Perturbation Theory in Celestical Mechanics

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Integrable Hamiltonian system

- Hamiltonian system :

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}(q,p), \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}(q,p) \qquad (j = 1, \ldots, k) \quad (1)$$

  where $q = (q_1, \ldots, q_k), p = (p_1, \ldots, p_k)$, $H : \mathbb{R}^{2k} \to \mathbb{R}$.

- Hamiltonian system (1) is integrable $\Longleftrightarrow$ there are k first integrals $F_1 (= H), F_2, \ldots, F_k$ such that $dF_1, \ldots, dF_k$ are linearly independent a.e. and that $\{F_i, F_j\} = 0$ for any $i, j = 1, \ldots, k$.
- Loosely speaking, the Liouville-Arnold theorem states that for an integrable Hamiltonian system, there are canonical variables (called action-angle variables)

$$(\theta, I) \in \mathbb{T}^k \times U (\subset \mathbb{R}^k) \mapsto (q,p) \in \mathbb{R}^{2k}$$

  such that the Hamiltonian depends only on I.

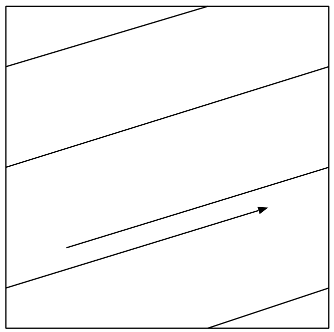
Integrable Hamiltonian system

Let $H(I)((\theta, I) \in \mathbb{T}^k \times U(\subset \mathbb{R}^k))$ be an integrable Hamiltonian. The canonical equations are

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial I}, \quad \frac{dI}{dt} = -\frac{\partial H}{\partial \theta}.$$

Since $\frac{\partial H}{\partial \theta} = 0$, I is a constant along any solution: $I \equiv I_0$. Therefore

$$\theta = \frac{\partial H}{\partial I}(I_0)t + \theta_0.$$



Perturbed system

Perturbed system:

$$H_\varepsilon(I,\theta) = h_0(I) + \varepsilon h_1(I,\theta;\varepsilon).$$

Twist condition:

$$\det(\mathrm{Hess}(h_0(I))) \neq 0.$$

KAM theory: under the twist condition, if $\omega = (\omega_1, \ldots, \omega_k) = \frac{\partial h_0}{\partial I}(I_0)$ is Diophantine:

$$\exists \tau > 0, \exists \gamma > k-1, \forall (l_1, \ldots, l_k) \in \mathbb{Z}^k \backslash \{0\}$$
$$\left| \sum_{j=1}^{k} l_j \omega_j \right| \geq \gamma(|l_1| + \cdots + |l_k|)^{-\tau},$$

the invariant torus with the frequency ω survives for small $\varepsilon > 0$.

Example (Isosceles three-body problem)

