Coop with Math Program

Rigorous numerics of global orbits for fast-slow systems

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Fast-slow system

$$(*)_\epsilon \qquad \begin{aligned} \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1 \end{aligned}$$

$$x \in \mathbb{R}^n : \text{fast}, \; y \in \mathbb{R}^k : \text{slow}, \; t \in \mathbb{R} : \text{time}$$

ex. FitzHugh-Nagumo

$$\begin{aligned} u_t &= \delta u_{xx} + f(u) - \lambda \\ \lambda_t &= \epsilon(u - \gamma\lambda) \end{aligned}$$

$$u(x, t) \mapsto u(x - \theta t)$$

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \delta^{-1}(\theta v - f(u) + \lambda) \\ \dot{\lambda} &= \epsilon\theta^{-1}(u - \gamma\lambda) \end{aligned}$$

Multiscale Problems in e.g. Materials Science, Life Science.

Fast-slow system

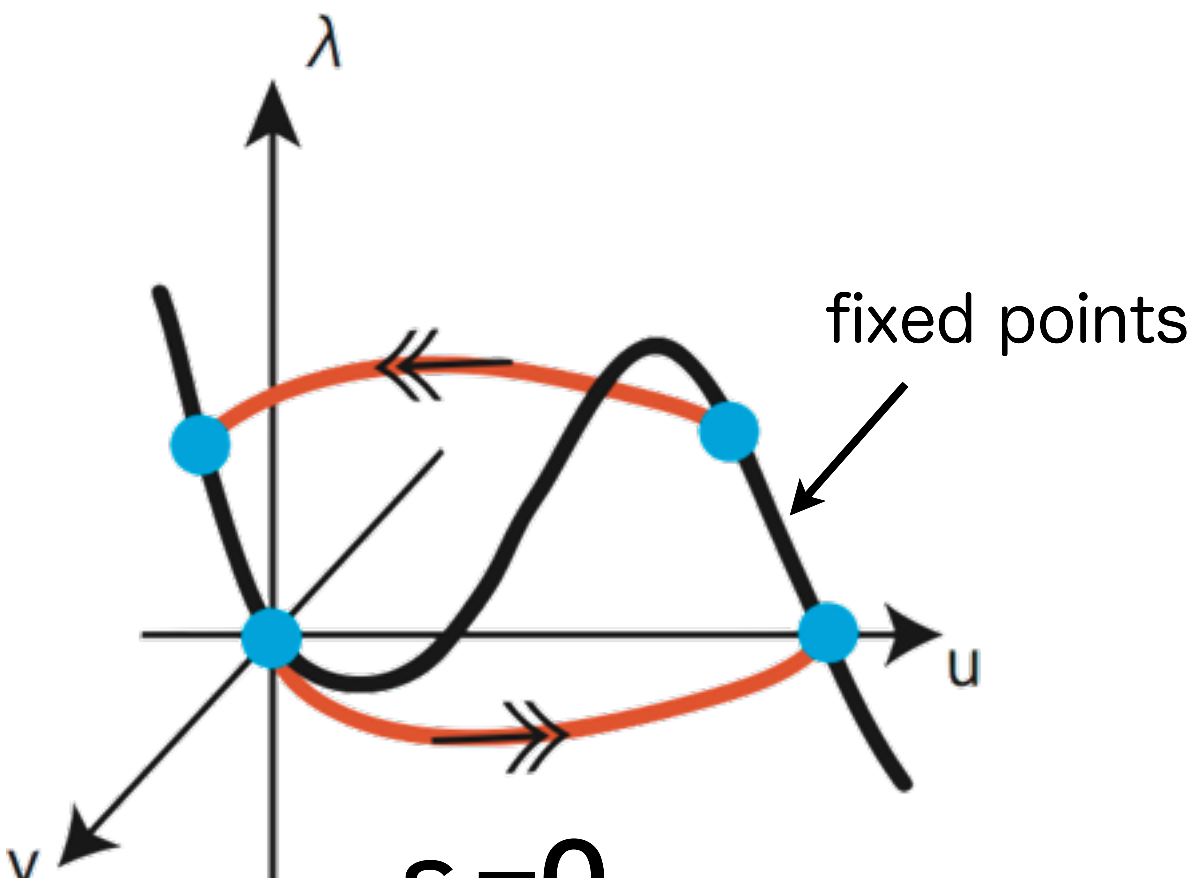
ex. FitzHugh-Nagumo

$$\dot{u} = v$$
$$\dot{v} = \delta^{-1}(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$

f : cubic nonlinearity
s.t. f(0) = f(1) = 0

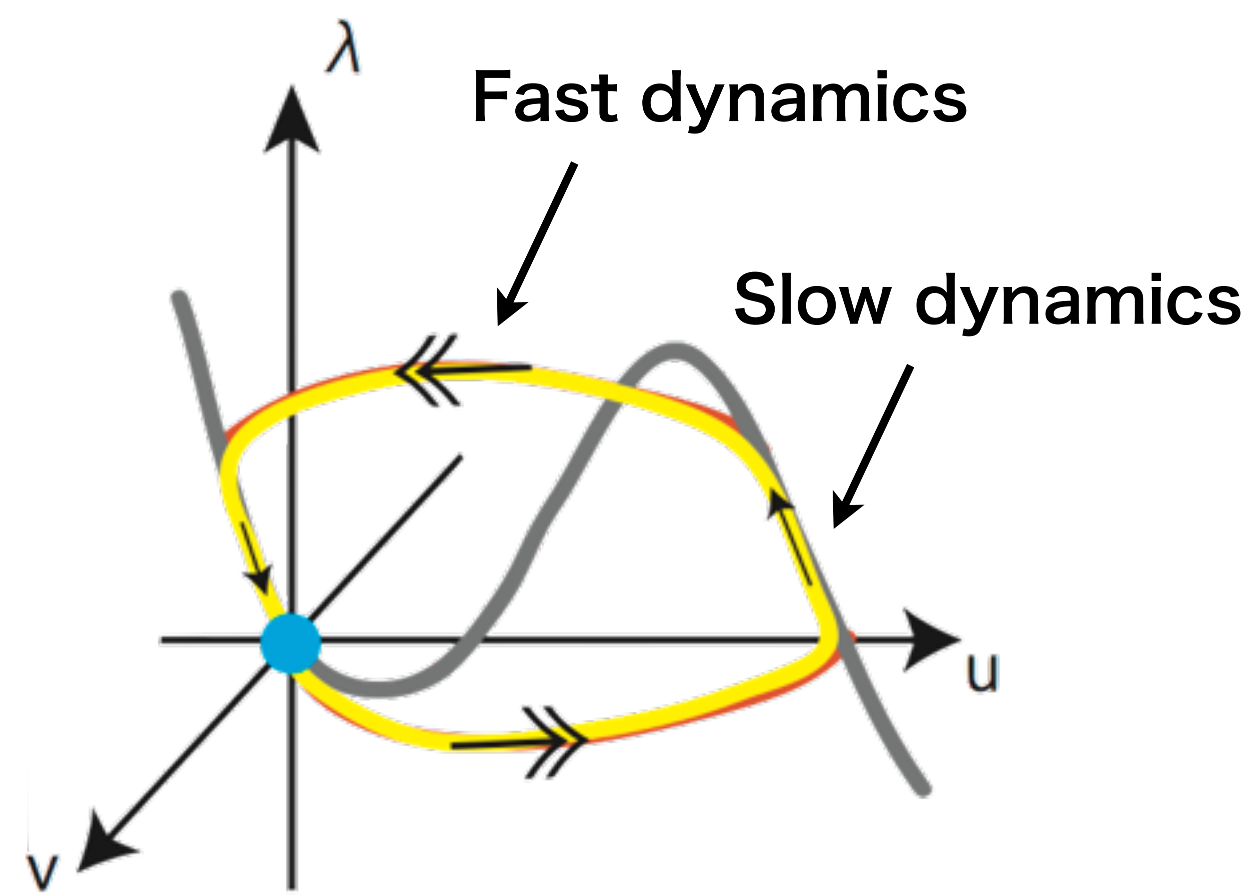
ε = 0 : {(u,v,λ)| v = 0, θv-f(u)+λ = 0} is a family of equilibria (nullcline)

ε > 0 : (0,0,0) is the only equilibrium.



ε =0

heteroclinic orbits and critical manifolds by nullclines



ε >0 : Sufficiently Small

homoclinic orbits

Fast-slow system

ex. FitzHugh-Nagumo

$$\dot{u} = v$$
$$\dot{v} = \delta^{-1}(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$

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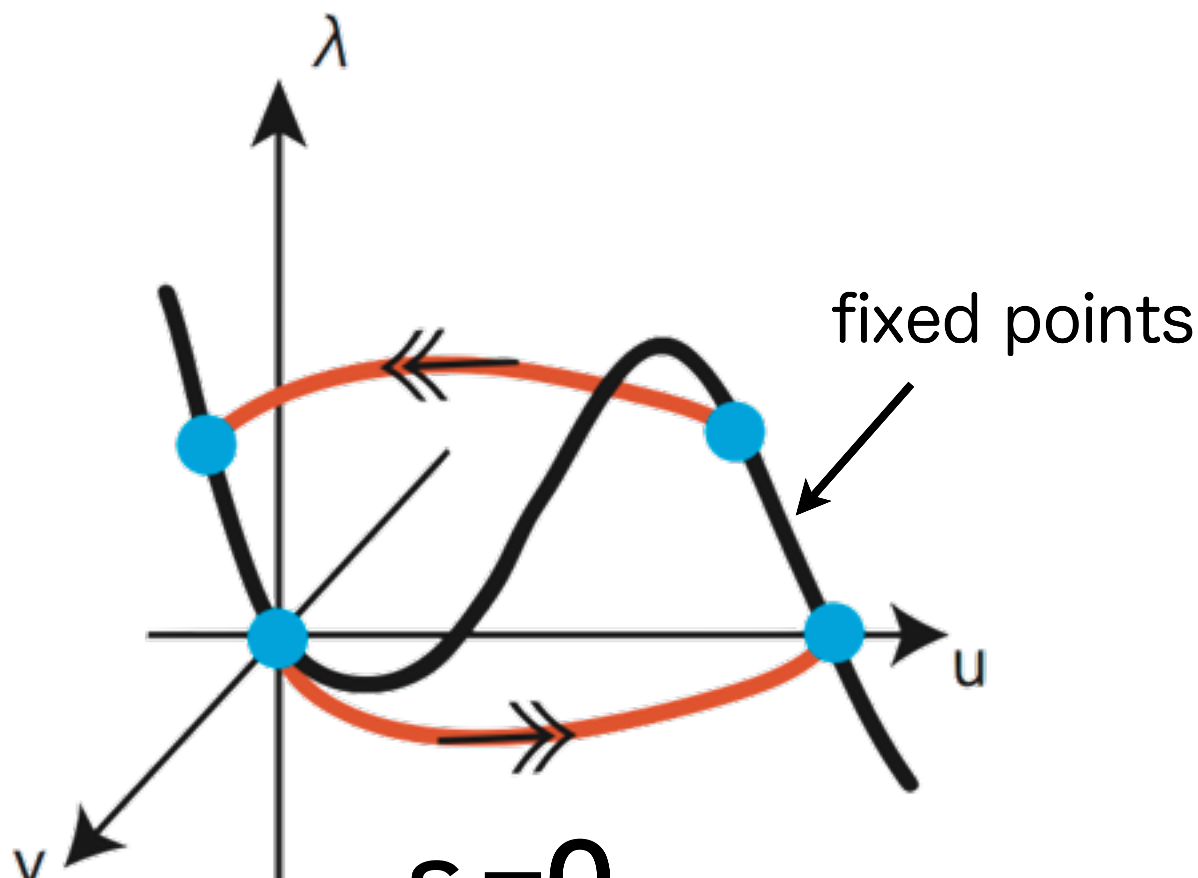
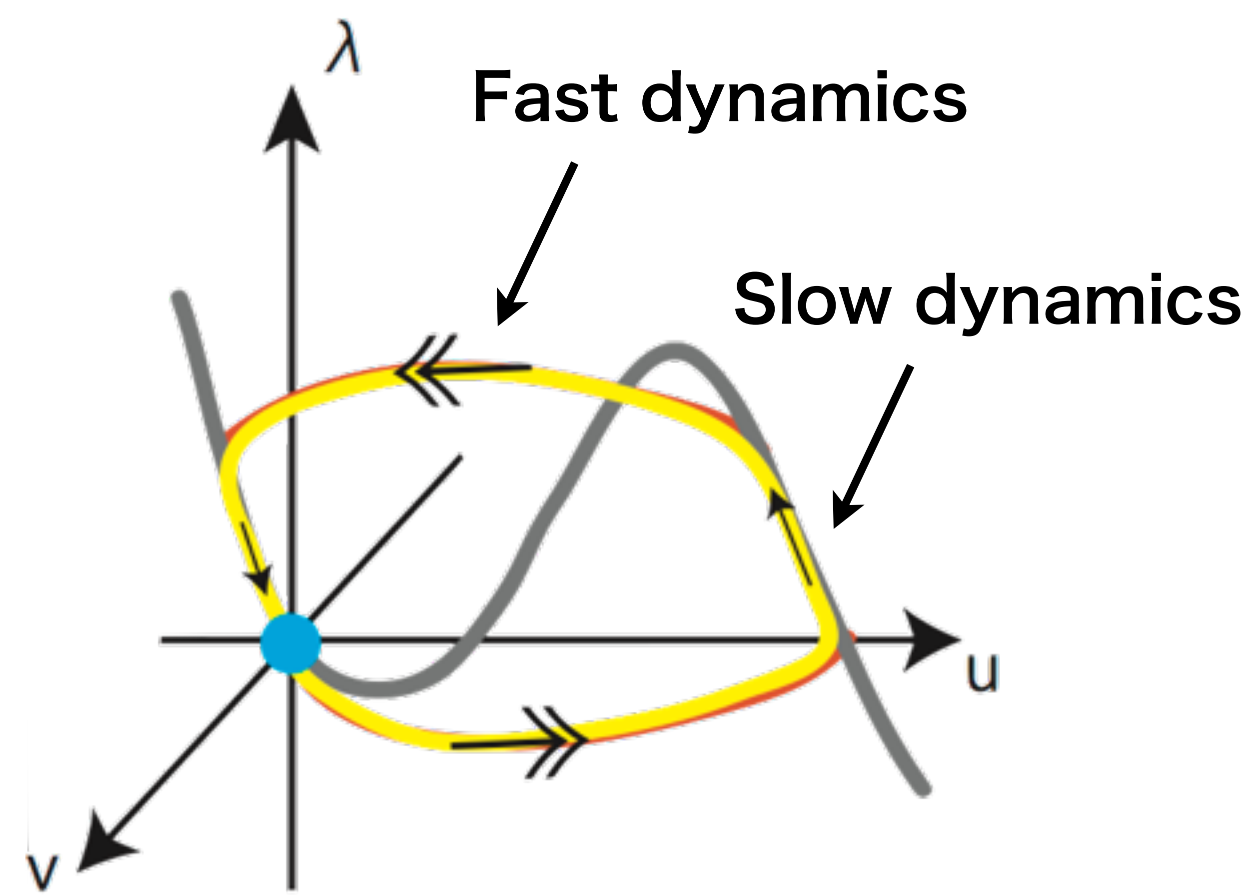


ε =0

heteroclinic orbits and critical manifolds by nullclines



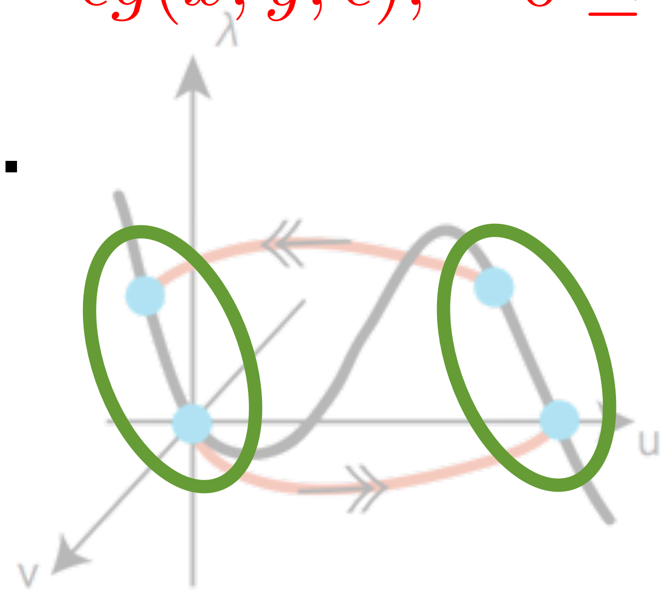
ε >0 : Given

homoclinic orbits ?

Goal : Produce the validation method for the existence of global orbits for **given ε as the continuation of singular limit orbits** for fast-slow systems.

$$\dot{x} = f(x, y, \epsilon)$$
$$\dot{y} = \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1$$

1.



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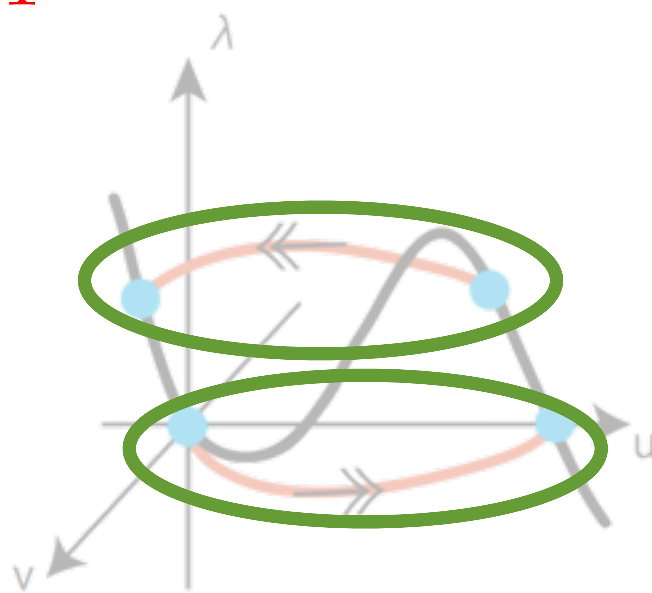

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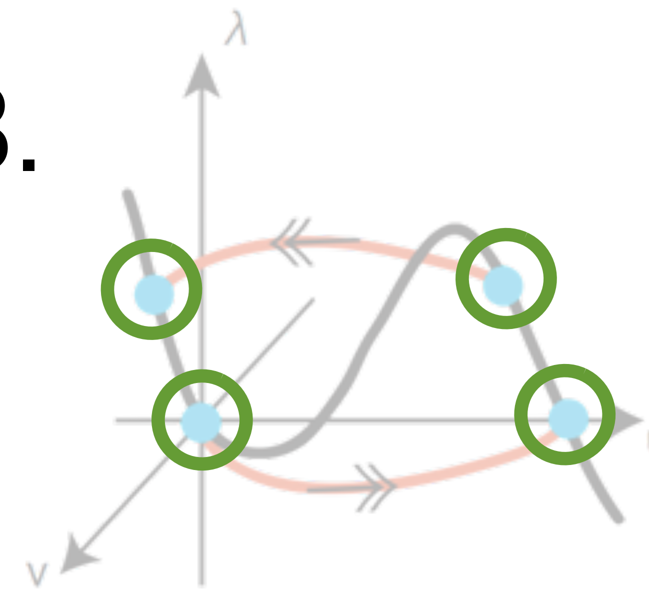


1. Slow Dynamics
2. Fast Dynamics
3. Matching

Key : Solve each scaled problem independently and match them.

