Vorticity and smoothness in incompressible viscous flows. Boundary value problems.

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Abstract

In these notes we give an overview on some known and new results on sufficient conditions for the regularity of the Navier-Stokes equations in terms of the direction of the vorticity. After recalling some known results we state a new theorem (the proof will appear in a forthcoming paper, in collaboration with L.C. Berselli) which establish that in regular domains Ω the solutions to the evolution Navier-Stokes equations under the slip-type boundary condition (2.15) must be smooth if the direction of the vorticity is 1/2-Hőlder continuous with respect to the space variables.

1 Introduction

In reference [11] it is proved that the solution of the evolution Navier–Stokes equations in the whole of $\mathbf{R}^3$ must be smooth if the direction of the vorticity is Lipschitz continuous with respect to the space variables. In reference [7] the above result is improved by showing that Lipschitz continuity may be replaced by 1/2-Hőlder continuity. We have some evidence that the 1/2 exponent is very difficult to improve. A next step is the extension of the above type of results to boundary value problems. In reference [5] the 1/2-Hőlder continuity sufficient condition is extended to solutions in the half-space $\mathbf{R}^3_+$ under the slip boundary condition. Note that in the half-space (more precisely, in any portion of flat boundary) this condition coincides with the well-known condition (2.15). In a forthcoming paper the author and L.C. Berselli prove that the above 1/2-Hőlder sufficient condition still holds for solutions to the Navier-Stokes equations in any arbitrary regular open set Ω under the boundary condition (2.15).
In these notes we give an overview on the above results.

2 Known and new results

In the sequel Ω denotes a bounded, connected, open set in $\mathbf{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{3,\alpha}$ for some $\alpha \in (0,1)$. We denote by $\underline{n}$ the unit outward normal to Γ. We do not introduce standard notation or notation whose meaning is clear from the context. We denote by $\| \cdot \|_p$ the canonical norm in the Lebesgue space $L^p := L^p(\Omega)$, $1 \leq p \leq \infty$.

$H^k := H^k(\Omega)$, k positive integer, denotes the classical Sobolev space. Scalar and vector function spaces are indicated by the same symbol.

Consider the evolution 3-D Navier–Stokes equations

$$
(2.1) \qquad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = 0 \quad \text{in } \Omega \times [0, +\infty), \\ \\ \nabla \cdot\, u = 0 \quad \text{in } \Omega \times [0, +\infty), \\ \\ u(x, 0) = u_0(x) \quad \text{in } \Omega\,. \end{cases}
$$

It is well known (under suitable boundary conditions if Γ is not empty) that there is at least one *weak solution* in $[0, +\infty)$ of the above problem and, for a suitable $\tau > 0$, a (unique) strong solution in $[0, \tau)$. It is not known, however, whether weak solutions are unique and whether strong solutions are global in time. We are interested in simple conditions on the vorticity ω,

$$\omega(x, t) = \nabla \times u(x, t)\,,$$

that guarantee the regularity of the solution. The following is a typical result (see [**?**]; see also [**?**]). Weak solutions are regular provided that

$$
(2.2) \qquad \omega \in L^p(0, T; L^q) \quad \text{for } \frac{2}{p} + \frac{n}{q} \leq 2, \quad 1 \leq p \leq 2\,.
$$

This result is an extension to values $p \leq 2$ of the classical condition

$$
(2.3) \qquad u \in L^p(0, T; L^s) \quad \text{for } \frac{2}{p} + \frac{n}{s} \leq 1, \quad 2 \leq p < \infty\,.
$$

This type of conditions have an analytical character. On the other hand, in references [11] and [7], a geometrical assumption is considered. Define the direction of the vorticity ξ as

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}$$

and denote by $\theta(x, y, t)$ the angle between the vorticity ω at two distinct points x and y at time t. In reference [11] the authors prove the following result.

Theorem 2.1. *(see [11]). Let be $\Omega = \mathbf{R}^3$ and u be a weak solution of* (2.1) *in $(0, T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. If*

$$
(2.4) \qquad \sin\theta(x, y, t) \leq\, c\, |x - \, y|
$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K, then the solution u is strong in $[0, T]$ and, consequently, is regular.

