Stochastic Characterization of Numerical Viscosity

Kohei Soga
(Keio University)

Dynamical Systems in Mathematical Physics
RIMS, Kyoto
2015/2/24

Introduction

Consider initial problems of Hamilton-Jacobi equations

$$(\mathrm{HJ})\begin{cases} v_t + H(x,t,v_x) = h & \text{in } \mathbb{R}^d \times (0,T] \qquad (h\text{: const.}), \\ v(x,0) = v^0(x) & \text{on } \mathbb{R}^d \quad (v^0 \in Lip). \end{cases}$$

Hamilton-Jacobi equations are closely related to

1. Classical mechanics (v is a generating function of a symplectic transform);
2. Optimal control theory (v is a cost function);
3. Inviscid fluids (v_x is an entropy solution);
4. Hamiltonian systems (a characteristic curve γ^* of v is an orbit and graph(v_x) is an invariant set);
5. Classical KAM theory, weak KAM theory;

   ⋮

(HJ) must be considered in the class of weak sol. "viscosity solutions".

Aim. Obtain approximation techniques for the viscosity sol. of (HJ) by which we can approximate all of v, v_x, γ^*.

Two Approximation techniques:

- the vanishing viscosity method (VVM), i.e., add νv_{xx} and $\nu \to 0+$,
- the finite difference method (FDM), i.e., $f'(x) \sim \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

$\to$ Stochastic approach to VVM.
(Fleming '69) Numerically inaccessible

$\to$ Stochastic approach to FDM under hyperbolic scaling.
(Soga [1]-[5]) Numerically accessible

Why "stochastic" in FDM? $\to$ "Numerical viscosity"

Stochastic approach is different from the standard frameworks, yielding new results.

---

[1] Soga ('14), Nonlinear Analysis.
[2] Soga ('14), Mathematics of Computation.
[3] Soga, submitted.
[4] Soga, submitted.
[5] Soga, preprint.

Assumptions for $H(x,t,p)$:

(H1) $H(x,t,p) : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, C^2, $H_{pp} > 0$, superlinear w.r.t. p.

(H2) The Legendre transf. L of H w.r.t. p satisfies $|L_{x^j}| \le \alpha(|L|+1)$.

$$L(x,t,\xi) = \sup_{p \in \mathbb{R}^d} \{p \cdot \xi - H(x,t,p)\}.$$

(H3) All the derivatives of H up to the second order are bdd. on $\mathbb{R}^d \times \mathbb{R} \times K$ for each $K \subset\subset \mathbb{R}^d$.

Under (H1)-(H3)

Convergence & error estimate for v, v_x, γ^* in FDM are not trivial, where the standard framework seems hopeless.

Hyperbolic PDE

$$\begin{cases} u_t(x,t) + a u_x(x,t) = 0, \\ u(x,0) = u_0(x) \text{ on } \mathbb{R}. \end{cases}$$

$$\Rightarrow \qquad u(x,t) = u_0(x - at).$$

Add a parabolic term

$$\begin{cases} u_t^\nu(x,t) + a u_x^\nu(x,t) = \nu^2 u_{xx}^\nu(x,t), \\ u^\nu(x,0) = u_0(x) \text{ on } \mathbb{R}. \end{cases}$$

$$\Rightarrow \qquad u^\nu(x,t) = \int_{\mathbb{R}} \frac{1}{2\nu\sqrt{\pi t}} e^{-\frac{(x-at-y)^2}{4\nu^2 t}} u_0(y)dy$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} u_0(x - at + \sqrt{2}\nu y)dy.$$

$u^\nu \to u$ as $\nu \to 0$ (vanishing viscosity method).