Preceding works (examples)

Connecting Orbits + Rigorous Numerics

D. Wilczak, Found. Comput. Math. (2006), 495--535.

Rigorous numerics of horseshoes, Shi'lnikov orbits and N-pulse solutions via **covering relations**

J. Mireles-James, J.P. Lessard, J.B. van der Berg and K. Mischaikow, SIAM J. Math. Anal. 43(2011), 1557--1594.

Rigorous numerics of connecting orbits via **Radii Polynomials** + **Parametrization**

Singular Perturbation + Rigorous Numerics

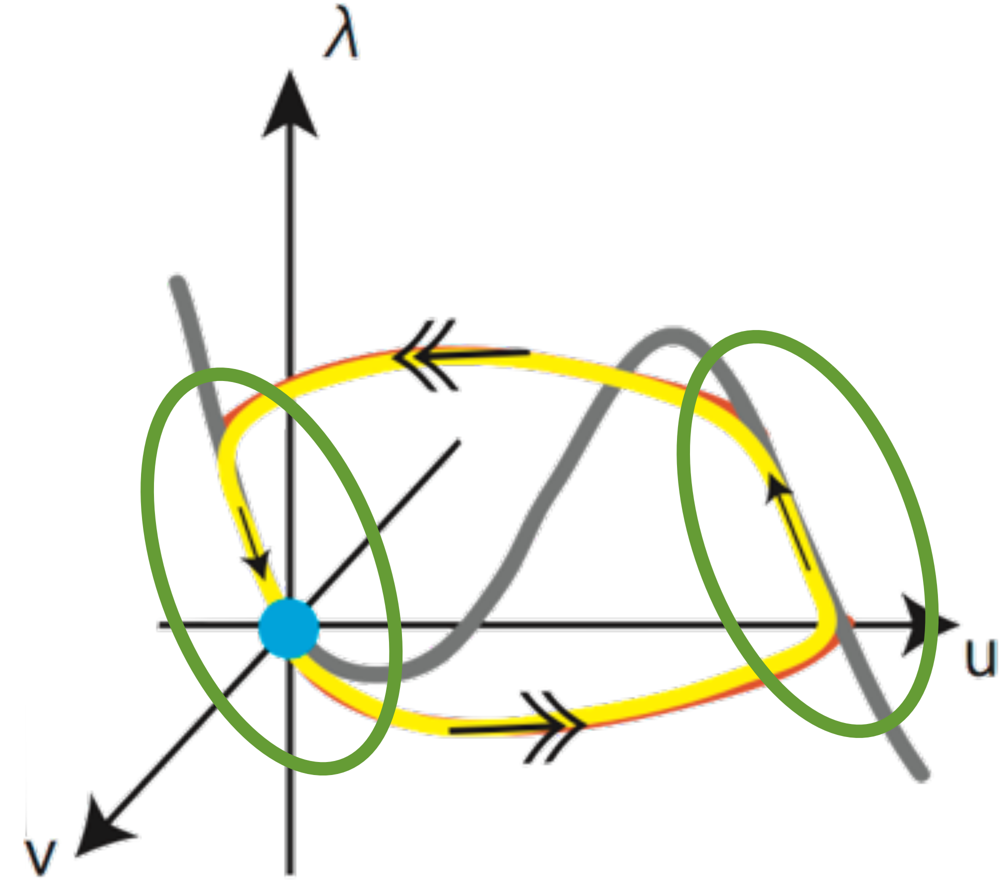
M. Gameiro, T. Gedeon, W. Kalies, H. Kokubu, K. Mischaikow and H. Oka, J., Dyn., Diff., Eq., 19 (2007), 623--654.

Singularly perturbed Conley index → horseshoes in fast-slow systems

("sufficiently close ε ")

Examples of interval arithmetics libraries : **INTLAB, PROFIL, CAPD**

1. Slow Dynamics
2. Fast Dynamics
3. Matching : “Covering-Exchange”
4. m-cones
5. Towards Validation -- overview
   (FitzHugh-Nagumo)



1. Slow Dynamics
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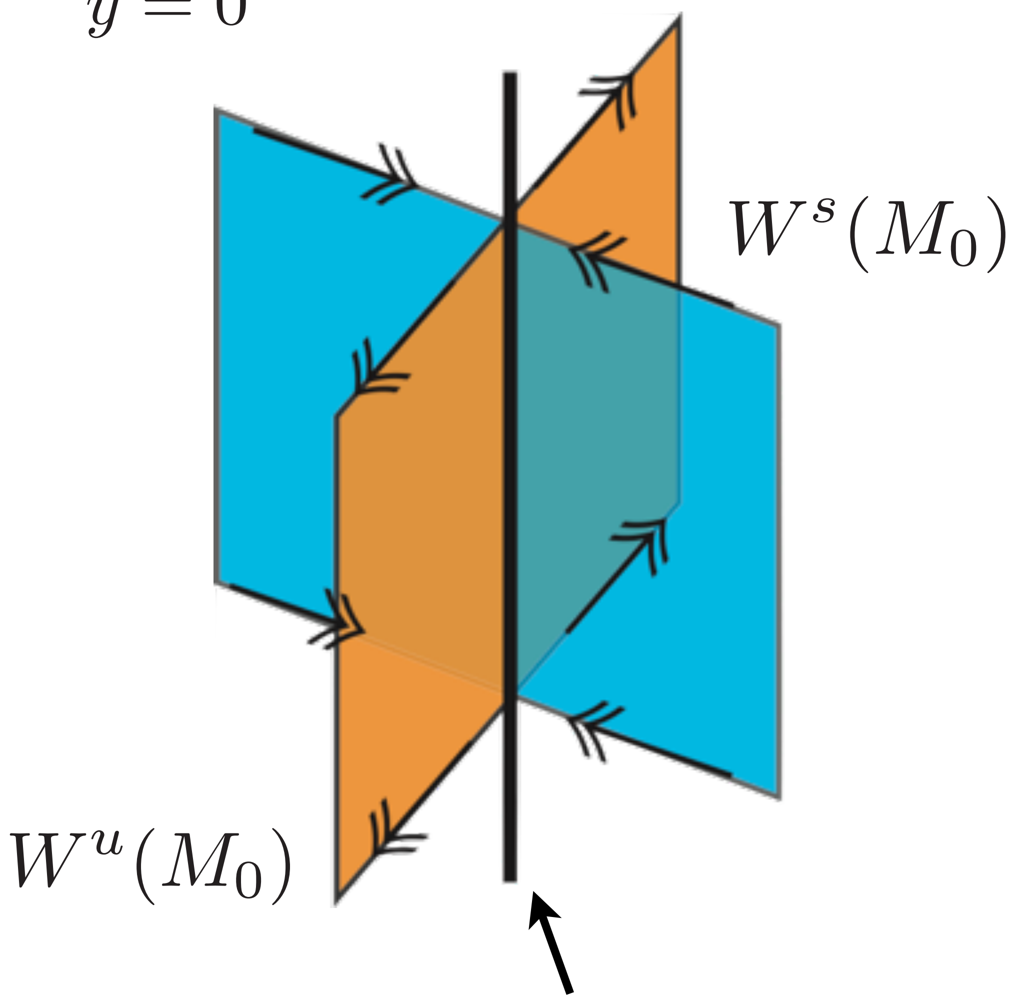
Slow manifold

ε =0

$$\dot{x} = f(x, y, 0)$$
$$\dot{y} = 0$$

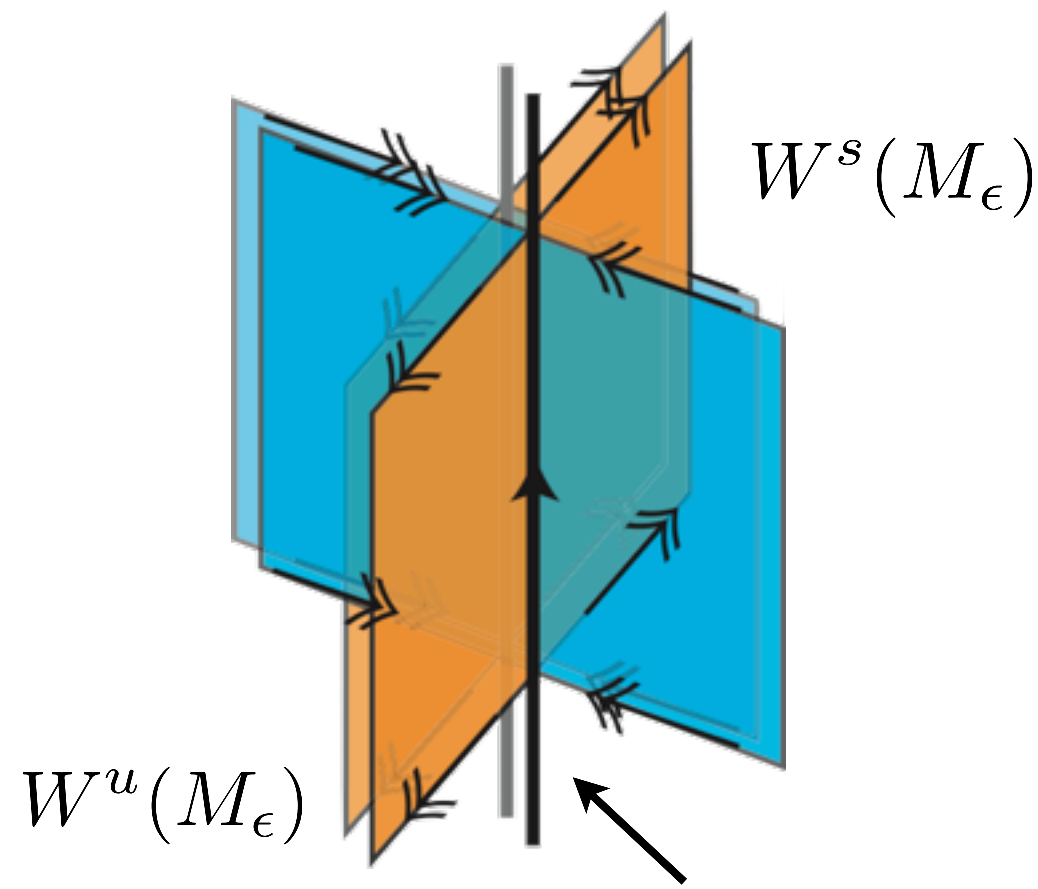


$$M_0 \subset \{f(x, y, 0) = 0\}$$

(invariant)

$\epsilon \in (0, \epsilon_0]$

$$\dot{x} = f(x, y, \epsilon)$$
$$\dot{y} = \epsilon g(x, y, \epsilon)$$



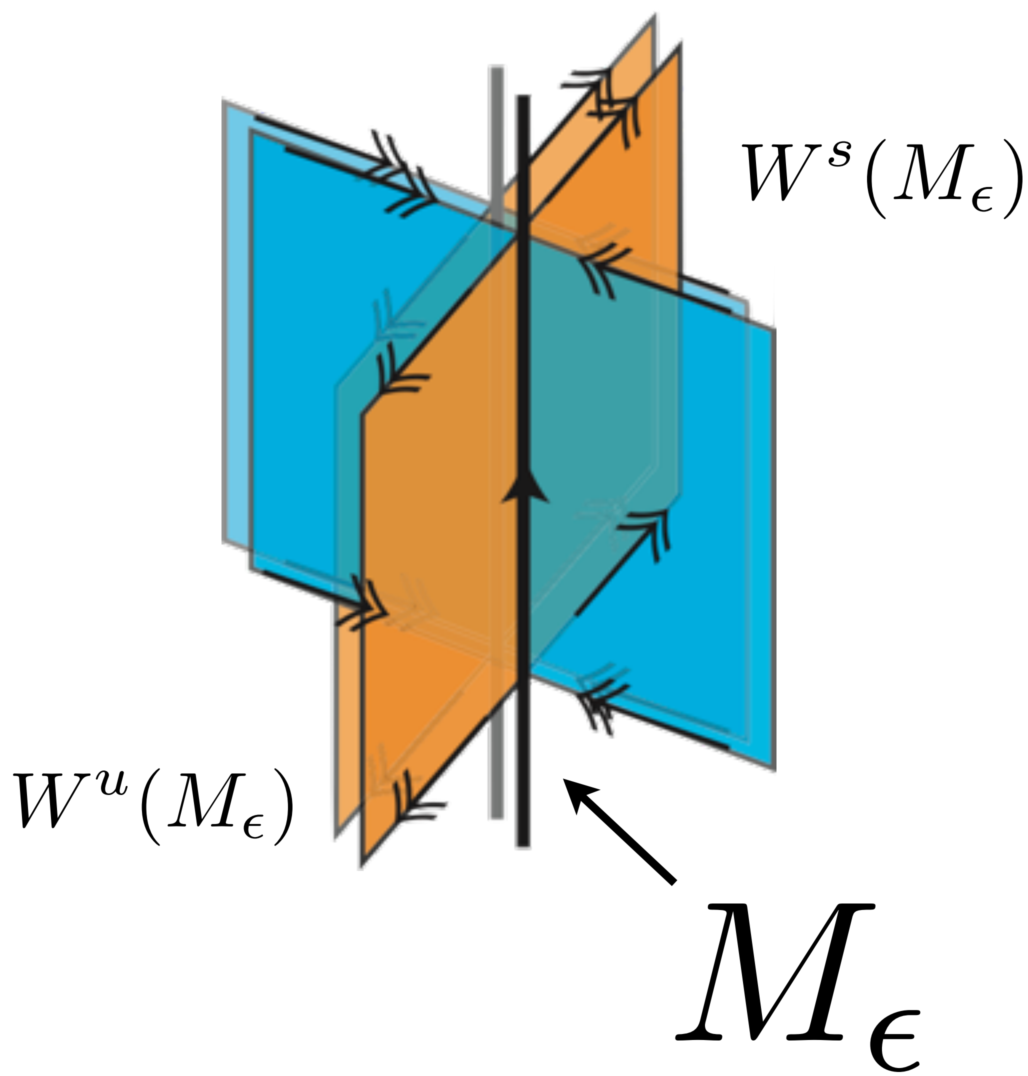
$$M_\epsilon$$

(locally invariant)

Slow manifold

$\epsilon \in (0, \epsilon_0]$

$$\dot{x} = f(x, y, \epsilon)$$
$$\dot{y} = \epsilon g(x, y, \epsilon)$$



(locally invariant)

Expression of Stable and Unstable Manifolds

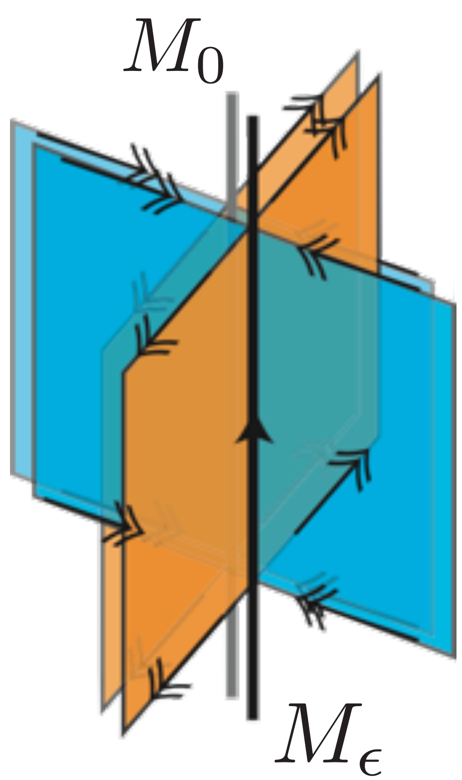
$$\lim_{t \to -\infty} x(t; \lambda) = p,$$
$$\lim_{t \to +\infty} x(t; \lambda) = q.$$

How can we verify **the infinite-time** behavior **mathematically** with **finitely many memories ?**

Where is the slow manifold ?

Is it really perturbed from M_0 ?

Which is the direction of (un)stable manifolds ?



Validation of slow manifolds

Invariant Manifold Theorem [Fenichel, 1979]

If the critical manifold M_0 is **normally hyperbolic** at $\varepsilon=0$, then for sufficiently small ε, $W^u(M_\epsilon)$ and $W^s(M_\epsilon)$ can be defined by graphs of smooth functions $b = h_u(a, y, \epsilon)$ and $a = h_s(b, y, \epsilon)$, respectively (a : fast unstable var., b : fast stable var.).