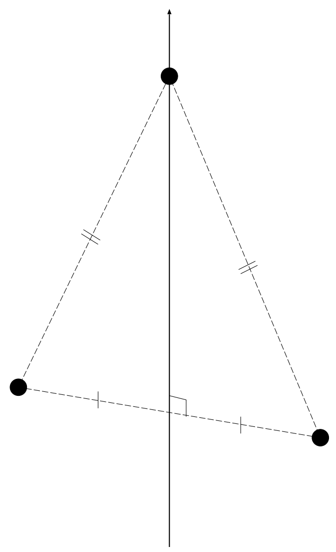


図 1　The isosceles 3-body problem

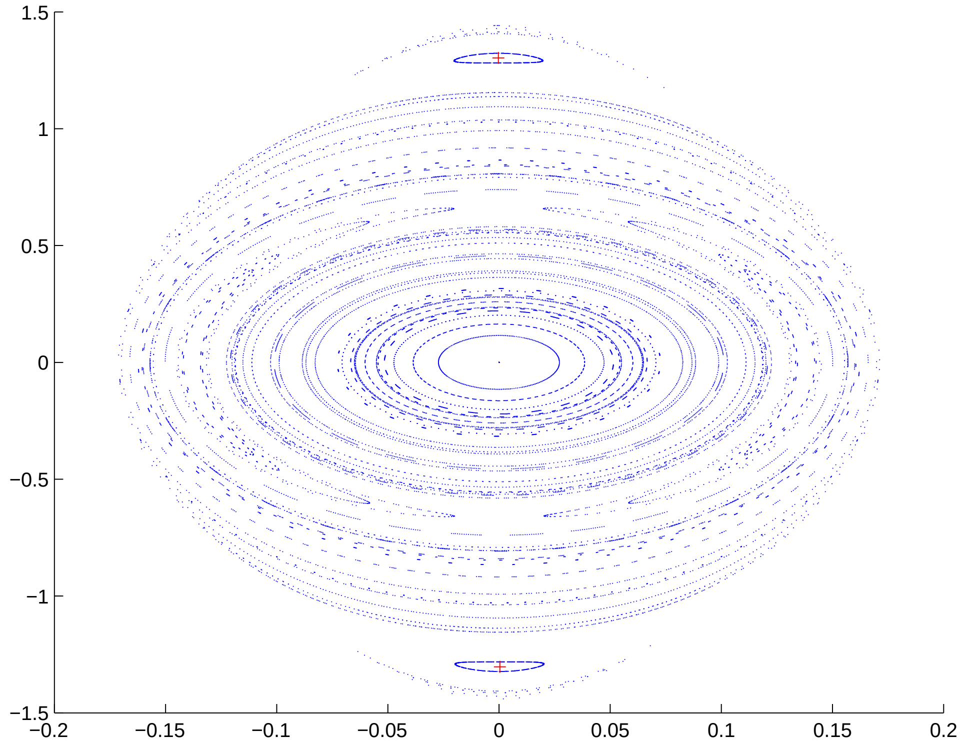


図 2　The Poincaré map

Ref: M. Shibayama, *RIMS Kôkyûroku Bessatsu*, **B13** (2009), 141-155.

Example (Kepler-type problem)

Kepler-type problem:

$$H_0 = \frac{1}{2}|p|^2 - \frac{1}{|q|^\alpha} \qquad (q, p \in \mathbb{R}^d).$$

(The original Kepler problem is the case $\alpha = 1$.)

This Hamiltonian is integrable.

For $\alpha \neq 1$ and $d = 2$, the twist condition is safisfied and hence KAM theorem can be applied. For the perturbed system H_ε, there are quasi-periodic solutions with Diophantine frequency.

In the case $\alpha = 1$, the Hamiltonian can be represented

$$H = -\frac{1}{2I_1^2}.$$

So the twist condition is not satisfied.

Example(Solar system)

Traditional problem is to show that almost all of quasi-periodic solutuions in the solar system survive. Its Hamiltonian is

$$H_\varepsilon(\theta, I) = -\sum_{k=1}^{n} \frac{\mu_k}{2I_{1k}^2} + \varepsilon f(\theta, I; \varepsilon)$$

where $I = (I_{11}, \ldots, I_{dn}) \in \mathbb{R}^{dn}, \theta = (\theta_{11}, \ldots, \theta_{dn}) \in \mathbb{T}^{dn}$. This does not satisfies the twist condition (very degenerate !). Arnold (1963) solved it for the case $d = 2, n = 2$. J. Féjoz (2004, Ergod. Th. & Dynam. Sys.) solved it for the case $d = 3, n \geq 2$.

Our problem

Consider a Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 + V(q) \quad (p,q \in \mathbb{R}^N)$$

where $V \in C^2(\mathbb{R}^N \backslash \{0\}, \mathbb{R}) \quad (V \sim -\frac{1}{|q|^\alpha})$.

Fix the energy $H(q,p) = h$.

We call $q(t)$ a generalized periodic solution with period T if

1. $q \in C(\mathbb{R}, \mathbb{R}^N)$ and T-periodic,
2. $D = \{t \in \mathbb{R} \mid q(t) = 0\}$ has zero measure,
3. $q \in C^2(\mathbb{R}\backslash D, \mathbb{R}^N)$ satisfies the canonical equations and the energy relation in $\mathbb{R}\backslash D$

Our goal is to show the existence of periodic solutions with prescribed energy for a perturabed system of Kepler-type problem.

Theorem

Let $N \geq 2$ and $V \in C^2(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$. Assume that there are $0 < a_1 < a_2, 0 < \alpha_1 < \alpha < \alpha_2 < 2$ such that

$$\frac{a_1}{|q|^\alpha} \leq -V(q) \leq \frac{a_2}{|q|^\alpha}, \qquad -\alpha_1 V(q) \leq \nabla V(q) \cdot q \leq -\alpha_2 V(q)$$
$$\nabla V(q) \to 0 \ (|q| \to \infty), \qquad |q|^3 \nabla V(q), |q|^4 \nabla^2 V(q) \to 0 \ (q \to 0)$$

Then for any $h < 0$, there is a generalized periodic solution with energy h. Let $T > 0$ be the minimal period. The number of collision is estimated as follows:

$$\#\{t \in [0, T) \mid q(t) = 0\} \leq f(a_1, a_2, \alpha, \alpha_1, \alpha_2).$$

Here

$$f(a_1, a_2, \alpha, \alpha_1, \alpha_2) = \frac{\pi a_2^{\frac{1}{\alpha}} \alpha^{\frac{3}{2}} (2-\alpha)^{\frac{2}{\alpha}} (2+\alpha_2)^{\frac{2+\alpha}{2\alpha}}}{2^{\frac{1}{\alpha}} a_1^{\frac{1}{\alpha}} \alpha_1 (2+\alpha)^{\frac{2+\alpha}{2\alpha}} (2-\alpha_2)^{\frac{2-\alpha}{2\alpha}} B\left(\frac{1}{2}, \frac{2+\alpha}{2\alpha}\right)}$$

and B is the Beta function.