Stochastic interpretation

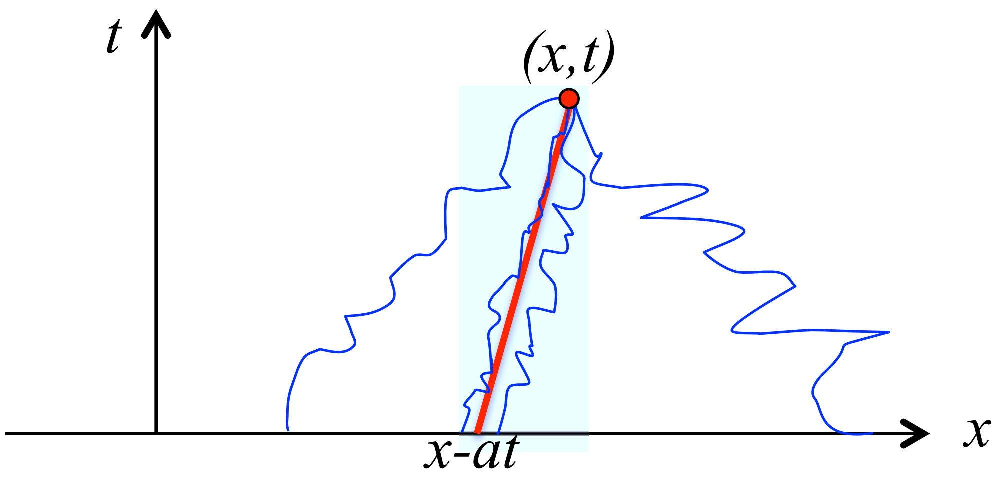
$u(x,t) = u_0(\gamma(0))$, where
$\gamma(s) = x - a(t-s)$: characteristic curve solving ODE

$$d\gamma(s) = a\,ds,\ \gamma(t) = x.$$

$u^\nu(x,t) = E[u_0(\gamma^\nu(0))]$, where
$\gamma^\nu(s) = x - a(t-s) + \sqrt{2}\nu B(t-s)$: stochastic process solving SDE

$$d\gamma^\nu(s) = a\,ds - \sqrt{2}\nu dB(t-s),\ \gamma^\nu(t) = x.$$

$\gamma^\nu \to \gamma$ and $u^\nu \to u$ as $\nu \to 0$ (law of large numbers).



Finite difference method from stochastic viewpoint

$$\frac{u_{m+1}^{k+1} - \frac{u_m^k + u_{m+2}^k}{2}}{\Delta t} + a\frac{u_{m+2}^k - u_m^k}{2\Delta x} = 0, \;\; u_m^0 = u_0(x_m).$$

$$\begin{aligned} u_{m+1}^{k+1} &= \left(\frac{1}{2} + \frac{a}{2}\lambda\right)u_m^k + \left(\frac{1}{2} - \frac{a}{2}\lambda\right)u_{m+2}^k \quad (\lambda := \Delta t/\Delta x) \\ &= \left(\frac{1}{2} + \frac{a}{2}\lambda\right)^2 u_{m-1}^{k-1} + 2\left(\frac{1}{2} + \frac{a}{2}\lambda\right)\left(\frac{1}{2} - \frac{a}{2}\lambda\right)u_m^{k-1} + \left(\frac{1}{2} - \frac{a}{2}\lambda\right)^2 u_{m+2}^{k-1} \\ &= \cdots = \sum_n P(x_n)u_0(x_n) = \sum_\gamma \mu(\gamma)u_0(\gamma^0), \end{aligned}$$

P: binomial distribution on $(2\Delta x)\mathbb{Z}$, if $|a\lambda| < 1$, with

average $= x_{m+1} - at_{k+1}$, variance $= (1 - a^2\lambda^2)t_{l+1}\frac{\Delta x^2}{\Delta t}$,

$\mu(\gamma)$: prob. density of sample path γ of the corresponding random walk.

By hyperbolic scaling limit $\Delta x, \Delta t \to 0$, $\lambda = \Delta t/\Delta x = O(1)$

$P \rightharpoonup \delta(x - at)$, $\gamma \to \gamma(s) = x - a(t - s)$ (law of large numbers).

Thus $u_m^k \to u(x,t)$ as $\Delta x, \Delta t \to 0$ with $\lambda = O(1)$.