Diagonalize at a point

$$\dot{a} = Aa + F_1(a, b, y, \epsilon)$$
$$\dot{b} = Bb + F_2(a, b, y, \epsilon)$$
$$\dot{y} = \epsilon g(a, b, y, \epsilon)$$

$\mathrm{Spec}(A) \subset \{\mathrm{Re}\lambda > 0\},\ \mathrm{Spec}(B) \subset \{\mathrm{Re}\lambda < 0\}$

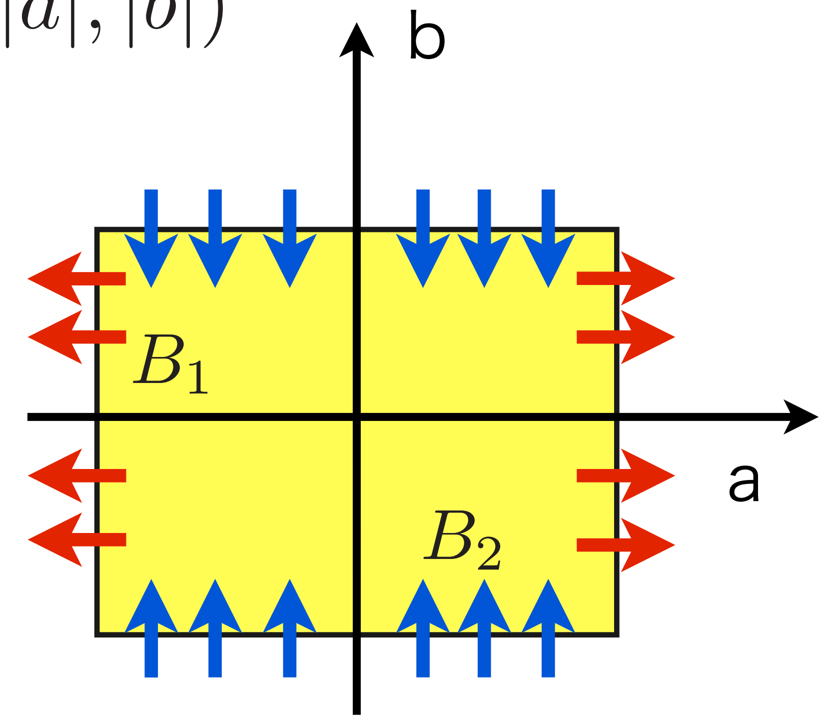
$F_1, F_2 = o(|a|, |b|)$

$K \subset \mathbb{R}^k$: cpt, convex

$B = B_1 \times B_2 \subset \mathbb{R}^n$: cpt, convex s.t.

$f(x, y, \epsilon) \cdot \nu_{\partial B_1} > 0$ on $\partial B_1 \times B_2 \times K \times [0, \epsilon_0]$,

$f(x, y, \epsilon) \cdot \nu_{\partial B_2} < 0$ on $B_1 \times \partial B_2 \times K \times [0, \epsilon_0]$



(**Fast-saddle-type Block**. a : unstable coord., b : stable coord.)

Validation of slow manifolds

$K \subset \mathbb{R}^k$: cpt, convex

$B = B_1 \times B_2 \subset \mathbb{R}^n$: cpt, convex

$$\dot{a} = Aa + F_1(a,b,y,\epsilon)$$
$$\dot{b} = Bb + F_2(a,b,y,\epsilon)$$
$$\dot{y} = \epsilon g(a,b,y,\epsilon)$$

Thm. [M. cf. Jones (1995) Theorem 4**]**

Define **Maximal Singular Values** of matrices :

$$\sigma^s_{\mathbb{A}_1} : \mathbb{A}_1(z) = \left(\frac{\partial F_1}{\partial a}(z)\right),\ \sigma^s_{\mathbb{A}_2} :\ \mathbb{A}_2(z) = \left(\frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z)\right),$$

$$\sigma^s_{\mathbb{B}_1} : \mathbb{B}_1(z) = \left(\frac{\partial F_2}{\partial a}(z)\right),\ \sigma^s_{\mathbb{B}_2} :\ \mathbb{B}_2(z) = \left(\frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right)$$

$$\sigma^s_{g_1} : g_1(z) = \left(\frac{\partial g}{\partial a}(z)\right),\ \sigma^s_{g_2} :\ g_2(z) = \left(\frac{\partial g}{\partial b}(z) \quad \frac{\partial g}{\partial y}(z) \quad \frac{\partial g}{\partial \eta}(z)\right)$$

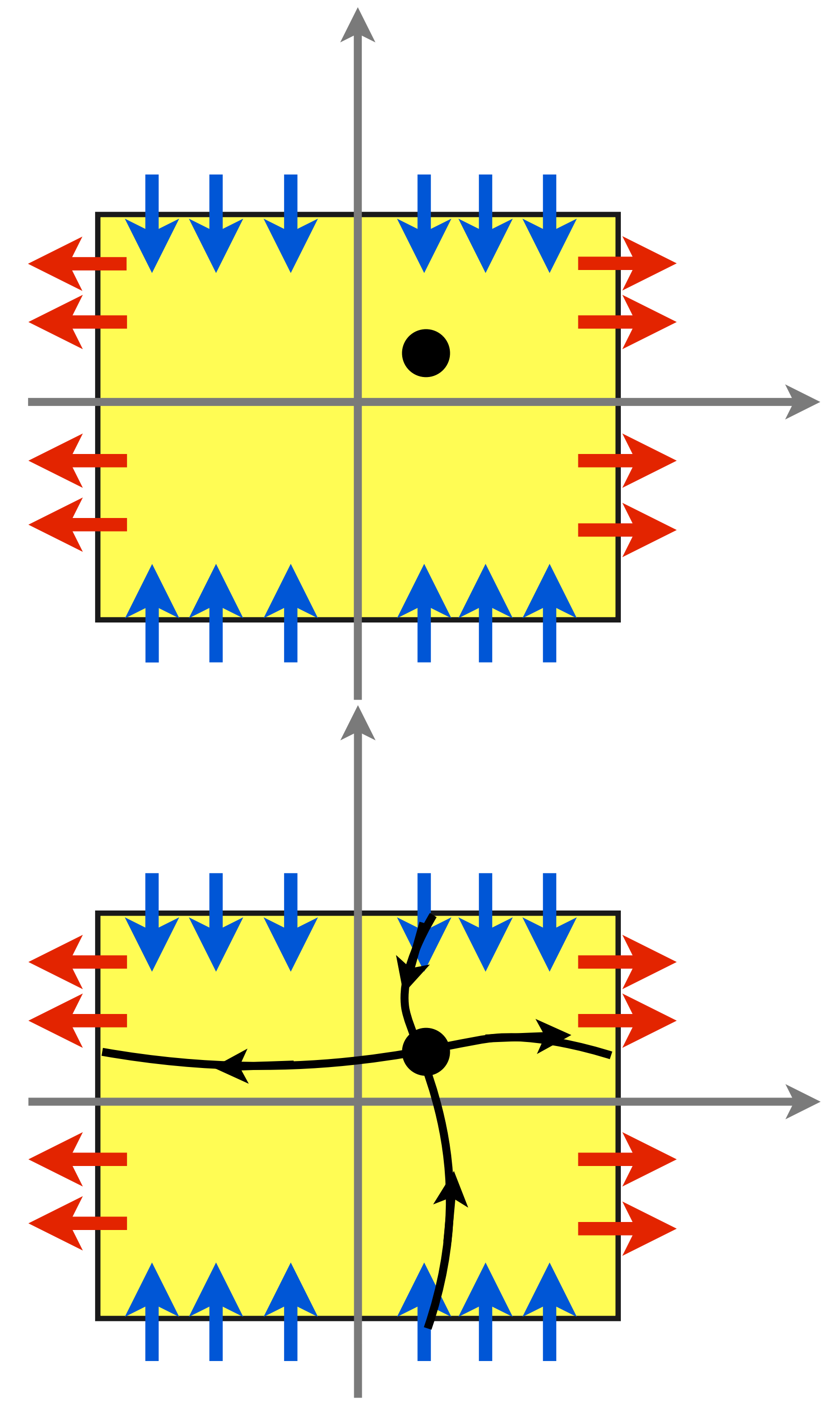
Assume the following inequalities（**stable cone conditions**）：

$$\inf \mathrm{Spec}(A) - \left(\sup \sigma^s_{\mathbb{A}_1} + \sup \sigma^s_{\mathbb{A}_2}\right) > 0,$$

$$\inf \mathrm{Spec}(A) + \inf |\mathrm{Spec}(B)| - \left\{\sup \sigma^s_{\mathbb{A}_1} + \sup \sigma^s_{\mathbb{A}_2} + \sup \sigma^s_{\mathbb{B}_1} + \sup \sigma^s_{\mathbb{B}_2} + \epsilon_0 \left(\sup \sigma^s_{g_1} + \sup \sigma^s_{g_2}\right)\right\} > 0,$$

Then **for all** $\epsilon \in [0, \epsilon_0]$ $W^s(M_\epsilon) \cap (B \times K)$ can be represented by the graph of a Lipschitz function on $B_2 \times K$. The similar statement holds for $W^u(M_\epsilon) \cap (B \times K)$. The slow manifold M_ϵ is the k-dimensional submanifold in $B \times K$ can be represented by their intersection. In particular, M_0 is normally hyperbolic.

Validation of slow manifolds



Fast-saddle-type blocks :

Slow manifold exists somewhere in the block.

The size of this block corresponds to the rigorous error between approximate and rigorous slow manifolds.

Cone conditions :

(Un)stable manifolds of slow manifolds have graph representations on (un)stable coordinates in blocks.

Exit contains a point of unstable manifolds.
Entrance contains a point of stable manifolds.

Rigorous bound of manifolds can be explicitly estimated via rigorous numerics !

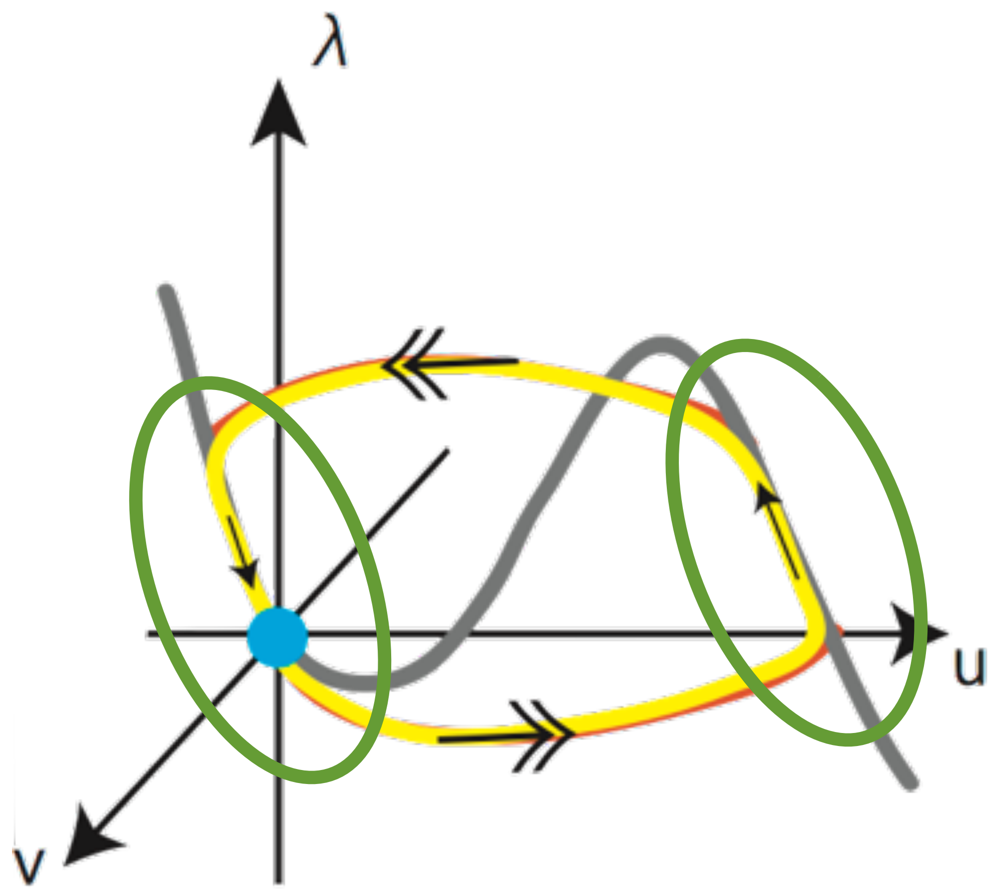
Requirements : inner product and singular values.

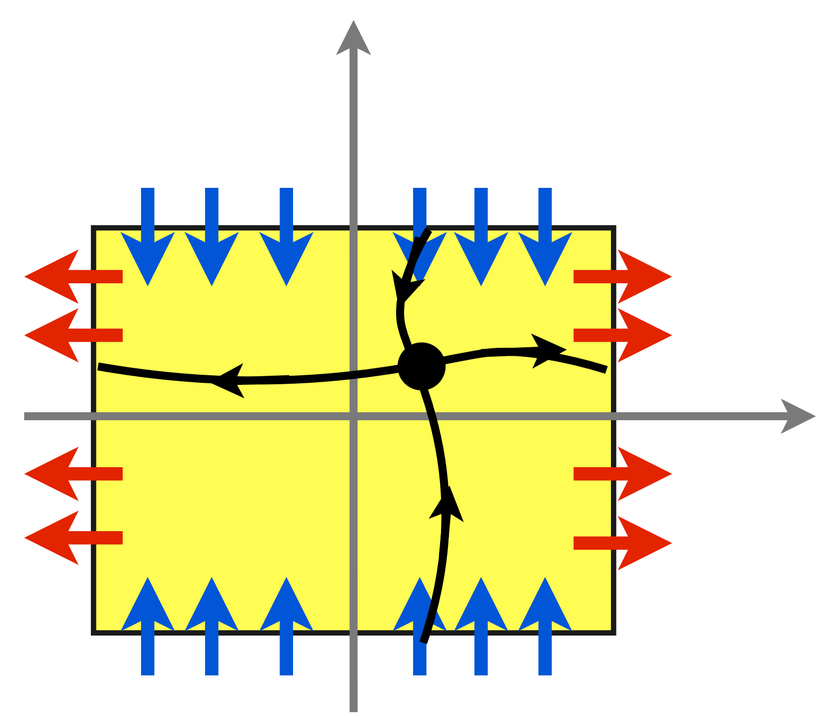
Towards rigorous numerics

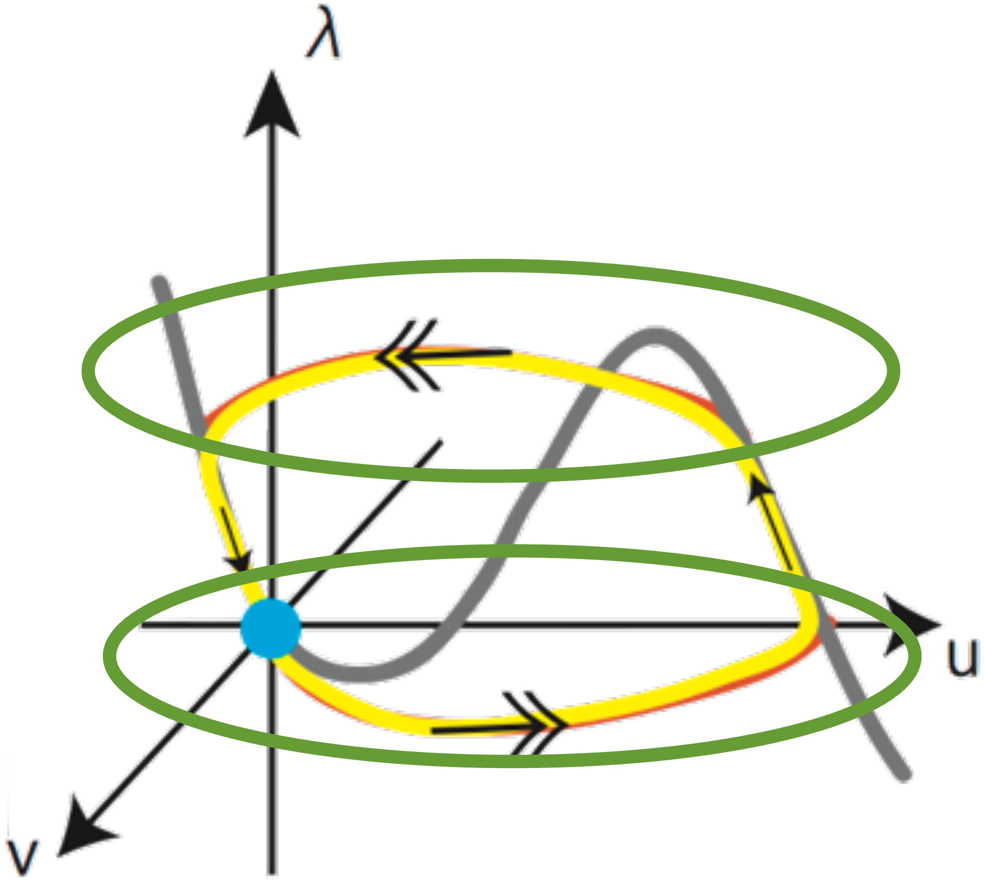
Key. Fast-saddle-type block, Cone condition

Blocks : Zgliczynski-Mischaikow (FoCM, 2001)
Cone condition, construction of Lyapunov functions :
Ref. : Zgliczynski (2009), M. (NOLTA, 2013)

Lyapunov function + Implicit Function Theorem → normal hyperbolicity







1. Slow Dynamics
2. **Fast Dynamics**
3. Matching : “Covering-Exchange”
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)

Covering relations

Def. [h-sets, Zgliczynski-Gidea (2002)]

h-set is the 4-tuple of the following :

$N \subset \mathbb{R}^n$: A compact set

$u(N), s(N) \in \mathbb{Z}_{\geq 0}$ s.t. $u(N) + s(N) = n$

$c_N : \mathbb{R}^n \to \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$: A homeomorphism s.t.