Corollary

Assume the same properties as in the theorem. For any $\alpha \in (1,2)$, there is $\delta > 0$ satisfying the following: if $a_1 \leq a_2 < (1+\delta)a_1, 0 < \alpha_2 - \alpha_1 < \delta$, then the obtained solution has no collision, and hence is a classical solution.

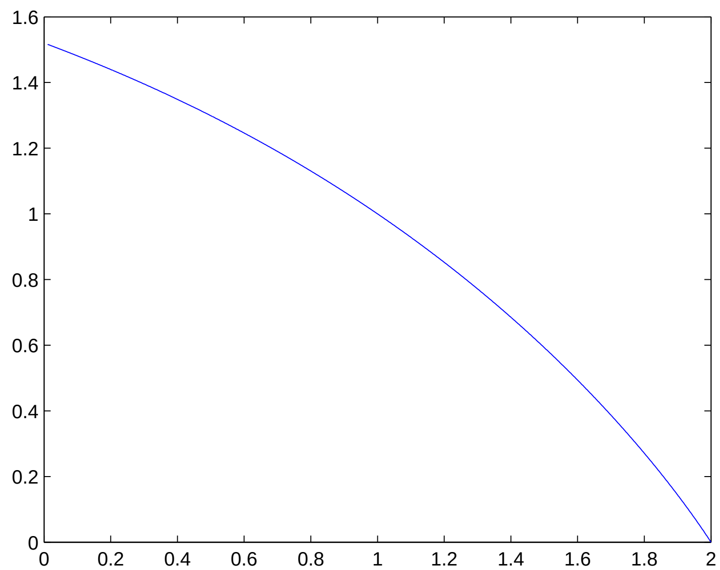


図 3　Graph of $f(1,1,\alpha,\alpha,\alpha)$.

History of the prescribed-energy problem

The existence problem of a periodic solution on the energy surface:

$$S_h = \{(q,p) \mid H = h\}.$$

- Weinstein(1978): S_h is compact and convex
- Rabinowitz (1978): S_h is compact and star-shaped
- Viterbo(1987): S_h is a compact contact manifold (variational proof)
- Hofer & Zehnder(1987): S_h is a compact contact manifold (geometric proof)
- Hofer(1993): S_h is diffeomorphic to $\mathbb{S}^3$.
- Tanaka (1993): natural Hamiltonian $H = \frac{1}{2}|p|^2 + V(q)$ with singular potential like $\frac{1}{|q|^\alpha}$ for $\frac{4}{3} < \alpha < 2(N=3), \quad 1 < \alpha < 2(N \geq 4)$.
- Our result extends his resut to the case $1 < \alpha < 2(N \geq 2)$.

Proof

The prescribed-energy problem is represented by the variational problem with respect to the functional

$$I(u) = \frac{1}{2}\int_0^1 \left|\frac{du}{d\tau}\right|^2 d\tau \int_0^1 h - V(u(\tau))d\tau.$$

By letting

$$T = \sqrt{\frac{\frac{1}{2}\int_0^1 \left|\frac{du}{d\tau}\right|^2 d\tau}{\int_0^1 h - V(u)d\tau}}$$

for a critical point $u(\tau)$ of I, $q(t) = u(t/T)$ is a solution with energy h. The domain Λ of I is defined by

$$E = \{u(\tau) \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N); u(\tau+1) = u(\tau)\}$$
$$\Lambda = \{u(\tau) \in E; u(\tau) \neq 0 \text{ for all } \tau\}.$$

Proof

Consider the case of $N = 2$. We take $\rho_R(\tau)$ defined by

$$\rho_R(\tau) = R(\cos 2\pi\tau, \sin 2\pi\tau).$$

Take small $R_0 > 0$ and large R_1. Let

$$Q = \{\eta \in C([R_0, R_1], \Lambda) \mid \eta(R_0) = \rho_{R_0}, \eta(R_1) = \rho_{R_1}\}.$$

