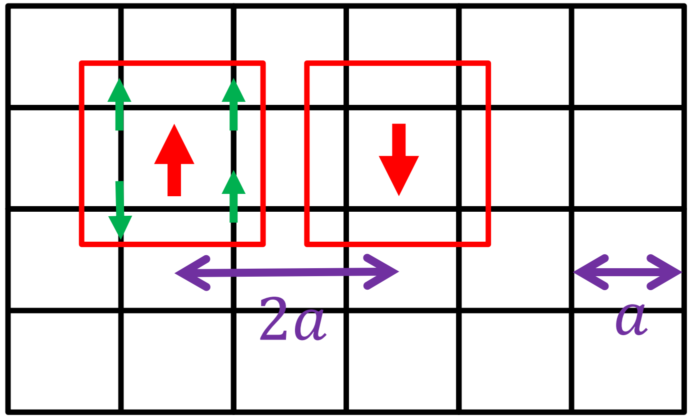
※ we can restrict ourself to the case with # of $\phi^A \leq 2$

$\because$ $\quad (\phi^A)^A = \phi, \quad \mathrm{tr}_N\left(\phi_1\cdots\phi_n\right) = \mathrm{tr}_N(\phi_n^A\cdots\phi_1^A)$

6. Fixed point analysis

$(m_N^2, \tilde{m}_N^2, \kappa_N^{(a)}) \to (m_{N-1}^2, \tilde{m}_{N-1}^2, \kappa_{N-1}^{(a)})$: how to get universal quantities?

<u>Scale transformation</u>



block spin transf.: to recover the original a

scale transf. $p \to bp$ $(a \to b^{-1}a)$

present case: unique scale $\rho_N \to$ NC $\alpha_N^2 \simeq \rho_N^2/N^2$

RG $N \to N-1$ increases noncommutativity

$\to$ recover the original noncommutativity via $\rho_{N-1} = (1+1/N)\rho_N$

$\to \alpha_{N-1} \simeq \rho_{N-1}/N = \alpha_N$ i.e. scale transf. $\quad \boldsymbol{b_N = \frac{1/\rho_{N-1}}{1/\rho_N}} = 1 + \frac{1}{N}$

scaling dimension: response to scale transf.

$\to$ scaling dim. $= d \Leftrightarrow$ eigenvalue of RG transf. near fp. $= b_N^d \simeq 1 + d/N$

Moyal plane limit (different large-N limit): Chu-Madore-Steinacker '92

stereographic projection: $x_+ = 2\rho_N \hat{J}_+ \left(\rho_N - \hat{J}_3\right)^{-1}, \;\; x_- = 2\rho_N \left(\rho_N - \hat{J}_3\right)^{-1} \hat{J}_-$

$N \to \infty$ with $\theta = 2\rho_N^2 / N$: fixed and restrict $\hat{J}_3 = -\rho_N + \mathcal{O}(1/\sqrt{N})$

near "south pole"

$\rightarrow [x_1,\; x_2] = -i\theta \;\; (x_\pm = x_1 \pm i x_2)$

In this case, $\frac{\rho_N^2}{N} = \frac{\rho_{N-1}^2}{N-1} \rightarrow b_N = 1 + \frac{1}{2N}$

we can describe field theories on fuzzy sphere and Moyal plane simultaneously and how to read scaling dimensions there is quite evident!!

RGE

$$m_{N-1}^2 = b_N^2 \Big[m_N^2 + 2(2N-1)\Delta_N \Big(\kappa_N^{(0)} + \Big(\frac{1}{4} - \frac{1}{2}(-1)^N\Big)\kappa_N^{(1)} + \frac{1}{2}\Big(1-(-1)^N\Big)\kappa_N^{(2\alpha)} \Big)$$

$$-2(2N-1)^2\Delta_N^3 b_N^2 \rho_N^2 \Big(2\kappa_N^{(0)} + \Big(\frac{1}{2} - (-1)^N\Big)\kappa_N^{(1)} + (1-(-1)^N)\kappa_N^{(2\alpha)} \Big)$$

$$\times \Big(\kappa_N^{(0)} - (-1)^N \kappa_N^{(1)} + \kappa_N^{(2\alpha)} + \kappa_N^{(2\beta)} \Big) \Big]$$

$$\kappa_{N-1}^{(0)} = b_N^2 \Big[\kappa_N^{(0)} - 2(2N-1)\Delta_N^2 b_N^2 \rho_N^2 \Big(\kappa_N^{(0)} + \Big(\frac{1}{4} - \frac{1}{2}(-1)^N\Big)\kappa_N^{(1)} + \frac{1}{2}\Big(1-(-1)^N\Big)\kappa_N^{(2\alpha)} \Big)^2 \Big]$$

...

