type-two polynomial time and restriced lookahead

Bruce Kapron and **Florian Steinberg**

INRIA

July 5, 2018

Oracle Turing machines

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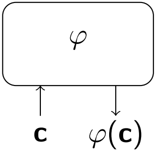

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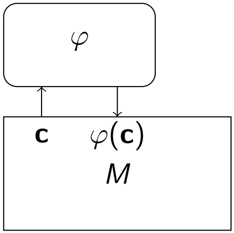

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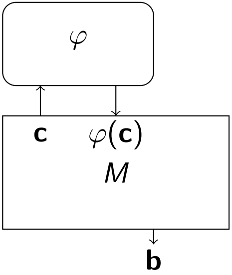

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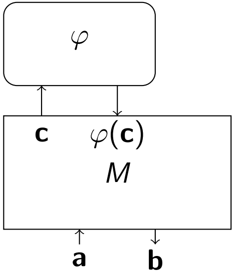

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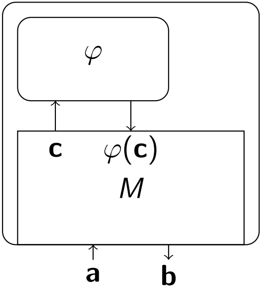

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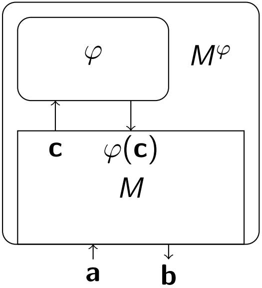

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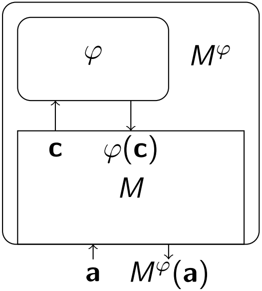

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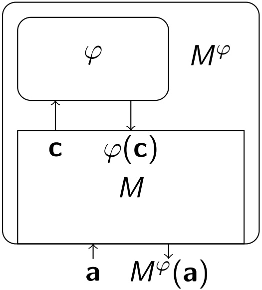

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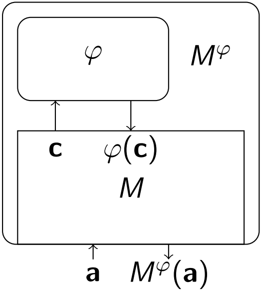

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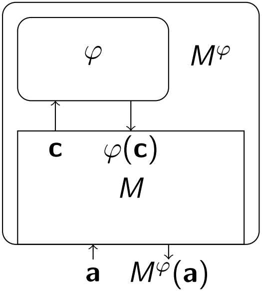

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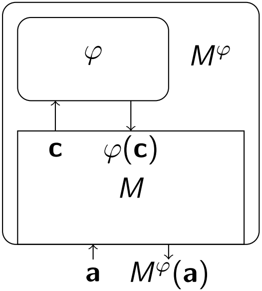

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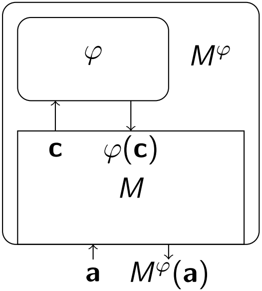

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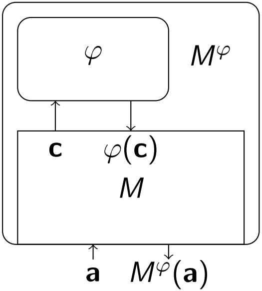

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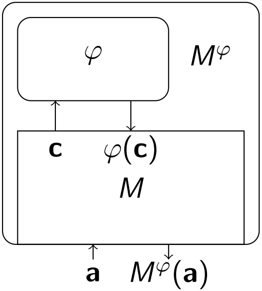

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Second-order polynomials...

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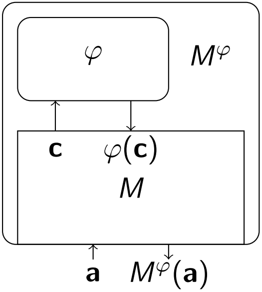

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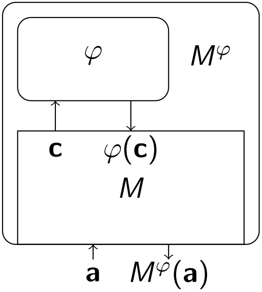

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Corollary

Preserves polynomial-time computability.

Comments

Original definition: bounded recursion scheme.