In [7] the authors improve the above result by showing that

$$
(2.5) \qquad \sin\theta(x, y, t) \leq\, c\, |x - \, y|^{1/2}
$$

is sufficient to guarantee the regularity of weak solutions. More precisely

Theorem 2.2. *(see [7]). Let be* $\Omega = \mathbf{R}^3$ *and* u *be a weak solution of* (2.1) *in* $(0,T)$ *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$. *Assume that for some* $\beta \in [1/2, 1]$ *and* $g \in L^a(0,T; L^b)$, *where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2}, \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right], \tag{2.6}$$

one has

$$\sin\theta(x,y,t) \leq g(t,x)|x-y|^\beta \tag{2.7}$$

in the region where the vorticity at both points x *and* y *is larger than an arbitrary fixed positive constant* K. *Then the solution* u *is strong in* $[0,T]$ *and, consequently, is regular.*
In particular (2.5) *alone is a sufficient condition for regularity.*

In [3] we assume that $\beta \in [0, 1/2]$ and show sufficient condition for the regularity of weak solutions that involves, simultaneously, the magnitude and the direction of the vorticity. More precisely,

Theorem 2.3. *(see [3]). Let* u *be a weak solution of* (2.1) *in* $(0,T)$ *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$. *Let* $\beta \in [0, 1/2]$ *and assume that* (2.21) *holds in the region where the vorticity at both points* x *and* y *is larger than an arbitrary fixed positive constant* K. *Assume, moreover,* (2.18),(2.19). *Then the solution* u *is strong in* $[0,T]$ *and, consequently, is regular. In particular* (2.5) *alone is a sufficient condition for regularity.*

The proof of Theorem 2.3 follows that given in [7].
As remarked in reference [5], in the assumptions made in references [11], [7] and [3] the quantity $\sin\theta(x,y,t)$ can be everywhere replaced by

$$|(\widehat{x-y}, \xi(x))\, Det(\widehat{x-y}, \xi(y), \xi(x))|\,, \tag{2.8}$$

as follows immediately from the proofs. We simply opt for replacing the above quantity by $\sin\theta(x,y,t)$. We use the notation $\widehat{z} = \frac{z}{|z|}$.
Clearly the above quantity can be replaced by any upper bound as, for instance,

$$|\cos\psi(x,y,t)|\, \sin\phi(x,y,t)$$

where $\psi(x,t)$ denotes the angle between $\xi(x,t)$ and $x-y$, and $\phi(x,y,t)$ denotes the angle between $\xi(y,t)$ and the plane generated by $\xi(x,t)$ and $x-y$.

A fundamental open problem remains the improvement of the best exponent β for which the assumption (2.21) guarantees the regularity of the solutions *without* any other additional hypotheses. The proof given in reference [3] formally leads us to believe that the sharpness of the regularity exponent $\beta = 1/2$ corresponds to that of the classical sufficient condition (2.3). Consequently, the above improvement appears quite difficult to obtain.

Another central problem is the extension of the theory to boundary value problems. In [5] it is proved that the above 1/2-Hőlder assumption still remains a sufficient condition for regularity under the Navier, or slip, boundary condition in the half-space $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3 : \, x_3 > 0\}$. Let us introduce the slip boundary condition, see (2.13), in the general case of an open set Ω in $\mathbf{R}^3$. Denote by

$$T = -p\, I + \nu(\nabla u + \nabla u^T)$$

the stress tensor, and set $\underline{t} = T \cdot \underline{n}$. Hence

$$T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \tag{2.9}$$

and

$$t_i = \sum_{k=1}^{n} T_{ik} n_k. \tag{2.10}$$

Also consider the linear operator $\underline{\tau}$,

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}\,, \tag{2.11}$$

the components of which are given by

$$\tau_i(u) = \nu \sum_{k=1}^{n} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k - 2\nu \left[\sum_{k,l=1}^{n} \frac{\partial u_l}{\partial x_k} n_k n_l \right] n_i. \tag{2.12}$$

Note that $\underline{\tau}(u)$ is tangential to the boundary.