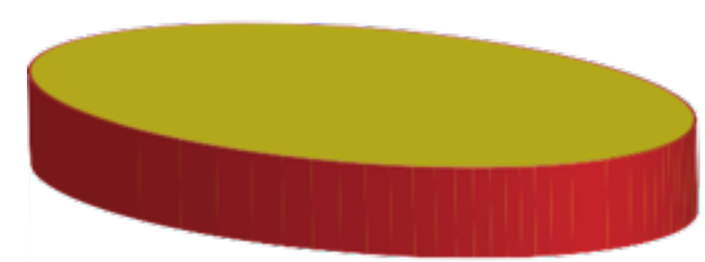
$$c_N(N) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

$u(N)$-dim. unit closed ball centered at the origin, radius 1

$$N_c := \overline{B_{u(N)}} \times \overline{B_{s(N)}},$$
$$N_c^- := \partial\overline{B_{u(N)}} \times \overline{B_{s(N)}},$$
$$N_c^+ := \overline{B_{u(N)}} \times \partial\overline{B_{s(N)}},$$
$$N^- := c_N^{-1}(N_c^-), \quad N^+ := c_N^{-1}(N_c^+).$$

□

Ex. : u(N)=1, s(N)=2



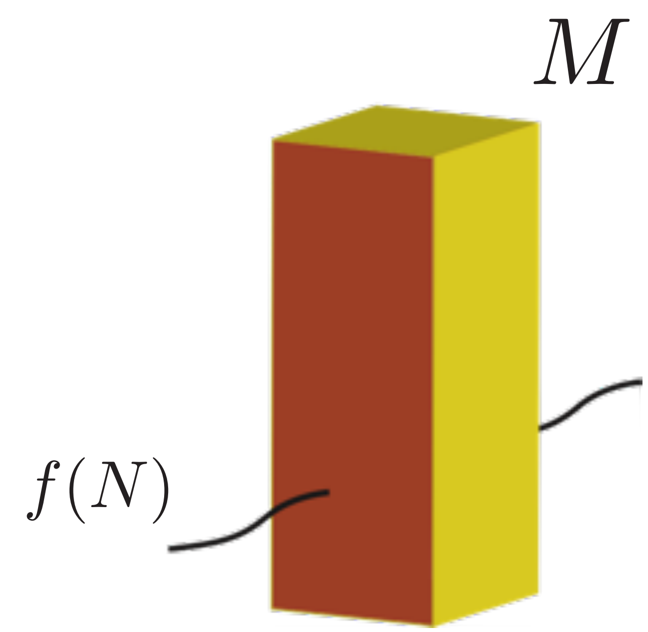
Ex. : u(N)=2, s(N)=1

Covering relations

Def. [Covering Relation, Zgliczynski-Gidea (2002)]

N, M : h-sets, $f : N \to \mathbb{R}^{\dim M}$ $\quad u(N) = u(M)$

Define $N \overset{f}{\Longrightarrow} M$ (**N f-covers M**) by



Ex. : u=1

1. There is a homotopy $h : [0,1] \times N_c \to \mathbb{R}^{\dim M}$ such that

$$h_0 = f_c, \qquad f_c := c_M \circ f \circ c_N^{-1},$$

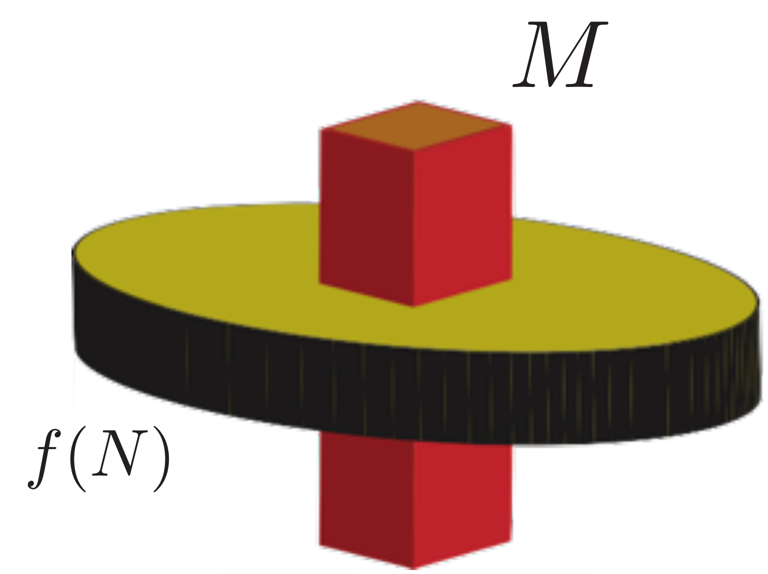
$$h([0,1], N_c^-) \cap M_c = \emptyset,$$

$$h([0,1], N_c) \cap M_c^+ = \emptyset,$$

2. There is a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$h_1(p,q) = (A(p), 0),$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1) \qquad \square$$



Ex. : u=2

Covering relations

Thm. [Zgliczynski-Gidea (2002), Wilczak (2006) etc.]

Let $\{M_k\}_{k=1}^n$: sequence of h-sets, $u(M_1) = u(M_2) = \cdots = u(M_k)$

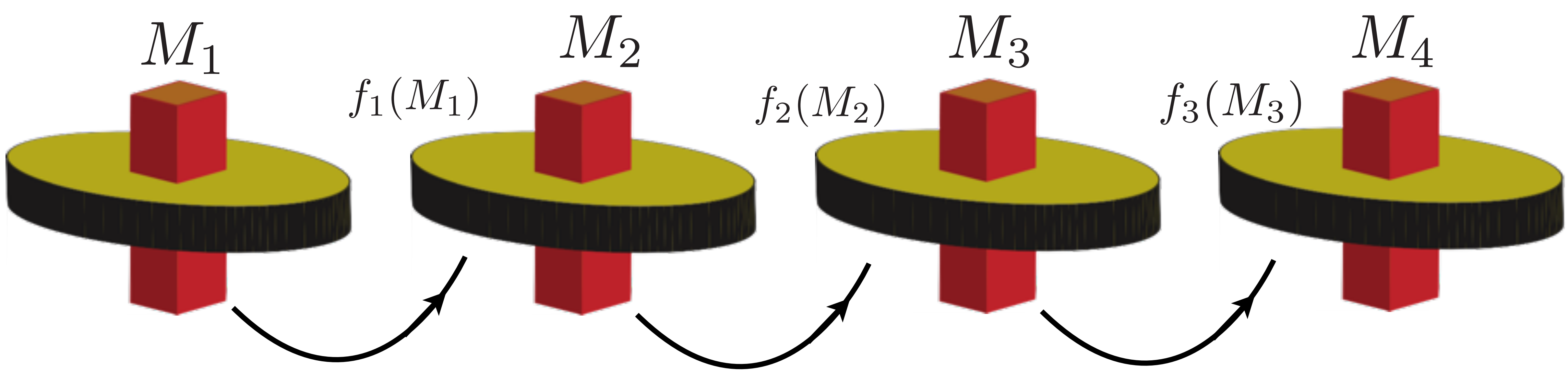
$$f_k : M_k \to \mathbb{R}^{\dim M_{k+1}} : \text{continuous}$$

Assume $M_1 \overset{f_1}{\Longrightarrow} M_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} M_k$.

Then

$$\exists x \in M_1 \text{ s.t. } f_i \circ \cdots \circ f_1(x) \in \mathrm{int} M_{i+1}, \quad i = 1, \cdots, k-1.$$

$\square$



Covering relations

Thm. [Zgliczynski-Gidea (2002), Wilczak (2006) etc.]

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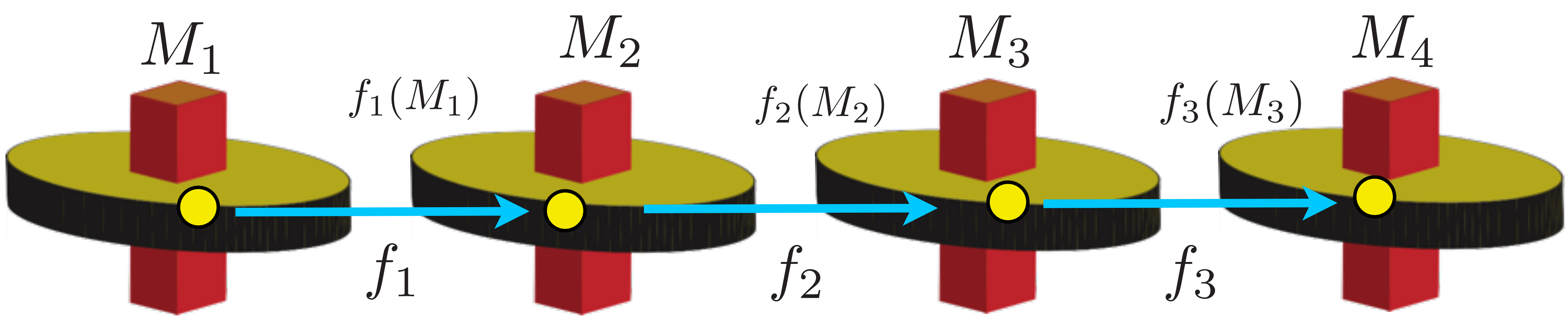
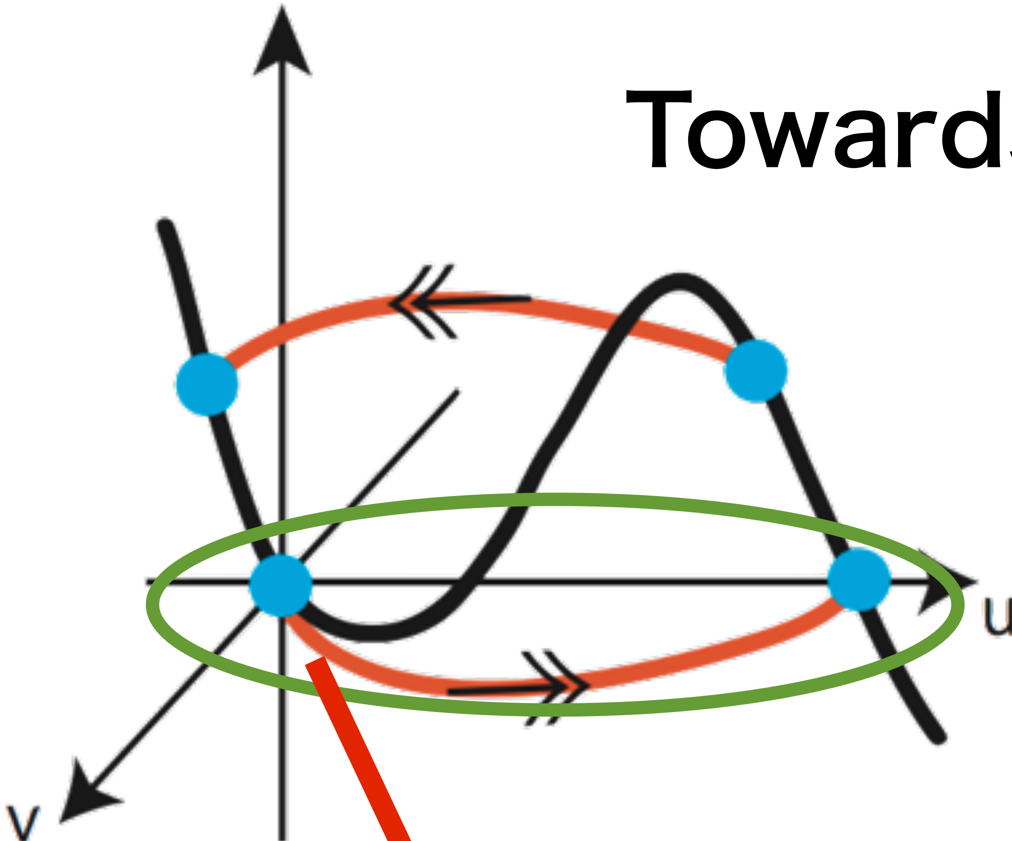
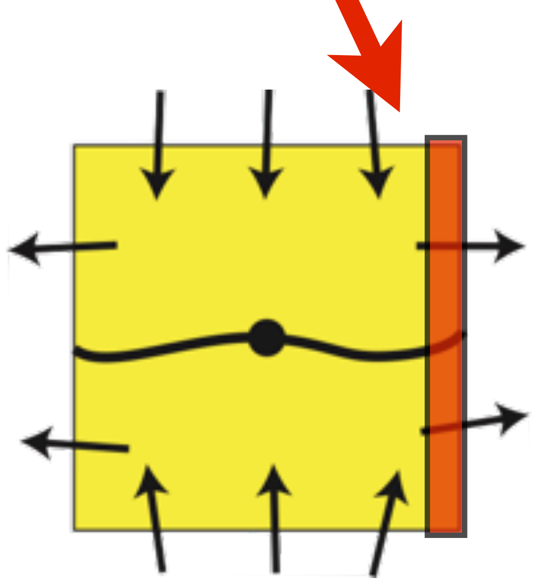


Towards rigorous numerics



ODE Solver : Lohner method.
cf. Zgliczynski (2002)

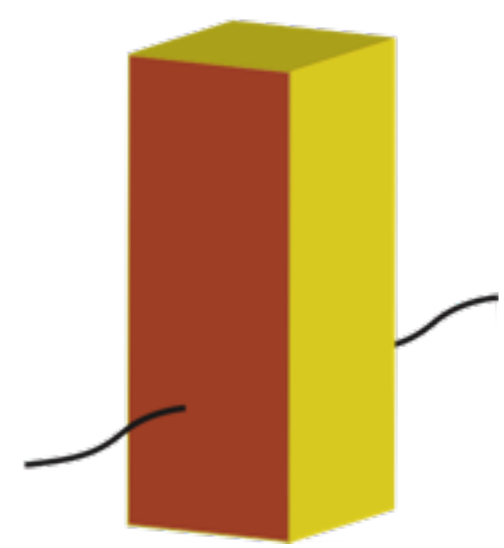
Covering Relation : Zgliczynski-Gidea (2002), Wilczak (2006), Zgliczynski (2009).



fast-saddle-type block

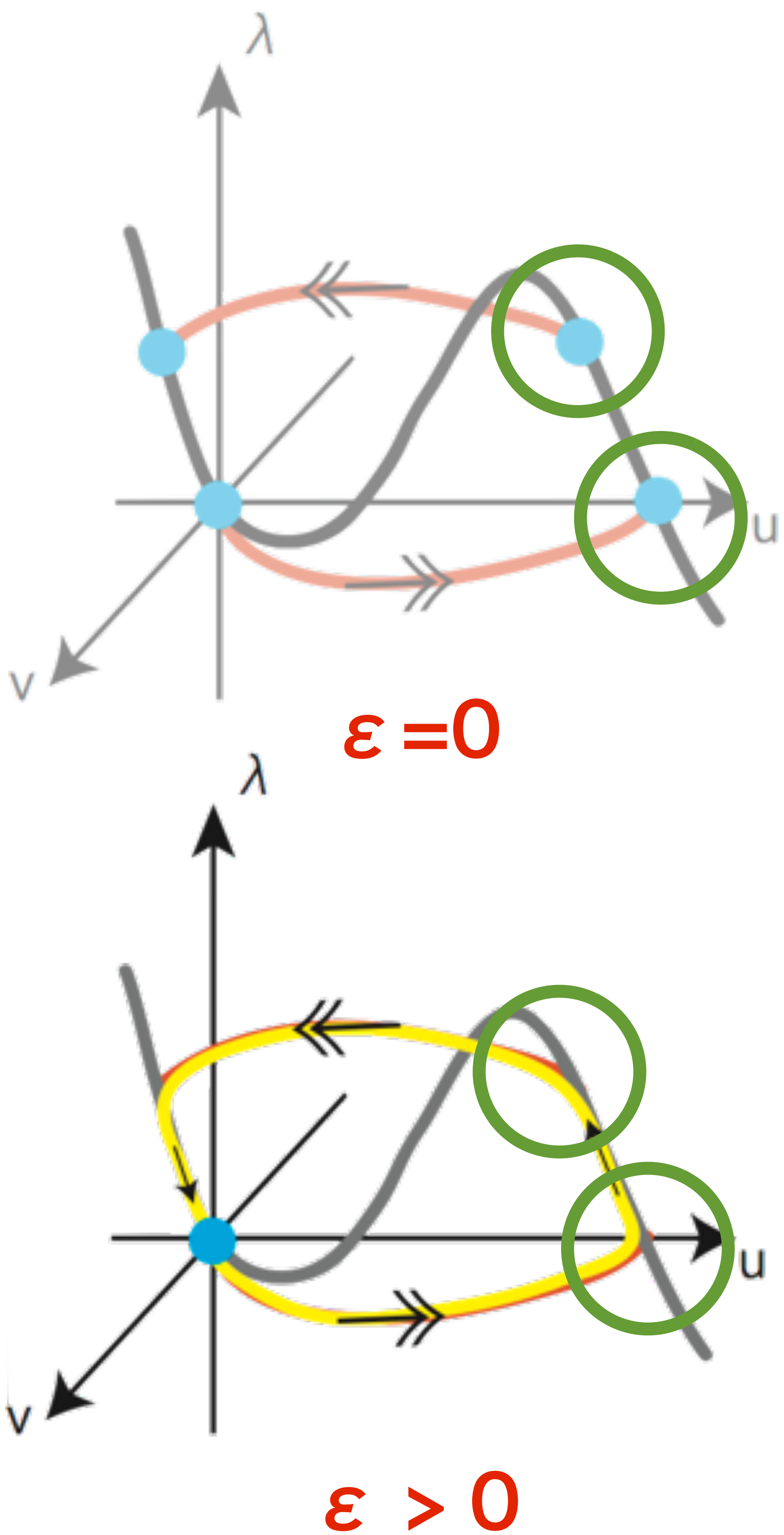
Solve ODE with the red face as the initial data ⇔ "Computation of unstable manifold"

other fast-saddle-type block

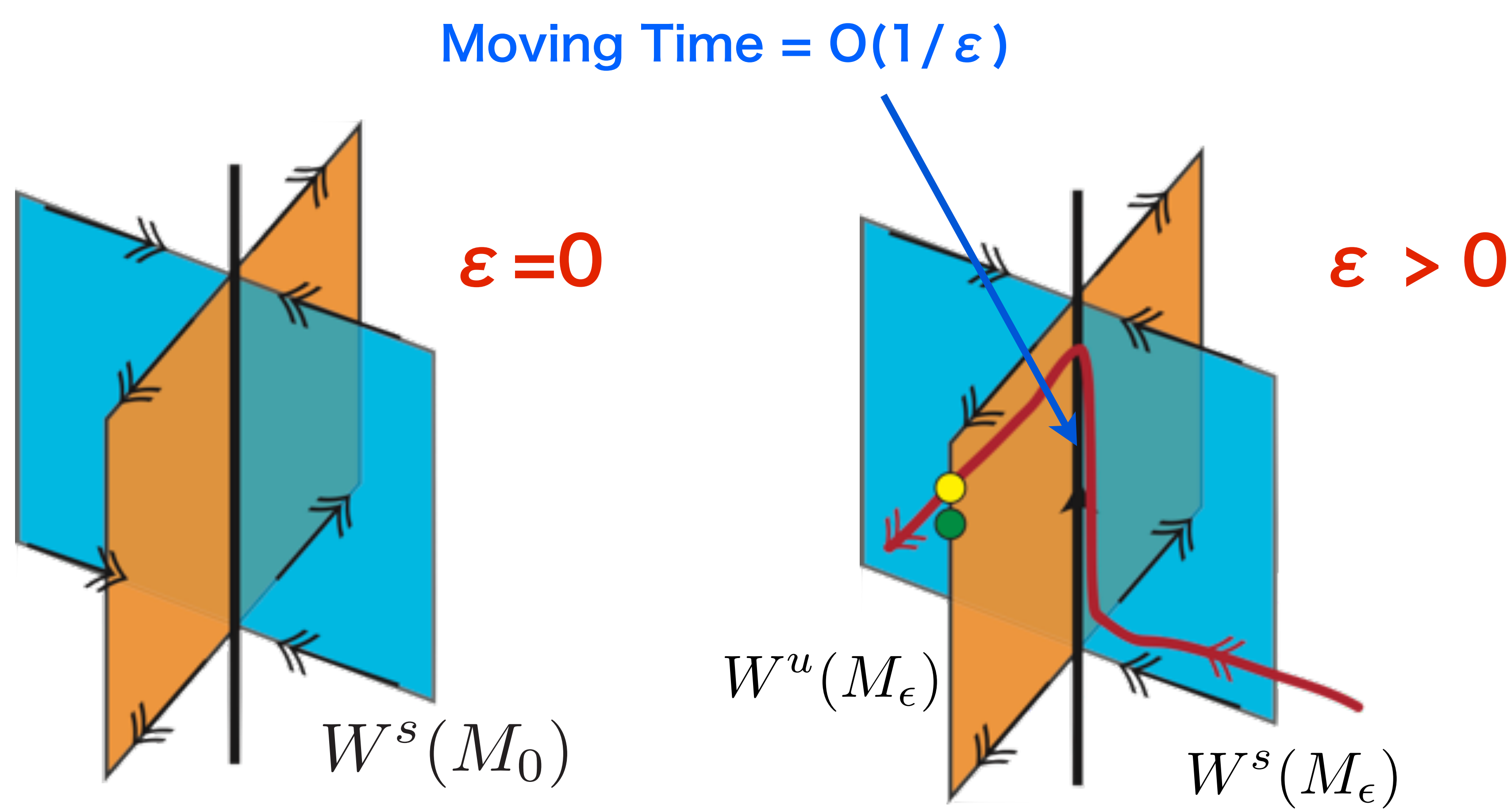


Covering relations for verifying orbits

“Matching”