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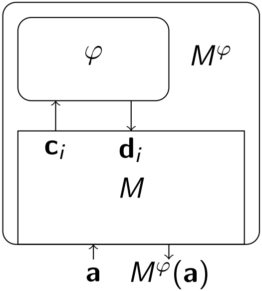
Theorem (Kawamura, S.)

Clocking is impossible.

Oracle polynomial-time



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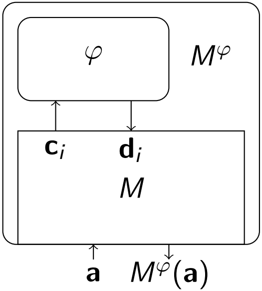

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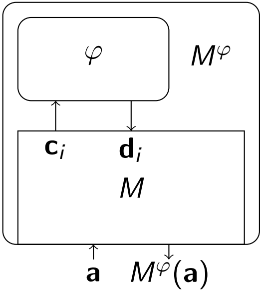

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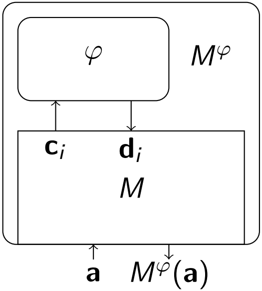

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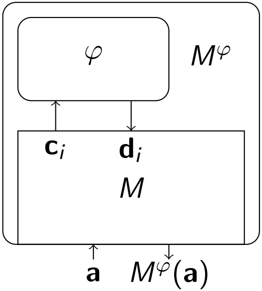

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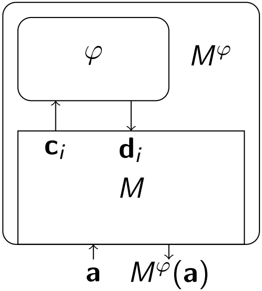

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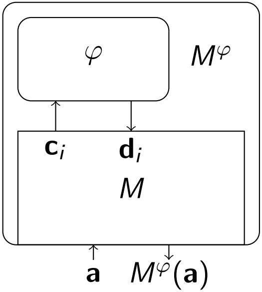

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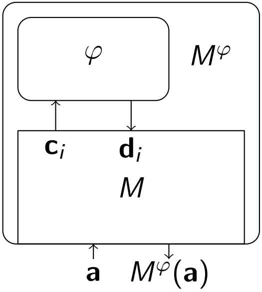

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Theorem

M runs in time $P \rightsquigarrow n \mapsto P(l_n, n)$ is step-count.

Opt but not pt

Example

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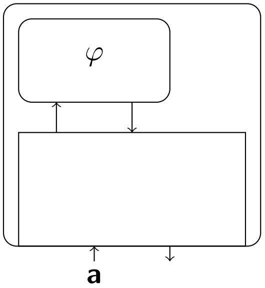

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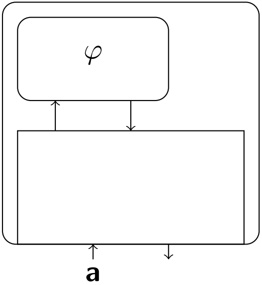

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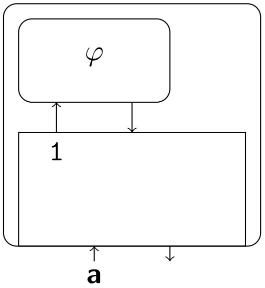

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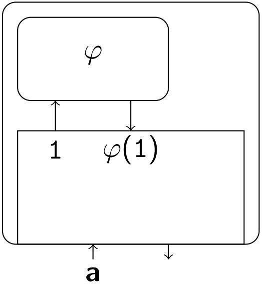

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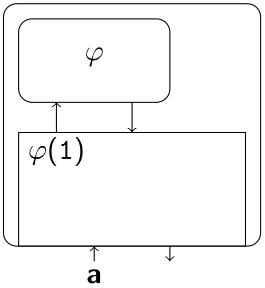

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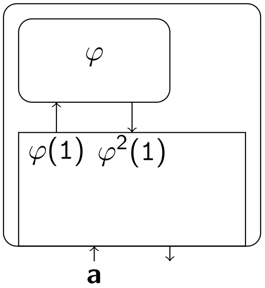

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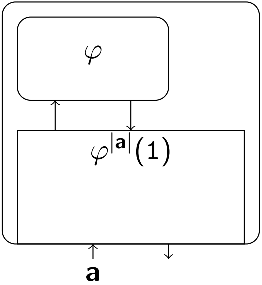