The slip boundary condition reads

$$\begin{cases} (u \cdot \underline{n})_{|\Gamma} = 0, \\ \\ \underline{\tau}(u)_{|\Gamma} = 0\,. \end{cases} \tag{2.13}$$

This boundary condition (2.13) was proposed by Navier, see [17]. We point out that this condition, and similar ones, are an appropriate model for many important flow problems. Besides the pioneering mathematical contribution [22] by Solonnikov and Ščadilov, this boundary condition has been considered by many authors. See, for instance, [1], [4], [9], [12], [13], [16], [18], [19], [23] and references therein.

In the half-space the slip boundary condition has the form (2.13)

$$\begin{cases} u_3 = 0, \\ \\ \nu \dfrac{\partial u_j}{\partial x_3} = 0, \quad 1 \leq j \leq 2. \end{cases} \tag{2.14}$$

It is worth noting, and immediate to verify, that in the half space (or, more generally, in any flat portion of the boundary Γ) the above slip boundary condition coincides with another well known boundary condition (see for instance [2] and [10]), namely

$$\begin{cases} (u \cdot n)_{|\Gamma} = 0, \\ \\ (\omega \times n)_{|\Gamma} = 0\,. \end{cases} \tag{2.15}$$

Theorem 2.4. *Let*

$$u_0 \in \{ v \in H^1(\mathbf{R}^3_+) : (\nabla \cdot v = 0 \quad \textit{and} \quad v_3(x_1, x_2, 0) = 0 \} \tag{2.16}$$

and let u be a weak solution of the Navier-Stokes equations (2.1) *where $\Omega = \mathbf{R}^3_+$, endowed with the boundary condition* (2.14)*. Let $\beta \in [0, 1/2]$ and assume that, for almost all $t \in]0, T[$,*

$$\sin \theta(x, y, t) \leq c|x - y|^\beta . \tag{2.17}$$

Moreover, suppose that

$$\omega \in L^2(0, T; L^r), \tag{2.18}$$

where

$$r = \frac{3}{\beta + 1}. \tag{2.19}$$

Then the solution u is strong in $[0, T]$ and, consequently, is regular. In particular the solution is regular if (2.5) *holds. In this case* (2.18) *is superfluous.*

The last claim follows from the fact that weak solutions satisfy (2.18) for $r = 2$.
In a forthcoming paper [8] we extend the above result to arbitrarily regular open sets Ω by considering the extension (2.15) of (2.14). More precisely, we prove the following result.

Theorem 2.5. *Let Ω be a regular bounded open set and*

$$u_0 \in V = \{ v \in H^1(\Omega) : (\nabla \cdot v)_{|\Omega} = 0 \quad \textit{and} \quad (v \cdot n)_{|\Gamma} = 0 \} . \tag{2.20}$$

Let u be a weak solution in $[0, T) \times \Omega$ of the Navier-Stokes equations (2.1) *under the boundary condition* (2.15)*.*

Assume that, for a.a. $t \in (0, T)$ the assumption (2.5) *holds. Then the solution u is strong in $[0, T]$, i.e.,*

$$u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) . \tag{2.21}$$

A fundamental tool in proving the Theorem 2.4 is the use of both the Green and the Neumann functions for $\mathbf{R}^3_+$, as suggested by the fact that in (2.14) the components of the velocity are not mixed. Moreover, localization is not needed. On the contrary, in order to prove the Theorem 2.5 we need to localize the problem near any point x_0 which lies on the boundary itself or even near the boundary. This leads to non trivial problems and to a deep study of the Green function associate to our boundary value problem. A fundamental tool in the proof are the sharp results on Green matrices for general boundary value problems proved by V. Solonnikov in his outstanding works [20] and [21]. See also [15].
Similar ideas have been applied in reference [6] for the non-slip boundary condition

$$u = 0 \quad on \quad \Gamma , \tag{2.22}$$

by appealing to the Green function for Ω. In this case the problems connected to the Green function are easier to treat than in the other cases referred above. The fundamental estimates concerning the non linear term $(\omega \cdot \nabla)\, u \cdot \omega$ are proved. However a new obstacle (due to the specific boundary condition (2.22)) appears, and regularity under the *sole* assumption $\sin\theta(x,y,t) \leq c|x-y|^{1/2}$ (or even $\sin\theta(x,y,t) \leq c|x-y|$) remains an open, challenging, problem.

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