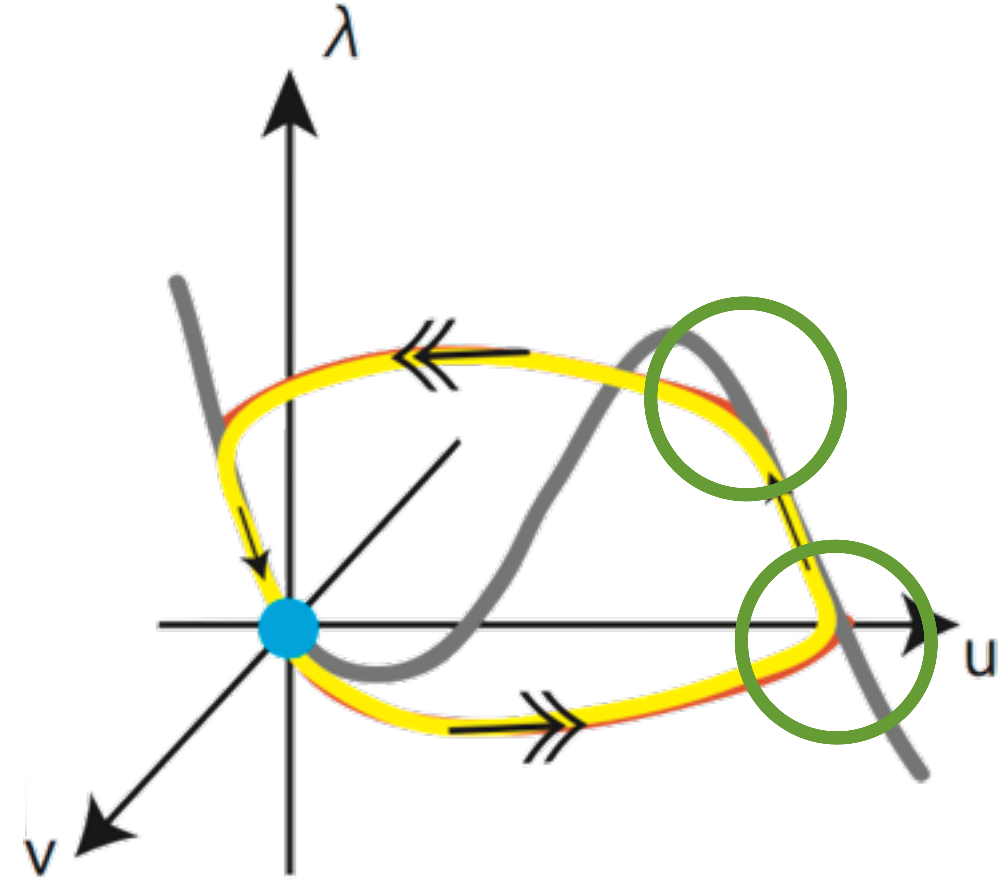


Is there a point **in a neighborhood** of heteroclinic orbits, **near** slow manifolds and another fast jump ?



Mathematically known :

Exchange Lemma (Jones-Kopell 1994, etc.)



1. Slow Dynamics
2. Fast Dynamics
3. **Matching : “Covering-Exchange”**
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5. Towards Validation -- overview (FitzHugh-Nagumo)

Covering-Exchange property

$$(*)_\epsilon \qquad \begin{aligned} \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1 \end{aligned}$$

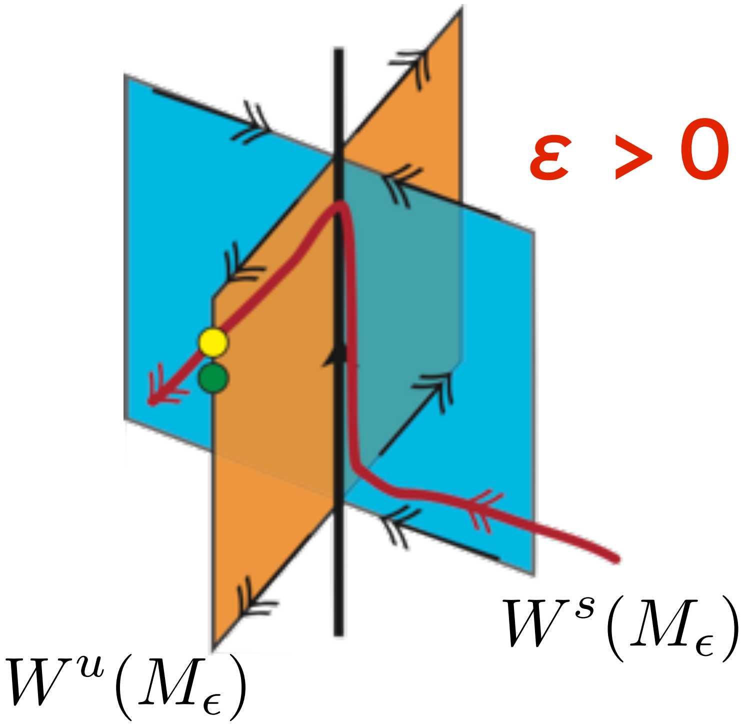
$$x \in \mathbb{R}^n : \text{ fast}, \; y \in \mathbb{R}^k : \text{ slow}, \; t \in \mathbb{R} : \text{ time}$$

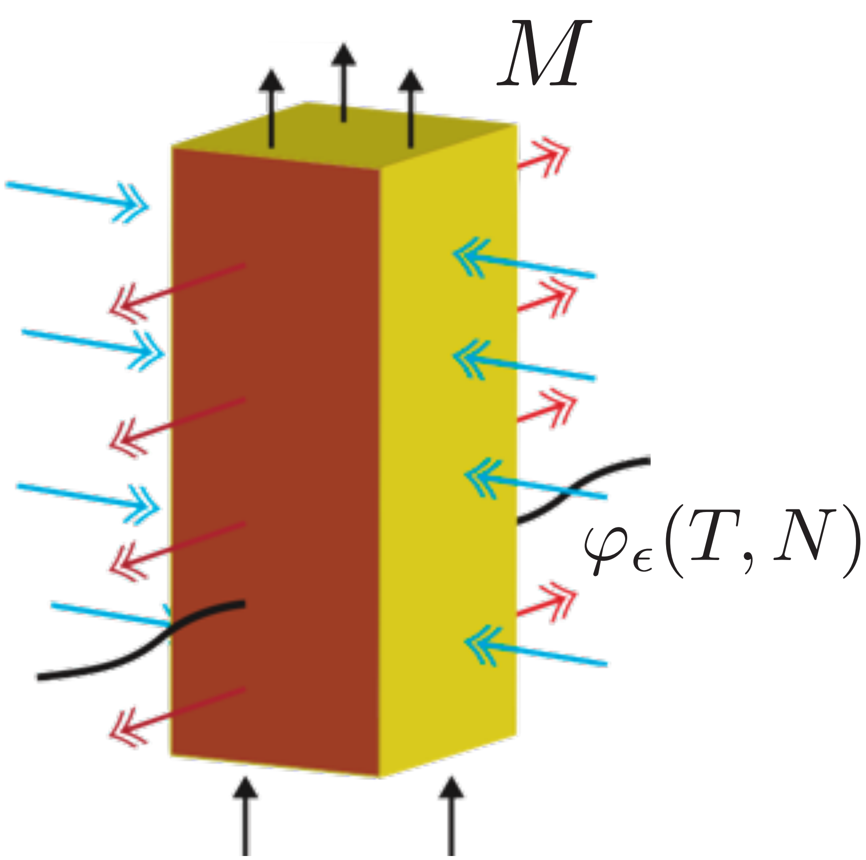
From now on assume the following :

$\dot{y} = \epsilon g(x, y, \epsilon)$ can be represented by

$$\begin{aligned} y &= (w, \theta_1, \cdots, \theta_{k-1}) \in \mathbb{R}^k, \\ \dot{w} &= \epsilon g_1(x, y, \epsilon), \\ \dot{\theta}_i &= 0. \end{aligned}$$

Covering-Exchange property





Def. (Covering-Exchange)

$N \subset \mathbb{R}^{u+s+k}$: h-set, $M \subset \mathbb{R}^{u+s+k}$: $(u+s+k)$-dim. h-set

We say that N satisfies the **covering-exchange property (CE)** with respect to M for $(*)_\epsilon$ if

1. M is a fast-saddle-type block.
2. M satisfies stable and unstable cone conditions.
3. For $q \in \{\pm 1\}$

   $$q \cdot g_1(x, y, \epsilon) > 0 \text{ in } M.$$

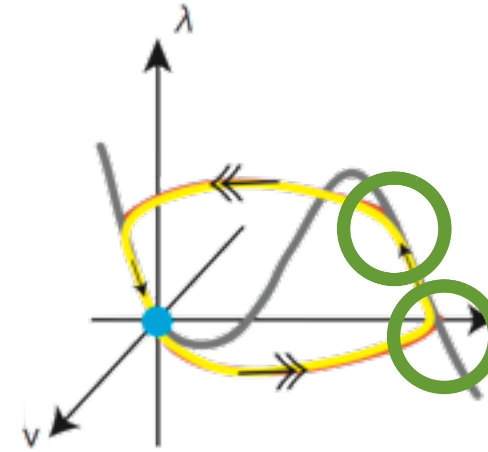
4. Letting φ_ϵ be the flow of $(*)_\epsilon$, for some T > 0

   $$N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} M.$$

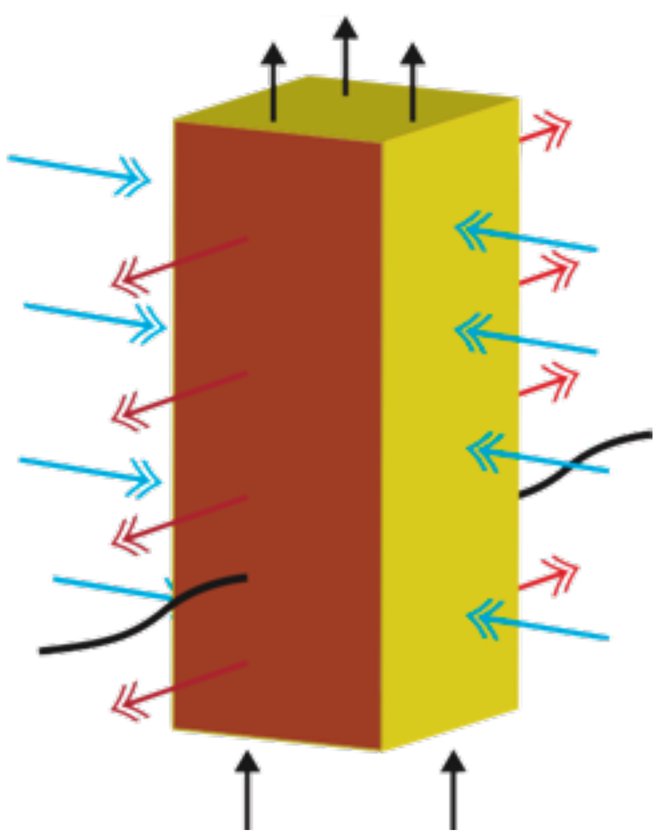
We say the pair (N,M) a **covering-exchange pair**.

Covering-Exchange property

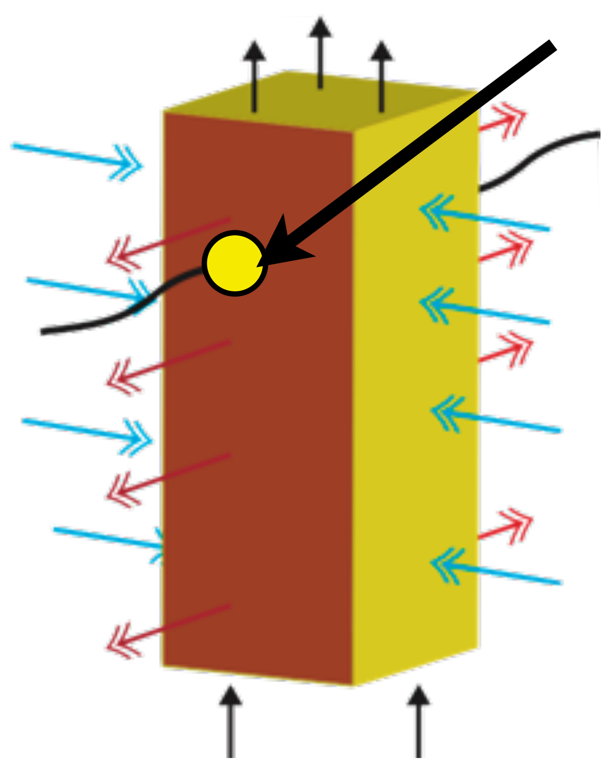
Dynamics of Covering-Exchange pairs



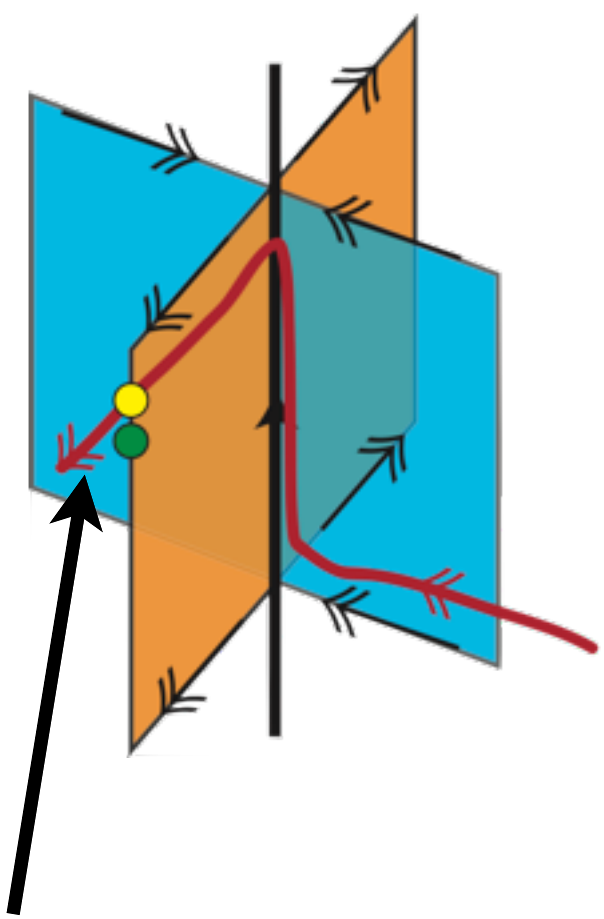
1. M is a fast-saddle-type block.
2. M satisfies stable and unstable cone conditions.
3. For $q \in \{\pm 1\}$ $\quad q \cdot g_1(x, y, \epsilon) > 0$ in M.
4. Letting φ_ϵ be the flow of $(*)_\epsilon$, for some T > 0, $N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} M$.



After sufficiently long time...