We can get a generalized solution attaining

$$c = \inf_{\eta \in Q} \max_{R \in [R_0, R_1]} I(\eta(R)).$$

Proof (estimate of the minimax value)

From the inequality

$$-\frac{a_2}{|q|^\alpha} \le V(q),$$

we get

$$\begin{aligned} I(\rho_R) &= \frac{1}{2}\int_0^1 \left|\frac{d\rho_R}{d\tau}\right|^2 d\tau \int_0^1 h - V(\rho_R) d\tau \\ &\le \frac{1}{2}\int_0^1 \left|\frac{d\rho_R}{d\tau}\right|^2 d\tau \int_0^1 h + \frac{a_2}{|\rho_R|^\alpha} d\tau \\ &= 2\pi^2(hR^2 + a_2 R^{2-\alpha}). \end{aligned}$$

The maximum of $2\pi^2(hR^2 + a_2R^{2-\alpha})$ on $R \in [R_0, R_1]$ is $\pi^2 \alpha a_2^{\frac{2}{\alpha}} (\frac{2-\alpha}{-2h})^{\frac{2-\alpha}{\alpha}}$.
Therefore the minimax values is no more than this value:

$$c \le \pi^2 \alpha a_2^{\frac{2}{\alpha}} (\frac{2-\alpha}{-2h})^{\frac{2-\alpha}{\alpha}}. \tag{2}$$

Proof (estimate of collision paths)

Assume $\gamma(t)$ is a generalized solution with collisions at $0 \leq \tau_1 < \tau_2 < \cdots < \tau_k < 1$. We can assume $\tau_1 = 0$. Let T be the period. Let $0 = T_1 < T_2 < \cdots < T_k < T$ be the collision times, i. e. $T_i = \tau_i T$, and let $T_{k+1} = T$.

$$
\begin{aligned}
I(\gamma) &= \frac{1}{2}\int_0^1 \left|\frac{d\gamma}{d\tau}\right|^2 d\tau \int_0^1 h - V(\gamma) d\tau \\
&= \frac{1}{2}\int_0^T \left|\frac{d\gamma}{dt}\right|^2 dt \int_0^T h - V(\gamma) dt \\
&= \left(\frac{1}{2}\sum_{i=1}^{k}\int_{T_i}^{T_{i+1}} \left|\frac{d\gamma}{dt}\right|^2 dt\right)\left(\sum_{i=1}^{k}\int_{T_i}^{T_{i+1}} h - V(\gamma) dt\right)
\end{aligned}
$$

For the obtained generalized solution, we get

$$
\frac{1}{2}\int_0^{T_0} \left|\frac{d\gamma}{dt}\right|^2 dt = \int_0^{T_0} h - V(\gamma) dt \geq \frac{\alpha_1}{2+\alpha_2}\mathcal{A}_{T_0}.
$$

Here $\mathcal{A}_{T_0}$ is the Lagrangian action functional for the Kepler-type problem:

$$
\mathcal{A}_{T_0} = \int_0^{T_0} \frac{1}{2}|\dot{q}|^2 + \frac{a_1}{|a|^{\alpha}} dt.
$$

Proof (estimate of collision paths)

For generalized solutions, the energy value is related to the period as follow:

$$-hT_0 \geq -\frac{2-\alpha_2}{2}\int_0^{T_0} V(\gamma)dt \geq \frac{2-\alpha_2}{2+\alpha_2}\mathcal{A}_{T_0}.$$

We know the exact value of the minimizer for the collision paths:

$$\inf \mathcal{A}_{T_0} = \frac{2+\alpha}{2-\alpha}\,\frac{2^{\frac{\alpha}{\alpha+2}}\,a_1^{\frac{2}{\alpha+2}}}{\alpha^{\frac{2\alpha}{\alpha+2}}}(B(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}))^{\frac{2\alpha}{\alpha+2}}\,T_0^{\frac{2-\alpha}{2+\alpha}}.$$

By putting them together, we can estimate the value for the collision path:

$$I \geq \frac{\alpha_1^2}{(2+\alpha_2)^2}(\frac{2-\alpha_2}{2+\alpha_2})^{\frac{2-\alpha}{\alpha}}(\frac{2+\alpha}{2-\alpha})^{\frac{2+\alpha}{\alpha}}\frac{2a_1^{\frac{2}{\alpha}}}{\alpha^2}(B(\frac{1}{2},\frac{1}{\alpha}+\frac{1}{2}))^2(-h)^{-\frac{2-\alpha}{\alpha}}k^2 \quad (3)$$

From (2) and (3), we estimate the number of collisions:

$$k \leq f(a_1, a_2, \alpha, \alpha_1, \alpha_2).$$

This completes the proof.

Thank you very much.