- higher order terms are strongly suppressed: locality of our RG

  ∵ Δ_N : highly suppressed ← fuzzy sphere structure
- essentially same form as in field theory case ← nice similarity
- depend on whether N is even or odd (from CG coefficients)

Schematically, RGE takes form

$$x^{(i)}_{N-1} = f^{(i)}\left(x^{(1)}_N, \cdots, x^{(6)}_N\right) = \sum_i c^{(i)} x^{(i)}_N + \sum_{j,k} d^{(i)}_{jk} x^{(j)}_N x^{(k)}_N \qquad (i,j,k = 1,\cdots,6)$$

recursion relation

fixed pt. eqs.: $x^{(i)}_* = f^{(i)}\left(x^{(1)}_*, \cdots, x^{(6)}_*\right)$ (6 quadratic eqs.)

have different solutions for even/odd N $\rightarrow$ do not converge

However, if we construct two-step RG (i.e. keeping even/odd):

$$x^{(i)}_{N-2} = g^{(i)}\left(x^{(1)}_N, \cdots, x^{(6)}_N\right)$$

by using one-step RG twice, then they have the same fps.!

in spite of the fact that $g^{(i)}$'s are different for even/odd N

# of fixed pts.: 4 (including the Gaussian fp.: $x^{(i)}_* = 0$ for all i)

Around each fixed pt., we linearize the RG transf.:

$$\delta x^{(i)}_{N-2} = R_{ij}\,\delta x^{(j)}_{N}, \qquad \delta x^{(i)}_{M} \equiv x^{(i)}_{M} - x^{(i)}_{*}$$

6 eigenvalues= $b_N^{d_i} = (1+1/N)^{d_i} \simeq 1 + d_i/N \rightarrow$ scaling dim.= d_i

List:

① Gaussian FP.: $\alpha_N^2 m_*^2 = \alpha_N^2 \tilde{m}_*^2 = \alpha_N^2 \kappa_*^{(a)} = 0$, $d_i = 2$ for all i

canonical dimension

② three nontrivial FPs (Wilson-Fisher type):

$(\alpha_N^2 m_*^2,\ \alpha_N^2 \tilde{m}_*^2,\ \alpha_N^2 \kappa_*^{(0)},\ \alpha_N^2 \kappa_*^{(1)},\ \alpha_N^2 \kappa_*^{(2\alpha)},\ \alpha_N^2 \kappa_*^{(2\beta)})$

a. $(-0.50,\ 0.50,\ 0.42,\ -1.71,\ 0.21,\ 1.06)$

b. $(-0.23,\ 0.42,\ 0.23,\ -1.71,\ 0.46,\ 1.33)$

c. $(-0.48,\ 0.51,\ 0.06,\ 0.25,\ 0.12,\ 0.07)$

consistent with perturbation theory

Scaling dimensions of operators around nontrivial fixed pts.:

a. $(-2.65, 2.00, 2.00, 1.44 + 0.77i, 1.44 - 0.77i, 0.48)$
b. $(-2.34, 2.00, 2.00, 1.38 + 0.71i, 1.38 - 0.71i, -0.55)$
c. $(-2.66, 2.00, 2.00, 1.99, 1.88, 1.33)$

Observations:

1. all fixed pts. have two degenerate operators of dim. 2 $\rightarrow \delta m_N^2$ & $\delta \tilde{m}_N^2$ (in fact, 86% for a, b and 99% for c in eigenvec.)
2. complex scaling dimension??

Essentially we are considering two linear differential eqs.:

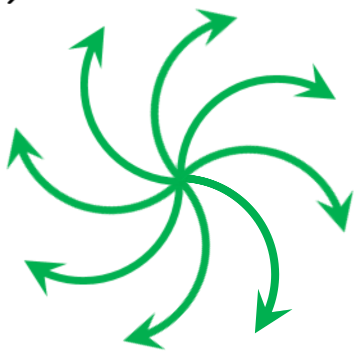
$$\delta \dot{x}^{(i)} = R_{ij}\delta x^{(j)}, \quad i,j = 1,2$$

R_{ij} has complex eigenvalues $\alpha \pm i\beta$

$$\rightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

β: periodicity, $\alpha > 1$: source → spiral source flow!

quite rare in RG flow



※ usually, in standard field theory case, we have independent operators with different quantum numbers → they never mix

present case: ϕ & ϕ^A have exactly the same quantum no. → mix!

8. Conclusions & discussions

- by use of fuzzy sphere structure, we bring the notion of energy in the space of matrices, based on which we develop large-N RG
  → local, similarity to the usual RG in field theory
- RG with rotational symmetry preserved
- nonplanar diagrams generate antipode matrices
  → we have to include them → RG with nonlocal interaction
- not only Gaussian, but nontrivial fixed pts. are found
  → existence of field theory on fuzzy sphere with maximal nonlocal int.
- Moyal plane limit: NO fixed points are found
  (fixed pts.. disagree for N even/odd)
  antipode transf. is not compatible with “near the south pole”