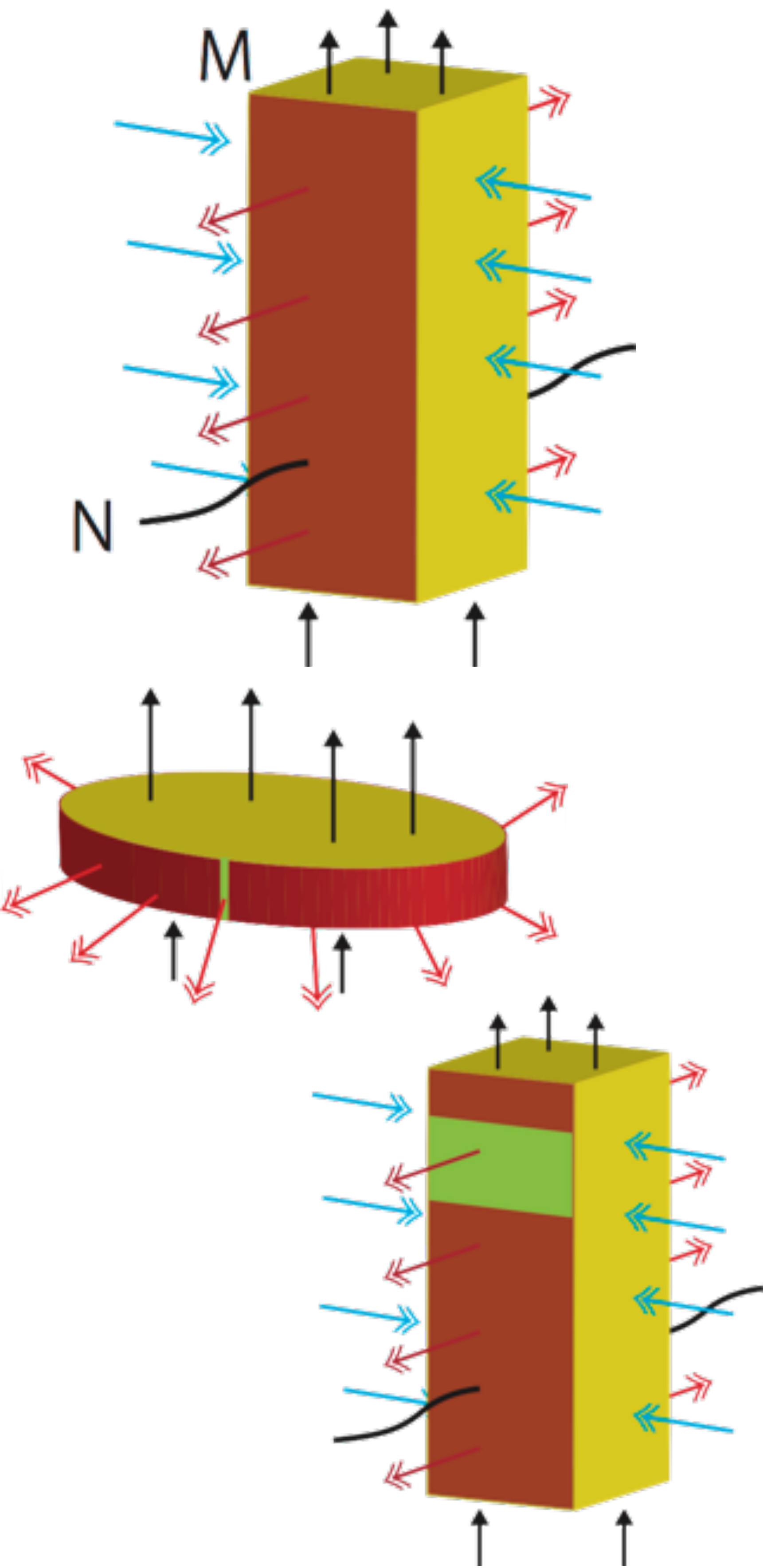


It comes from N.



Topologically describes orbits colored by red.

Fast-exit face and admissibility



Def. (Fast-exit face)

Define a **fast-exit face** of a fast-saddle-type block M by

$$M^a := c_M^{-1}\left(\{a\} \times \overline{B_s} \times (w^-, w^+) \times \prod_{i=2}^{k}[-1,1]\right), \quad a \in \partial B_u.$$

where $M_c = \overline{B_u} \times \overline{B_s} \times [-1,1] \times \prod_{i=2}^{k}[-1,1]$

Def. (admissibility)

$\tilde{M} \subset M$: h-set satisfying 1~3 of (CE) and $M_0 \subset M$: a fast-exit face are said to be **admissible in M** if

$$M_0 \cap \tilde{M} = \emptyset, \quad u(M_0) = u(\tilde{M}),$$

The $u(M_0)$ -component of M_0 contains w-coordinate.

If q=+1, $\inf \pi_w(M_0)_c - \sup \pi_w(\tilde{M})_c > 0.$

If q= -1, $\inf \pi_w(\tilde{M})_c - \sup \pi_w(M_0)_c > 0.$

Singular limit connecting orbits and their continuation

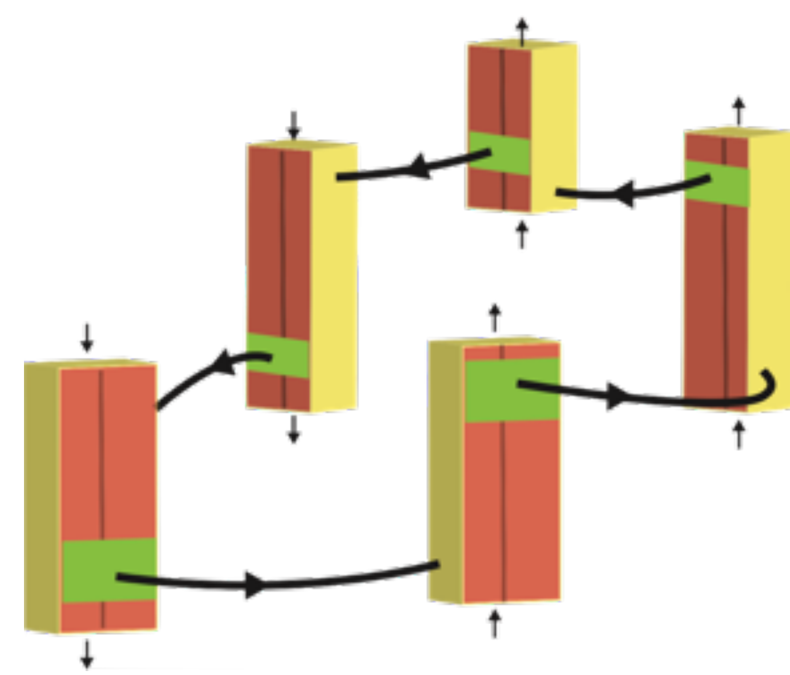
Thm. [M. cf. Jones (1995)**]**

For the fast-slow system $(*)_\epsilon$ assume that, for given $\epsilon_0 > 0$ and $\rho \in \mathbb{N}$ there is an ε ($\in [0, \epsilon_0]$)-parameter family of the following sets :

$\mathcal{S}_\epsilon^j$: (j=0,⋯, ρ) fast-saddle-type block which forms a covering-exchange pair with $\mathcal{F}_\epsilon^{j-1}$ ($\mathcal{F}_\epsilon^\rho$ if j = 0).

$\tilde{\mathcal{S}}_\epsilon^j$: (j=0,⋯, ρ) fast-saddle-type block which forms a covering-exchange pair with $\mathcal{F}_\epsilon^{j-1}$ and the pair $(\tilde{\mathcal{S}}_\epsilon^j, \mathcal{F}_\epsilon^j)$ forms an admissible pair in $\mathcal{S}_\epsilon^j$.

$\mathcal{F}_\epsilon^j$: (j=0,⋯, ρ) a fast-exit face of $\mathcal{S}_\epsilon^j$.



Then **for all** $\epsilon \in (0, \epsilon_0]$ there is a periodic orbit for $(*)_\epsilon$ which passes all $\mathcal{S}_\epsilon^j$.

Singular limit connecting orbits and their continuation

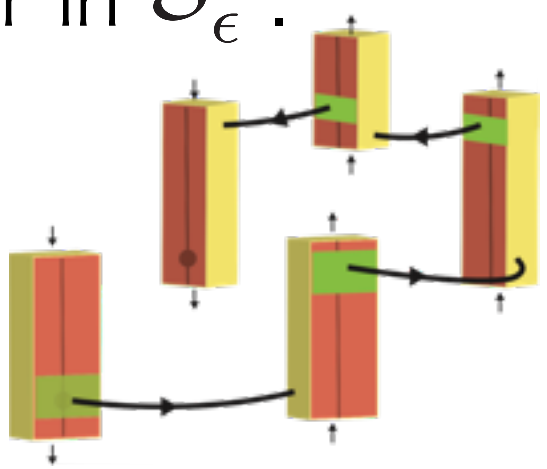
Thm. [M. cf. Jones (1995)**]**

For the fast-slow system $(*)_\epsilon$ assume that, for given $\epsilon_0 > 0$ and $\rho \in \mathbb{N}$ there is an ε ($\in [0, \epsilon_0]$)-parameter family of the following sets :

$\mathcal{S}_\epsilon^j$: (j=0,⋯, ρ) fast-saddle-type block
(j=1,⋯, ρ-1) fast-saddle-type block which forms a CE pair with $\mathcal{F}_\epsilon^{j-1}$.
(i=0, ρ) invariant sets $S_{\epsilon,u}, S_{\epsilon,s}$ are contained there, respectively.

$\tilde{\mathcal{S}}_\epsilon^j$: (j=0,⋯, ρ) fast-saddle-type block
(j=1,⋯, ρ) fast-saddle-type block which forms a CE pair with $\mathcal{F}_\epsilon^{j-1}$ and the pair $(\tilde{\mathcal{S}}_\epsilon^j, \mathcal{F}_\epsilon^j)$ forms an admissible pair in $\mathcal{S}_\epsilon^j$.

$\mathcal{F}_\epsilon^j$: (j=0,⋯, ρ-1) a fast-exit face of $\mathcal{S}_\epsilon^j$
(j=0) there is an intersection with $W^u(S_{\epsilon,u})$.



Then **for all** $\epsilon \in (0, \epsilon_0]$ there is a heteroclinic orbit for $(*)_\epsilon$ connecting $S_{\epsilon,u}$ and $S_{\epsilon,s}$ which passes all $\mathcal{S}_\epsilon^j$.

Singular limit connecting orbits and their continuation

Idea of the proof (in the case of Periodic orbits)

$$\Pi := (\tilde{\mathcal{S}}_\epsilon^0)_c \times (\mathcal{F}_\epsilon^0)_c \times (\tilde{\mathcal{S}}_\epsilon^1)_c \times (\mathcal{F}_\epsilon^1)_c \times \cdots \times (\tilde{\mathcal{S}}_\epsilon^\rho)_c \times (\mathcal{F}_\epsilon^\rho)_c$$
$$\subset \mathbb{R}^{d_s^0} \times \mathbb{R}^{d_f^0} \times \mathbb{R}^{d_s^1} \times \mathbb{R}^{d_f^1} \times \cdots \times \mathbb{R}^{d_s^\rho} \times \mathbb{R}^{d_f^\rho}.$$

→ **Prove that the mapping degree** $\deg(F_\epsilon, \Pi, 0)$ **of the map below can be defined and is nonzero :**

$$F_\epsilon \begin{pmatrix} (p_s^0, q_s^0) \\ (p_f^0, q_f^0) \\ (p_s^1, q_s^1) \\ (p_f^1, q_f^1) \\ \vdots \\ (p_s^\rho, q_s^\rho) \\ (p_f^\rho, q_f^\rho) \end{pmatrix} := \begin{pmatrix} (p_f^0, q_f^0) - \pi^0 \circ (P_\epsilon^0)_c(p_s^0, q_s^0) \\ (p_s^1, q_s^1) - (\varphi_\epsilon(T^0, \cdot))_c(p_f^0, q_f^0, (\pi^0)^c \circ (P_\epsilon^0)_c(p_s^0, q_s^0)) \\ (p_f^1, q_f^1) - \pi^1 \circ (P_\epsilon^1)_c(p_s^1, q_s^1) \\ (p_s^2, q_s^2) - (\varphi_\epsilon(T^1, \cdot))_c(p_f^1, q_f^1, (\pi^1)^c \circ (P_\epsilon^1)_c(p_s^1, q_s^1)) \\ \vdots \\ (p_f^\rho, q_f^\rho) - \pi^\rho \circ (P_\epsilon^\rho)_c(p_s^\rho, q_s^\rho) \\ (p_s^0, q_s^0) - (\varphi_\epsilon(T^\rho, \cdot))_c(p_f^\rho, q_f^\rho, (\pi^\rho)^c \circ (P_\epsilon^\rho)_c(p_s^\rho, q_s^\rho)) \end{pmatrix}.$$

↑

Components involving (un)stable manifolds are added in the case of heteroclinic orbits.

Towards rigorous numerics

Key. Covering-Exchange

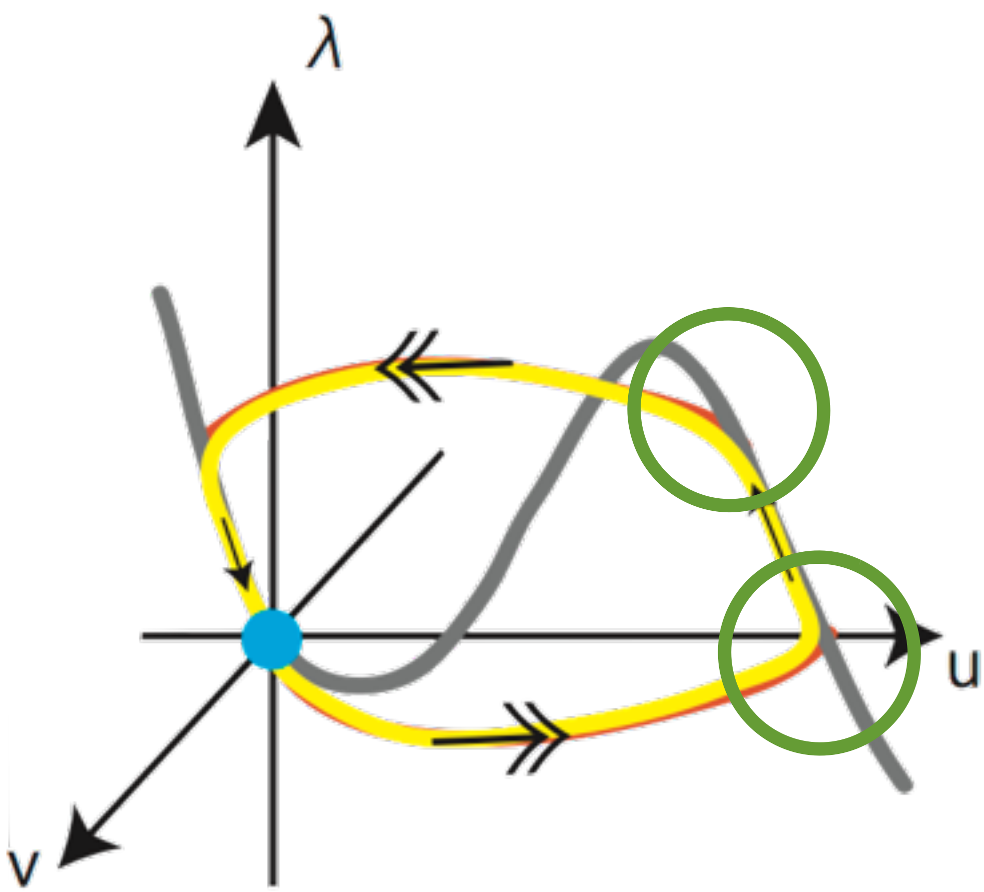
Blocks and Cone conditions : Already stated.

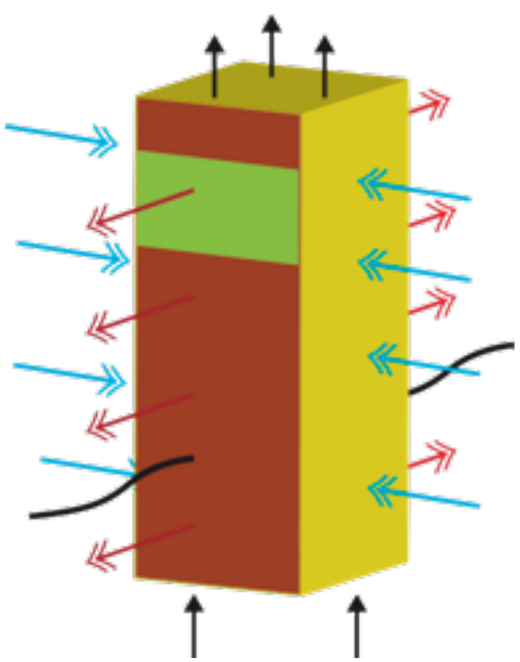
Covering Relation : Already stated.

Sign of vector fields : Easy !

Fast-exit face + Admissibility : Easy !

Nothing new for rigorous numerics !

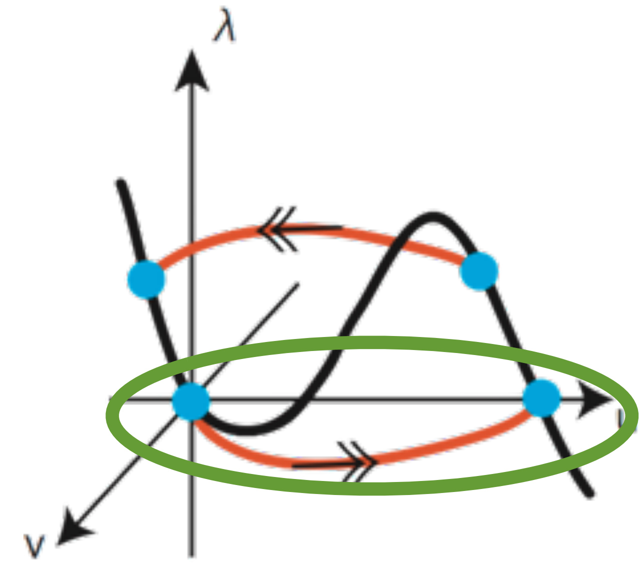




Practical Computations

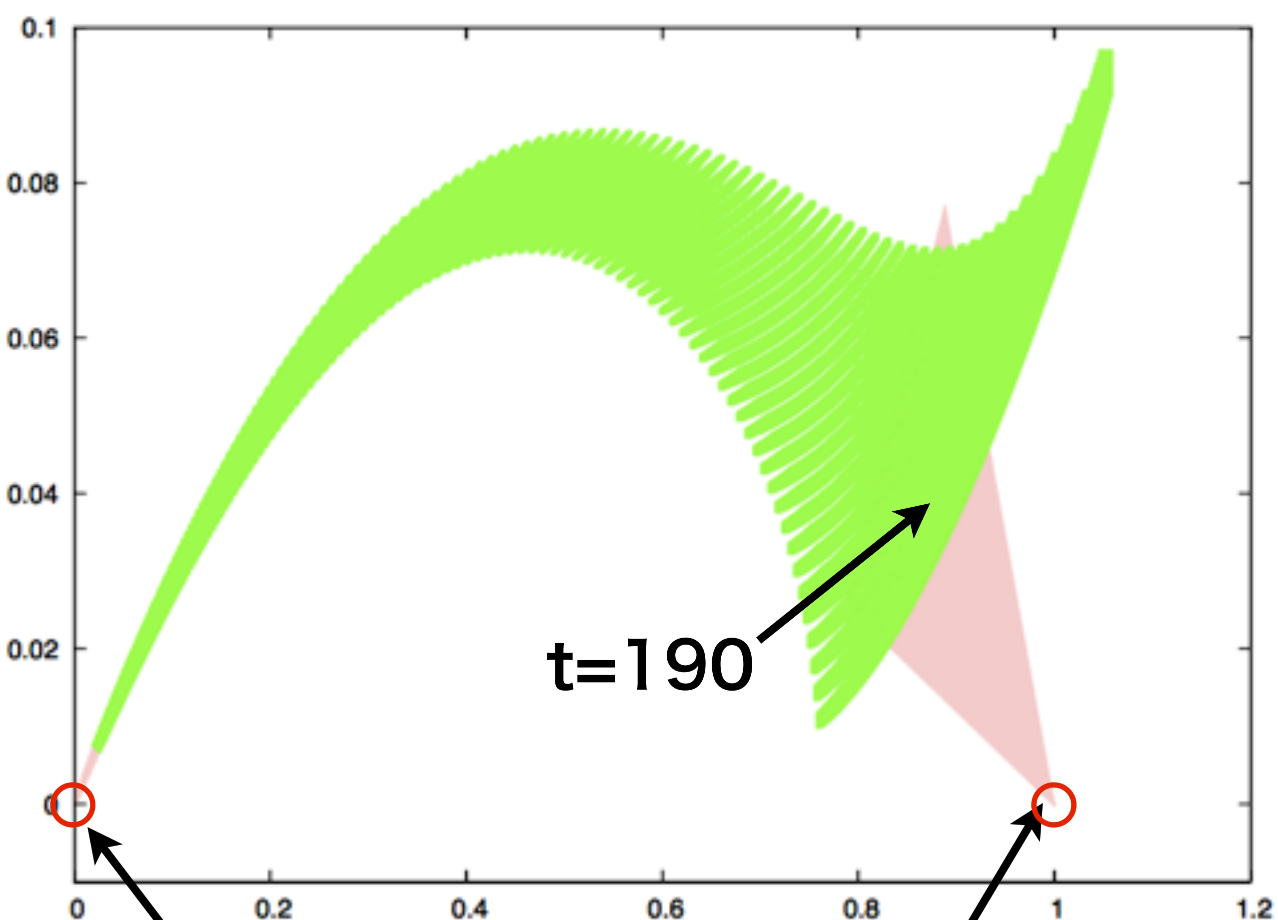
$$\dot{u} = v$$
$$\dot{v} = 0.2(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$

$$f(u) = u(u-0.2)(1-u),$$
$$\theta \in [0.947, 0.948],\ \epsilon \in [0, 10^{-5}]$$