Preliminaries

Representation formulas for v and v_x:

$$v(x,t) = \inf_{\gamma \in AC, \gamma(t)=x} \Big[\int_0^t L(\gamma(s), s, \gamma'(s))ds + v^0(\gamma(0)) \Big] + ht.$$

If (x,t): regular point of v (i.e. $\exists\, v_x(x,t)$) and γ^*: minimizer,

$$v_x(x,t) = \int_0^t L_x(\gamma^*(s), s, \gamma^{*\prime}(s))ds + v_x^0(\gamma^*(0)). \tag{1}$$

Each minimizer γ^* is a backward characteristic curve.

Even grid, odd grid and discretization:

$\Delta x, \Delta t > 0$: discretization parameters.

$G_{even} := \{m\Delta x \mid m \in \mathbb{Z}, m = even\}$, $G_{odd} := \{m\Delta x \mid m \in \mathbb{Z}, m = odd\}$.

$$\mathcal{G} := \bigcup_{k \geq 0} \big\{((G_{even})^d \times \{t_{2k}\}) \cup ((G_{odd})^d \times \{t_{2k+1}\})\big\},\ t_k := k\Delta t,$$

$$\tilde{\mathcal{G}} := \bigcup_{k \geq 0} \big\{((G_{odd})^d \times \{t_{2k}\}) \cup ((G_{even})^d \times \{t_{2k+1}\})\big\}.$$

$(x_m, t_k) = (x^1_{m_1}, \ldots, x^d_{m_d}, t_k) \in \mathcal{G}, \tilde{\mathcal{G}}$.

$(x_m, t_k), (x_{m+1}, t_{k+1})$ for points of $\mathcal{G}$,

$(x_{m+1}, t_k), (x_m, t_{k+1})$ for points of $\tilde{\mathcal{G}}$ with $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^d$.

Consider the sets with the standard basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$,

$$B^i_+ := \{\sigma_1 e_1 + \cdots + \sigma_d e_d \mid \sigma_i = +1, \sigma_j = \pm 1,\ j = 1, \ldots, d,\ j \neq i\},$$

$$B^i_- := \{\sigma_1 e_1 + \cdots + \sigma_d e_d \mid \sigma_i = -1, \sigma_j = \pm 1,\ j = 1, \ldots, d,\ j \neq i\},$$

$$B := \{\sigma_1 e_1 + \cdots + \sigma_d e_d \mid \sigma_j = \pm 1,\ j = 1, \ldots, d\} = B^i_+ \cup B^i_-,$$

$$b := \sharp B = 2^d,\ \bar{b} := \sharp B^{\pm}_i = 2^{d-1}.$$

For each $x_m \in (G_{even})^d$,

$\{x_{m+\omega}\}_{\omega \in B} \subset (G_{odd})^d$ forms the d-cube $\mathcal{C}_m$ with the centre x_m,

$\{x_{m+\omega}\}_{\omega \in B^i_\pm} \subset (G_{odd})^d$ form the two sides of $\mathcal{C}_m$, facing each other and orthogonal to e_i.

The same for each $x_m \in (G_{odd})^d$.

Let $v = v^k_{m+\mathbf{1}}$ denote a function $v : \tilde{\mathcal{G}} \ni (x_{m+\mathbf{1}}, t_k) \mapsto v^k_{m+\mathbf{1}} \in \mathbb{R}$.

Define spatial difference derivatives of $v^k_{m+\mathbf{1}}$ on $\mathcal{G}$ as

$$(D^i_x v^k)_m := \left\{ \Big(\overline{b}^{-1} \sum_{\omega \in B^i_+} v^k_{m+\omega} \Big) - \Big(\overline{b}^{-1} \sum_{\omega \in B^i_-} v^k_{m+\omega} \Big) \right\} \frac{1}{2\Delta x}, \quad i = 1, \ldots, d,$$

$$(D_x v^k)_m := \big((D^1_x v^k)_m, \ldots, (D^d_x v^k)_m \big).$$

Define time difference derivatives of $v^k_{m+\mathbf{1}}$ as

$$D_t v^{k+1}_m := \left\{ v^{k+1}_m - b^{-1} \sum_{\omega \in B} v^k_{m+\omega} \right\} \frac{1}{\Delta t}.$$