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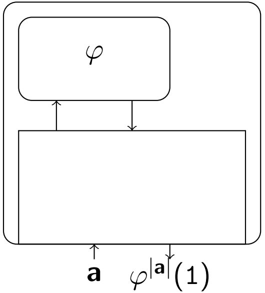

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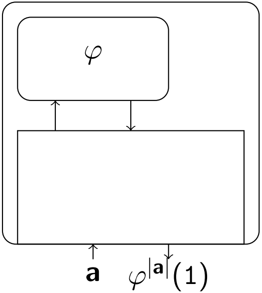

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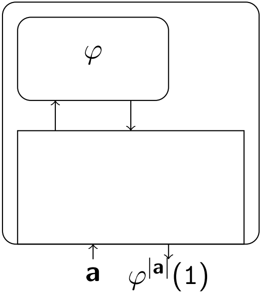

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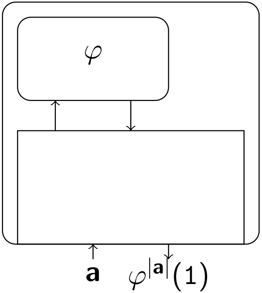
Then $|F(\varphi)| \simeq 2^{|\mathbf{a}|}$.



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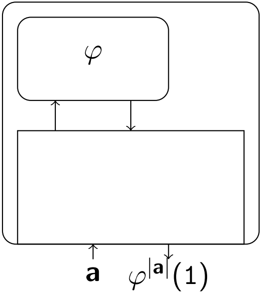

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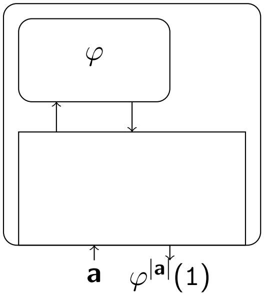
Essentially iteration operator.



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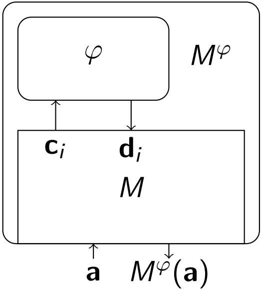
length and lookahead revisions



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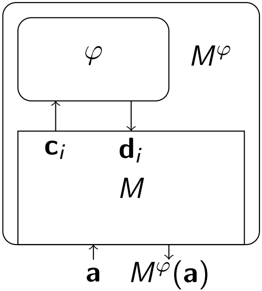

Definition

- finite length revision (spt).

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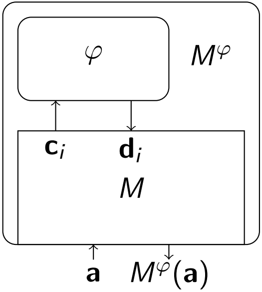

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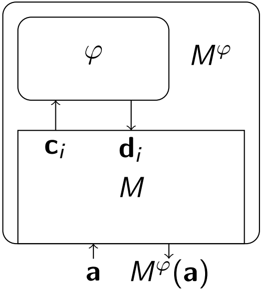

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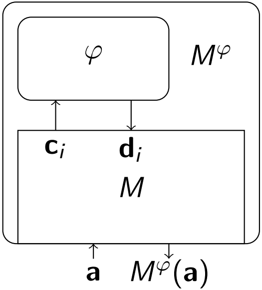

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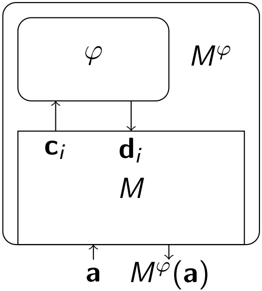

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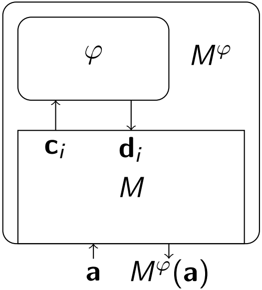
length and lookahead revisions



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- finite length revision (spt).
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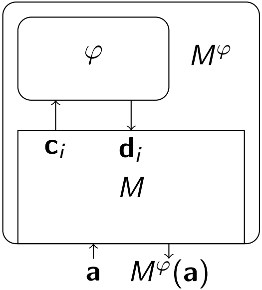

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Examples

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- $F(\varphi)(\mathbf{a}) := \max\{|\varphi(\mathbf{b})| \mid \mathbf{b} \subseteq \mathbf{a}\}$

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Lemma (Kapron, S.)

$\mathcal{R}$ is mpt.

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The lambda closure of mpt are the feasible functionals.

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Thanks!

Thank you for listening!