Total orbit : dt = 0.001, t = 0 ~ 190

$$\lambda \in [-0.00242308, 0.00242308]$$

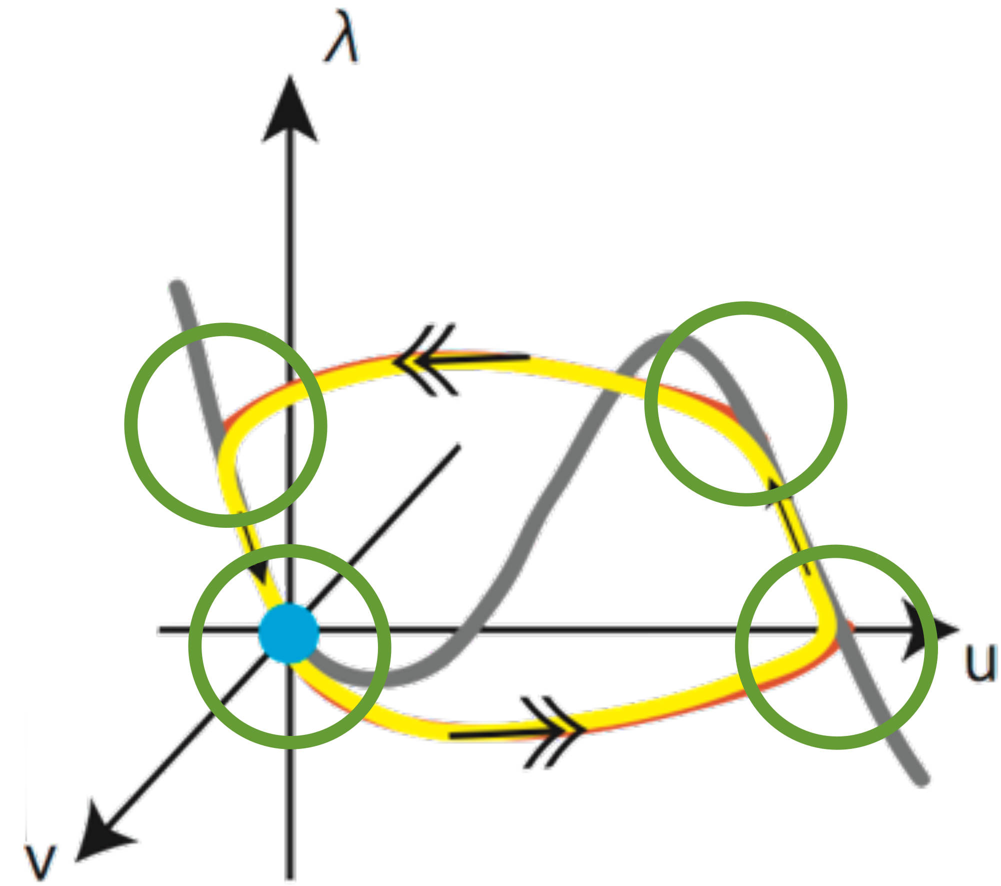


・Blocks are chosen small in order to get a good estimate of manifolds.

・Rigorous numerics encloses the error of global orbits in each step and become bigger and bigger !

Left : Enclosure of orbits is already larger than the block !

Validations without any ideas are so crazy !



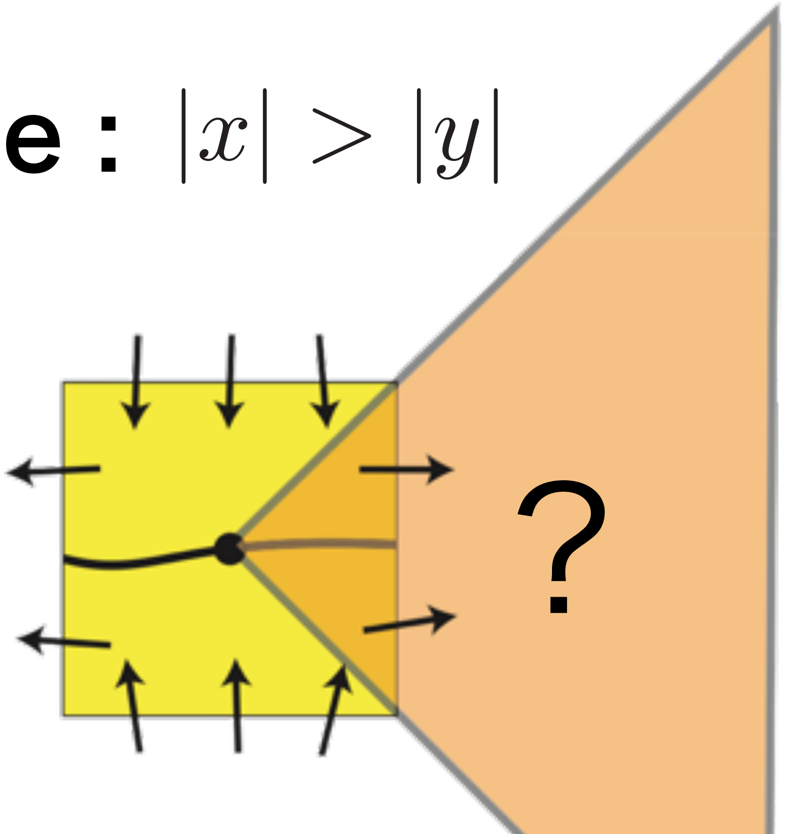
1. Slow Dynamics
2. Fast Dynamics
3. Matching : “Covering-Exchange”
4. m-cones
5. Towards Validation -- overview (FitzHugh-Nagumo)

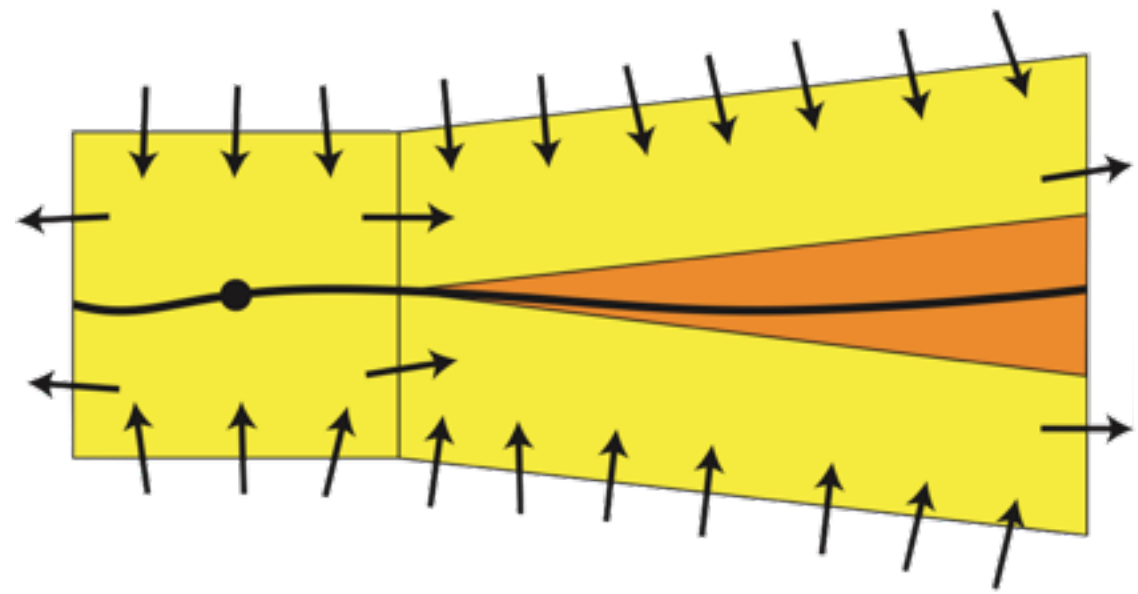
m-cones

Extend (un)stable manifolds making sharp cones.

cone : $|x| > |y|$

m-cone : $|x| > m|y|$





Isolating blocks

- Very small in general.
- Where the unstable manifold extends ?（cone：orange domain)
- Flow moves very slowly near fixed points

→ increase of computation costs.

Cones, m-cones

- Unstable manifold is contained in cones

→ Be cones sharper and raise the accuracy of the unstable manifold.

- Away from equilibria.
- **isolation is preserved.**

m-cones

Cone condition for fast-slow system.

Thm. [M. cf. Jones (1995) Theorem 4**]**

Define **Maximal Singular Values** of matrices :

$$\sigma^s_{\mathbb{A}_1} : \mathbb{A}_1(z) = \begin{pmatrix} \frac{\partial F_1}{\partial a}(z) \end{pmatrix}, \ \sigma^s_{\mathbb{A}_2} : \ \mathbb{A}_2(z) = \begin{pmatrix} \frac{\partial F_1}{\partial b}(z) & \frac{\partial F_1}{\partial y}(z) & \frac{\partial F_1}{\partial \eta}(z) \end{pmatrix},$$

$$\sigma^s_{\mathbb{B}_1} : \mathbb{B}_1(z) = \begin{pmatrix} \frac{\partial F_2}{\partial a}(z) \end{pmatrix}, \ \sigma^s_{\mathbb{B}_2} : \ \mathbb{B}_2(z) = \begin{pmatrix} \frac{\partial F_2}{\partial b}(z) & \frac{\partial F_2}{\partial y}(z) & \frac{\partial F_2}{\partial \eta}(z) \end{pmatrix}$$

$$\sigma^s_{g_1} : g_1(z) = \begin{pmatrix} \frac{\partial g}{\partial a}(z) \end{pmatrix}, \ \sigma^s_{g_2} : \ g_2(z) = \begin{pmatrix} \frac{\partial g}{\partial b}(z) & \frac{\partial g}{\partial y}(z) & \frac{\partial g}{\partial \eta}(z) \end{pmatrix}$$

Assume the following inequalities (**stable cone conditions**) :

$$\inf \mathrm{Spec}(A) - \left(\sup \sigma^s_{\mathbb{A}_1} + \sup \sigma^s_{\mathbb{A}_2}\right) > 0,$$

$$\inf \mathrm{Spec}(A) + \inf |\mathrm{Spec}(B)| - \left\{\sup \sigma^s_{\mathbb{A}_1} + \sup \sigma^s_{\mathbb{A}_2} + \sup \sigma^s_{\mathbb{B}_1} + \sup \sigma^s_{\mathbb{B}_2} + \epsilon_0 \left(\sup \sigma^s_{g_1} + \sup \sigma^s_{g_2}\right)\right\} > 0,$$

Then **for all** $\epsilon \in [0, \epsilon_0]$ $W^s(M_\epsilon) \cap (B \times K)$ can be represented by the graph of a Lipschitz function on $B_2 \times K$. The similar statement holds for $W^u(M_\epsilon) \cap (B \times K)$. The slow manifold M_ϵ is the k-dimensional submanifold in $B \times K$ can be represented by their intersection. In particular, M_0 is normally hyperbolic.

m-cones

Stable m-cone condition for fast-slow system.

Thm. [M., cf. M.-Yamamoto]

Let B, K as above.

Define **Maximal Singular Values** of matrices :

$$\sigma^{s,m}_{\mathbb{A}_1} : \mathbb{A}_1(z) = \left(\frac{\partial F_1}{\partial a}(z)\right), \quad \sigma^{s,m}_{\mathbb{A}_2} : \mathbb{A}_2(z) = \underline{m^{-1}}\left(\frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z)\right),$$

$$\sigma^{s,m}_{\mathbb{B}_1} : \mathbb{B}_1(z) = \underline{m}\left(\frac{\partial F_2}{\partial a}(z)\right), \quad \sigma^{s,m}_{\mathbb{B}_2} : \mathbb{B}_2(z) = \left(\frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right),$$

$$\sigma^{s,m}_{g_1} : g_1(z) = \underline{m}\left(\frac{\partial g}{\partial a}(z)\right), \quad \sigma^{s,m}_{g_2} : g_2(z) = \left(\frac{\partial g}{\partial b}(z) \quad \frac{\partial g}{\partial y}(z) \quad \frac{\partial g}{\partial \eta}(z)\right).$$

Assume the following inequalities（**stable m-cone conditions**）：

$$\inf \mathrm{Spec}(A) - \left(\sup \sigma^{s,m}_{\mathbb{A}_1} + \sup \sigma^{s,m}_{\mathbb{A}_2}\right) > 0,$$

$$\inf \mathrm{Spec}(A) + \inf |\mathrm{Spec}(B)| - \left\{\sup \sigma^{s,m}_{\mathbb{A}_1} + \sup \sigma^{s,m}_{\mathbb{A}_2} + \sup \sigma^{s,m}_{\mathbb{B}_1} + \sup \sigma^{s,m}_{\mathbb{B}_2} + \sigma\left(\sup \sigma^{s,m}_{g_1} + \sup \sigma^{s,m}_{g_2}\right)\right\} > 0,$$

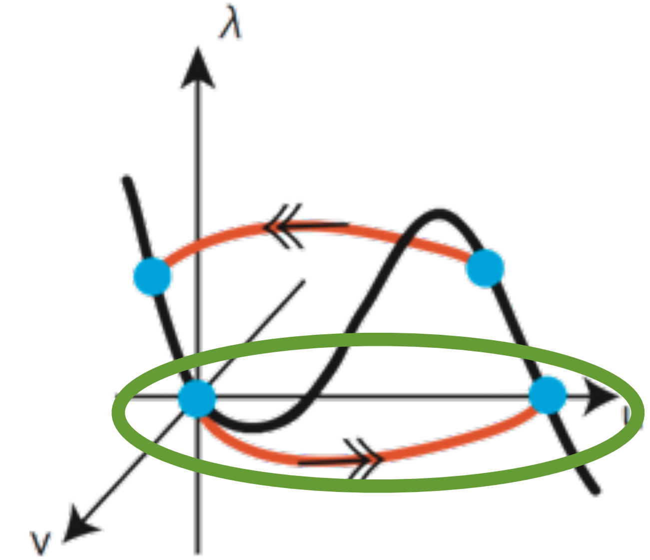
Then the function $M(t) := |\Delta a(t)|^2 - m^2|\Delta\zeta(t)|^2 \ (\zeta = (b,y))$ satisfies :

$M'(t) > 0.$ holds on the set $M(t) = 0$ as long as orbits stay B×K.