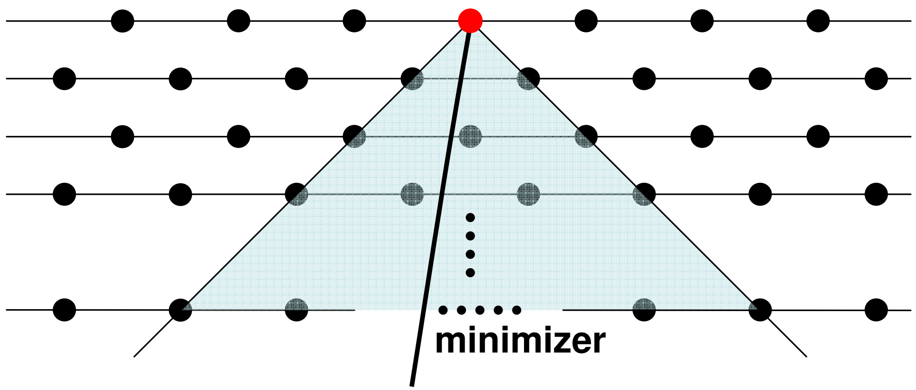
Our discretization of (HJ) is

$$(\mathrm{HJ})_\Delta \begin{cases} D_t v_m^{k+1} + H(x_m, t_k, (D_x v^k)_m) = h \quad \text{in } \tilde{\mathcal{G}}, \\ v_{m+\mathbf{1}}^0 = v^0(x_{m+1}). \end{cases}$$

Note that v_m^{k+1} is unknown and is determined by $\{v_{m+\omega}^k\}_{\omega\in B}$.

The diffusion effect of $(\mathrm{HJ})_\Delta$ at each grid point within Δt is characterized by $\mathcal{C}_m$.

Under hyperbolic scaling $0 < \lambda_0 \le \Delta t/\Delta x$, the propagation speed is finite.



Inhomogeneous controlled random walks in $\tilde{\mathcal{G}}$:

Consider backward random walks γ within $[0, t_{l+1}]$ which start from x_n at t_{l+1} and move by $\omega\Delta x$, $\omega \in B$ in each backward time step Δt:

$$\gamma = \{\gamma^k\}_{k=l',\cdots,l+1}, \quad \gamma^{l+1} = x_n, \quad \gamma^k = \gamma^{k+1} + \omega\Delta x.$$

More precisely, we set the following for each $(x_n, t_{l+1}) \in \tilde{\mathcal{G}}$ and $0 \le l$:

$$X_{k+1}^{l+1,n} := \{x_m \mid \ (x_m, t_{k+1}) \in \tilde{\mathcal{G}}, \ \| x_m - x_n \| \le \| \mathbf{1} \| \ (l-k)\Delta x\}, \ \ k \le l,$$

$$G = G^{l+1,n} := \bigcup_{0 \le k \le l} \left(X_{k+1}^{l+1,n} \times \{t_{k+1}\}\right) \subset \tilde{\mathcal{G}},$$

$$\xi : G \ni (x_m, t_{k+1}) \mapsto \xi_m^{k+1} \in ([-(d\lambda)^{-1}, (d\lambda)^{-1}])^d, \quad \lambda = \Delta t/\Delta x,$$

$$\rho : G \times B \ni (x_m, t_{k+1}; \omega) \mapsto \rho_m^{k+1}(\omega) := b^{-1}(1 - \lambda(\omega \cdot \xi_m^{k+1})) \in [0, 1],$$

$$\gamma : \{0, 1, \ldots, l+1\} \ni k \mapsto \gamma^k \in X_{k+1}^{l+1,n}, \ \gamma^k = \gamma^{k+1} + \omega\Delta x, \ \omega \in B,$$

Ω_n^{l+1} : the family of these γ.

$\{\rho_m^{k+1}(\omega)\}_{\omega \in B}$: a transition probability from (x_m, t_{k+1}) to points belonging to $\{(x_m + \omega\Delta x, t_k)\}_{\omega \in B}$.

ξ: control of γ ($\leftrightarrow$ drift term in SDE).

Define the density of each path $\gamma \in \Omega_n^{l+1}$ as

$$\mu(\gamma) := \prod_{0 \le k \le l} \rho^{k+1}_{m(\gamma^{k+1})}(\omega^{k+1}),$$

where $\omega^{k+1} := (\gamma^k - \gamma^{k+1})/2\Delta x$.