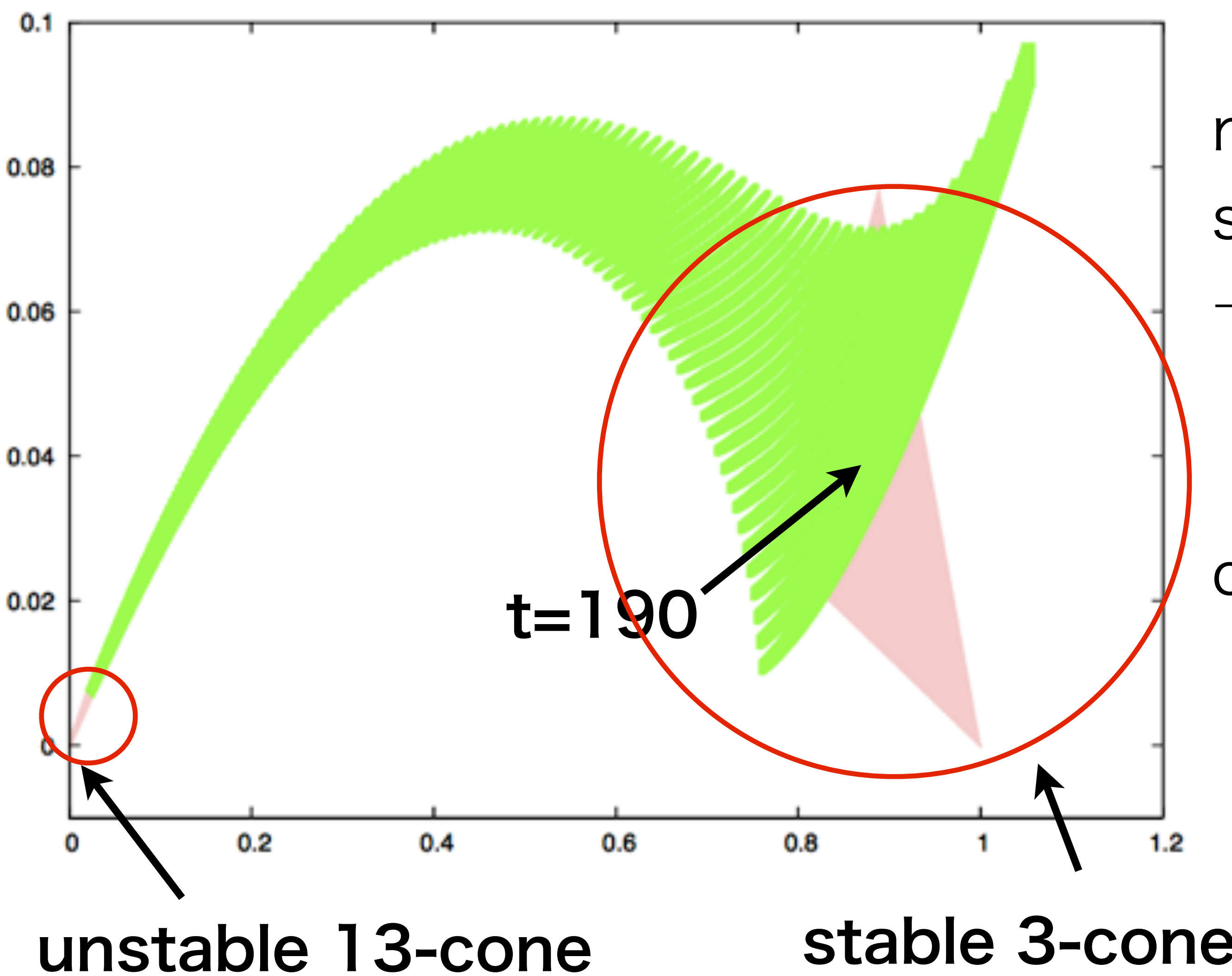
with m-cones ...

$$\dot{u} = v$$
$$\dot{v} = 0.2(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$

$$f(u) = u(u-0.2)(1-u),$$
$$\theta \in [0.947, 0.948], \ \epsilon \in [0, 10^{-5}]$$



$\lambda \in [-0.00242308, 0.00242308]$



Total orbit : dt = 0.001, t = 0 ~ 190

・Unstable m-cone : orbits leaves a neighborhood of slow manifolds in a short time.
→ prevent error accumulations

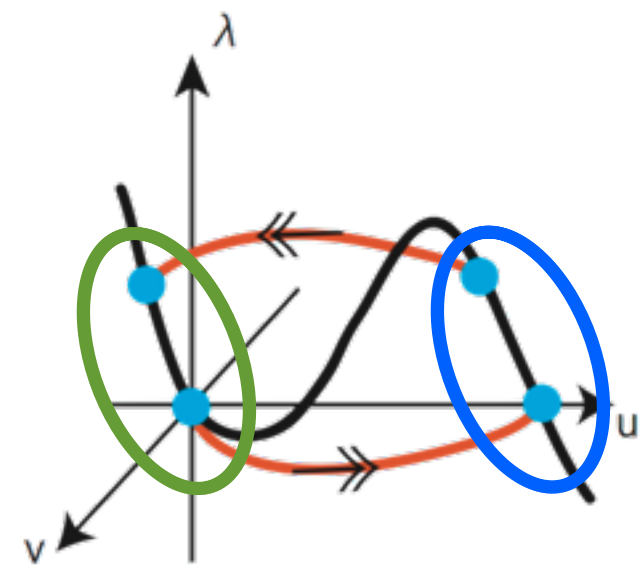
・Stable m-cone : blocks for verifying covering relations become larger.

Verifications become dramatically easy !!

1. Slow Dynamics
2. Fast Dynamics
3. Matching : “Covering-Exchange”
4. m-cones
5. **Towards Validation -- overview (FitzHugh-Nagumo)**

Homoclinic orbits of the FitzHugh-Nagumo system -- overview

$$\dot{u} = v$$
$$\dot{v} = 0.2(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$



$$f(u) = u(u-0.2)(1-u),$$
$$\theta \in [0.947, 0.948],\ \epsilon \in [0, 10^{-5}]$$

Computation environment

Library : CAPD (http://capd.ii.uj.edu.pl) 3.0
CPU : 1.6GHz Intel Core i5 (Macbook Air 2011 model)
Memory : 4GB 1333 MHz DDR3

1. 1st branch

We can construct fast-saddle-type blocks satisfying cone conditions for $\lambda \in [-0.0005, 0.1]$ around green branch.

2. 3rd branch

We can construct fast-saddle-type blocks satisfying cone conditions for $\lambda \in [-0.0005, 0.1]$ around blue branch.

Homoclinic orbits of the FitzHugh-Nagumo system -- overview

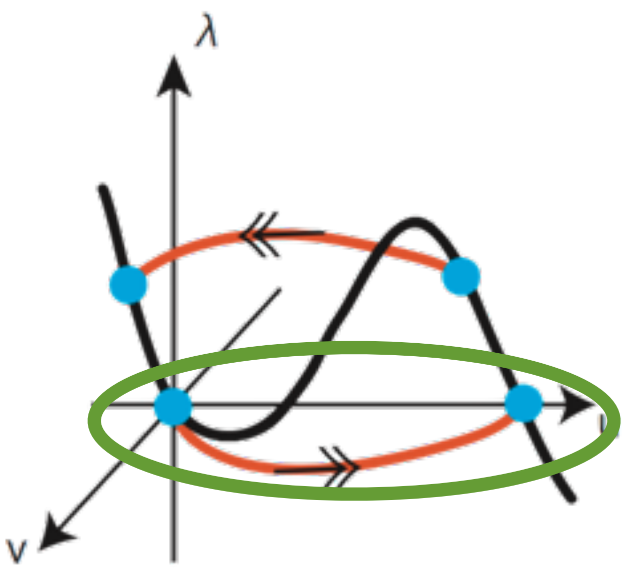
$\dot{u} = v$

$\dot{v} = 0.2(\theta v - f(u) + \lambda)$

$\dot{\lambda} = \epsilon\theta^{-1}u$

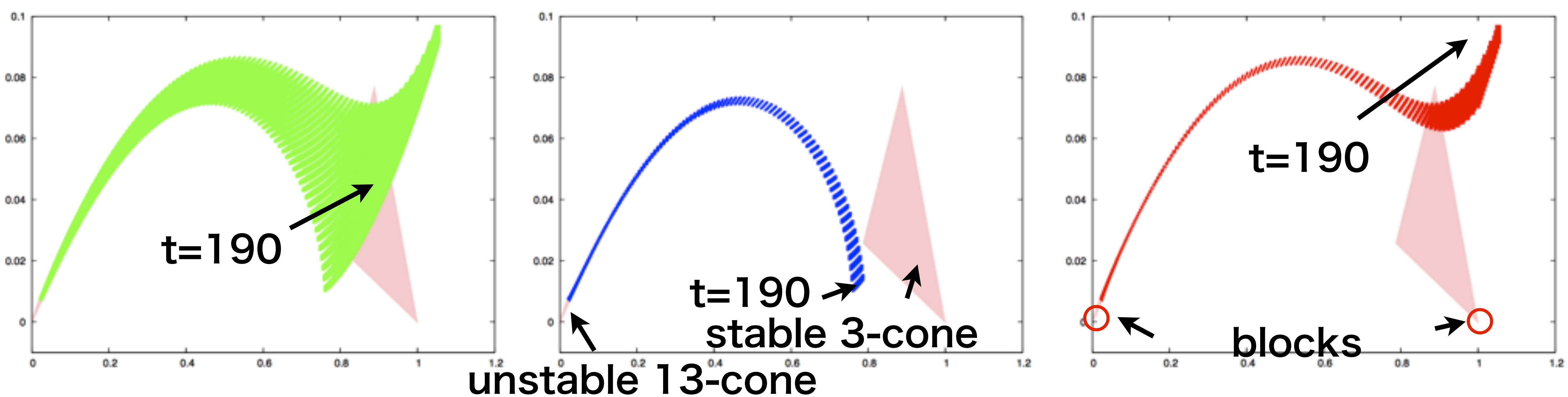
$$f(u) = u(u - 0.2)(1 - u),$$

$$\theta \in [0.947, 0.948],\ \epsilon \in [0, 10^{-5}]$$



Total orbit : dt = 0.001, t = 0 ~ 190

3. Fast trajectory from $(u, v, \lambda) \approx (0, 0, 0)$

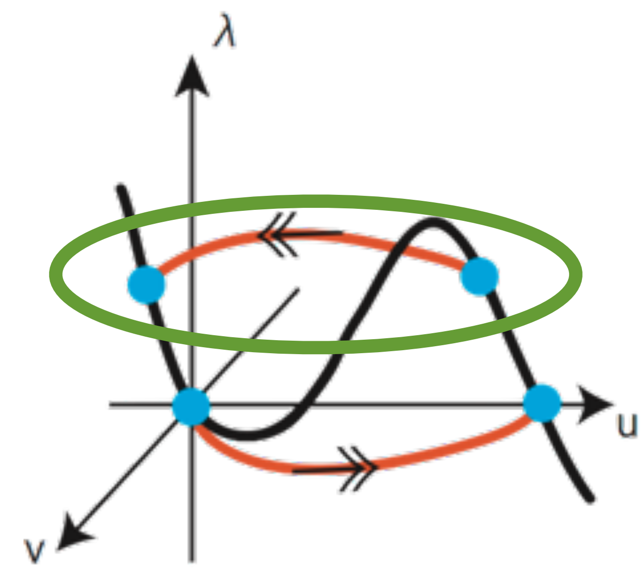


$\lambda \in [-0.00242308, 0.00242308]$ $\lambda = -0.00242308$ $\lambda = +0.00242308$

Homoclinic orbits of the FitzHugh-Nagumo system -- overview

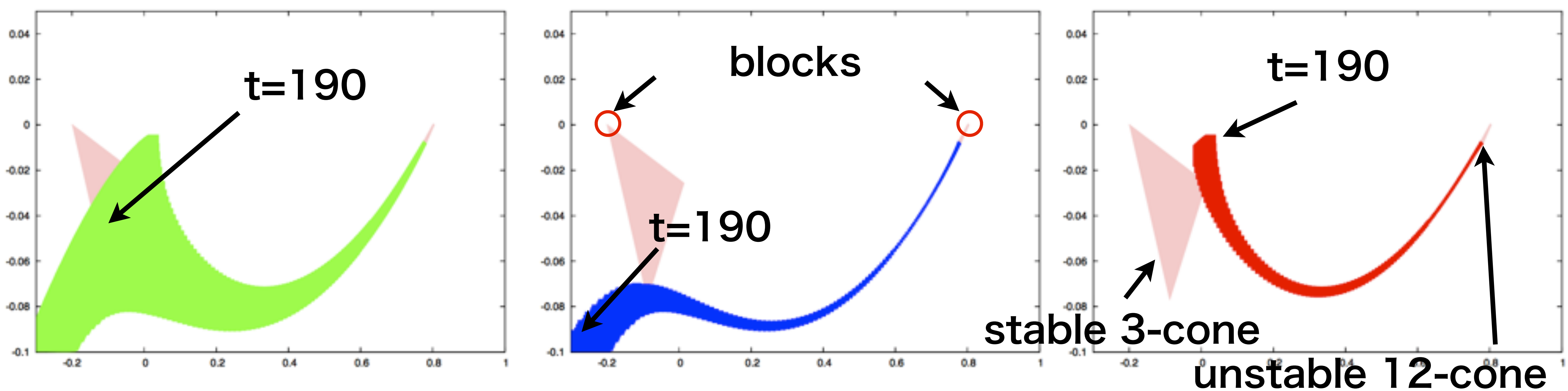
$$\dot{u} = v$$
$$\dot{v} = 0.2(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$

$$f(u) = u(u-0.2)(1-u),$$
$$\theta \in [0.947, 0.948],\ \ \epsilon \in [0, 10^{-5}]$$



Total orbit : dt = 0.001, t = 0 ~ 190

4. Fast trajectory from $(u, v, \lambda) \approx (0.8, 0, 0.0955)$

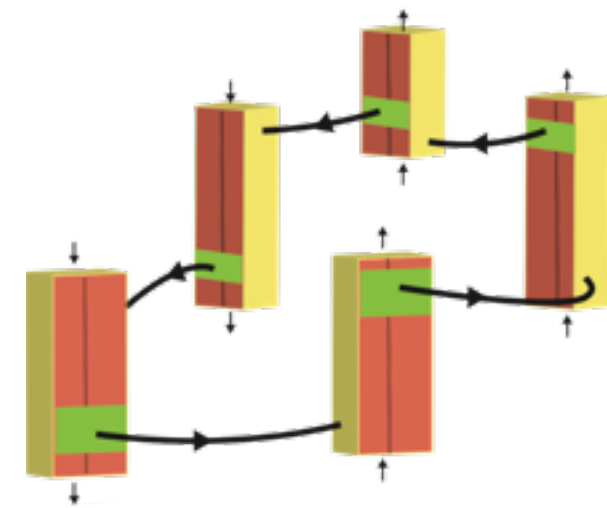


$\lambda \in [0.0929167, 0.0980833]$ $\lambda = 0.0929167$ $\lambda = 0.0980833$

Homoclinic orbits of the FitzHugh-Nagumo system -- overview

$$\dot{u} = v$$
$$\dot{v} = 0.2(\theta v - f(u) + \lambda)$$
$$\dot{\lambda} = \epsilon\theta^{-1}u$$

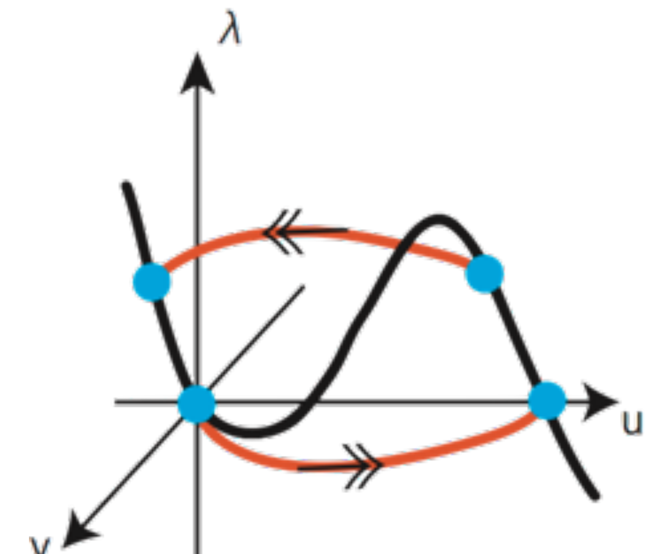
$$f(u) = u(u-0.2)(1-u),$$
$$\theta \in [0.947, 0.948],\ \epsilon \in [0, 10^{-5}]$$



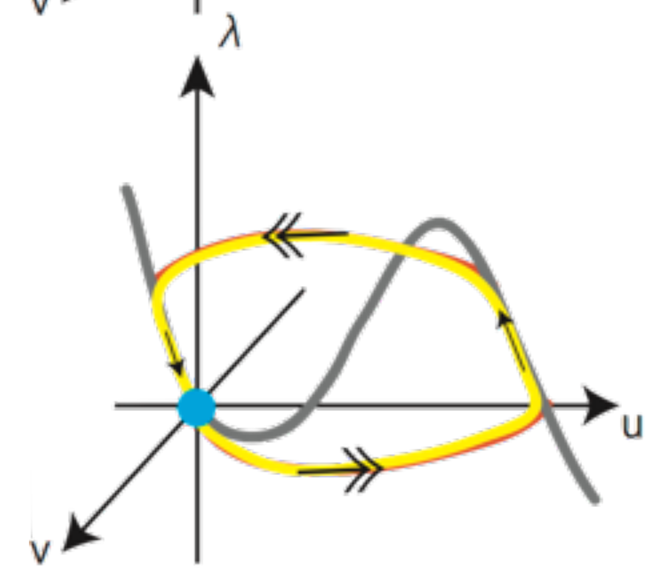
Computer Assisted Result [M.]

There exist the following trajectories of the FitzHugh-Nagumo system :

1. $\epsilon = 0$: A singular homoclinic orbit consisting of two components of nullcline and two heteroclinic orbits connecting them.



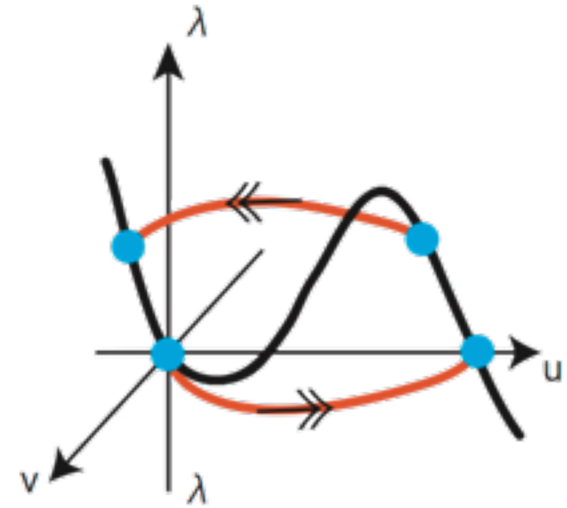
2. $\epsilon \in (0, 10^{-5}]$: homoclinic orbit of $(u, v, \lambda) = (0, 0, 0)$ as the continuation of the singular orbit obtained in 1.

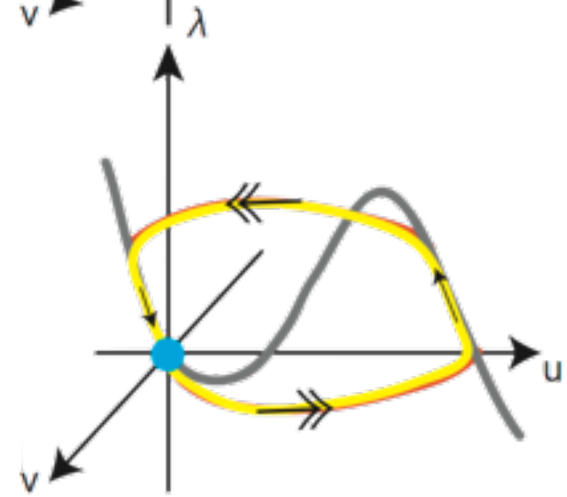


Conclusion

・**Slow Dynamics** : proposed a sufficient condition for validating slow manifolds and dynamics around them.

・**Matching** : topologically described the matching of dynamics in different time scales.





→ Sample validation of singular perturbation problem.

Periodic, Heteroclinic : computing.

Further directions :

・Other examples （ multi-slow variables ）

・Slow manifolds containing non-hyperbolic points like **fold points**

・Transversality (via Exterior Algebra)

Ex. : Double-pulse in the FitzHugh-Nagumo sys.
Guchenheimer-Kuehn, SIADS(2009) →