The density $\mu(\cdot) = \mu(\cdot;\xi)$ yields a probability measure of Ω, namely

$$prob(A) = \sum_{\gamma \in A} \mu(\gamma;\xi) \quad \text{for } A \subset \Omega_n^{l+1}.$$

The expectation with respect to this probability measure is denoted by $E_{\mu(\cdot;\xi)}$, i.e., for a random variable $f : \Omega^{l+1,l'} \to \mathbb{R}$

$$E_{\mu(\cdot;\xi)}[f(\gamma)] := \sum_{\gamma \in \Omega^{l+1,l'}} \mu(\gamma;\xi) f(\gamma).$$

Asymptotics of the probability measure of Ω for $\Delta \to 0$ under hyperbolic scaling is studied in Soga [1], [5].

Main Results [5]

Consider the stochastic action functional for each (x_n, t_{l+1})

$$E_n^{l+1}(\xi) := E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1} L(\gamma^k, t_{k-1}, \xi^k_{m(\gamma^k)})\Delta t + v^0(\gamma^0)\Big] + ht_{l+1}.$$

Thm. For each $T>0$, $\exists\,\lambda_1>0$ s.t. if $\lambda = \Delta t/\Delta x < \lambda_1$ then

1. $v_n^{l+1} = \inf_{\xi} E_n^{l+1}(\xi)$.
2. "inf" is attained by ξ^* which is bounded by $(d\lambda_1)^{-1}$.
3. ${\xi^*}_m^{k+1} = H_p(x_m, t_k, c + (D_x v^k)_m)$.
   In particular, $(D_x v^k)_m = L_\xi(x_m, t_k, {\xi^*}_m^{k+1})$ and this is bounded.

Let $\Delta x, \Delta t \to 0$ under hyperbolic scaling $0 < \lambda_0 \le \lambda = \Delta t/\Delta x < \lambda_1$.

v_Δ: linear interpolation of v^k_{m+1},

u_Δ: step function given by $(D_x v^k)_m$,

γ_Δ: linear interpolation of the minimizing random walk starting at (x_n, t_{l+1}) next to a point (x,t).

Thm. For $\Delta = (\Delta x, \Delta t) \to 0$,

1. $v_\Delta(x,t) \to v(x,t) = \inf_\gamma \Big[\int_0^t \{L(\gamma(s), s, \gamma'(s))\} ds + v^0(\gamma(0)) \Big] + ht.$
2. $|v_\Delta(x,t) - v(x,t)| \le \beta_1 \sqrt{\Delta x}$ on $\mathbb{T} \times [0,T]$.
3. $\gamma_\Delta \to \gamma^*$ unif. in probability for each regular point (x,t).
4. If v^0 is semiconcave, then
$$u_\Delta(x,t) \to v_x(x,t) = \int_0^t L_x(\gamma^*(s), s, \gamma^{*\prime}(s)) ds + v^0_x(\gamma^*(0)).$$
5. Except any "small" nbhd. of shocks (non-regular points of v), $u_\Delta \to v_x$ uniformly.

$*$ Semiconcavity assumption can be removed for $d = 1$.

Applications

$d = 1$ with the periodic setting [3], [4]:

- Time global stability of v^k_{m+1} and $(D_x v^k)_m$ with fixed $\Delta x, \Delta t$.
- Long time behaviors of v^k_{m+1} and $(D_x v^k)_m$ for $k \to \infty$.
- Existence of periodic sol. $\bar{v}^k_{m+1}$ and $(D_x \bar{v}^k)_m$ as well as the effective Hamiltonian $\bar{h}_{\Delta}$.
- Numerical methods of classical & weak KAM theory.
- Selection problems of $\mathbb{Z}^2$-periodic viscosity solutions and entropy solutions.

Future works

- Similar results to [3], [4] for $d > 1$.
- Diffusive scaling limit, i.e., $\Delta x, \Delta t \to 0$ with $\Delta x^2/\Delta t = O(1)$.
- Toward system of equations.