On some qualitative properties of stratified flows

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Abstract. A solution of the Cauchy problem for a system of an exponentially stratified fluid in the gravity field is obtained in the form of singular integrals. If the initial data have a specified smoothness, the solution is written in the form of integrals with weak singularities of the kernels. Both these forms of solutions enable exact L_p estimates to be obtained. We also establish the asymptotic behaviour as $t \to \infty$ and investigate the spectral properties of the corresponding operators.

1. Introduction

The objective of this paper is to study the qualitative properties of the solutions of the system which describes small motions of an exponentially stratified fluid in the homogeneous gravity field, such as existence, uniqueness and L_p-estimates. We obtain the solution in the form of singular integrals, taken in the Cauchy principal value sense, when singularities are removed by a ball, that is, isotropically. If the initial data have a specified smoothness, the solution is written in the form of integrals with weak singularities of the kernels. Both these forms of solutions enable exact L_p-estimates $(p > 1)$ to be obtained. This paper is inspired by the works [6],[7], where similar results were obtained for rotating (not stratified) fluid. The smoothness of the solution of stratified system for the particular case of the intrusion was studied in [8]. The isolated case of uniqueness for stratified fluid in a class of increasing functions was considered in [9].

We consider a system of equations in the form

$$\left\{ \begin{array}{c} \rho_* \frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x_1} = 0 \\ \rho_* \frac{\partial v_2}{\partial t} + \frac{\partial p}{\partial x_2} = 0 \\ \rho_* \frac{\partial v_3}{\partial t} + g\rho + \frac{\partial p}{\partial x_3} = 0 \\ \frac{\partial \rho}{\partial t} - \frac{N^2 \rho_*}{g} v_3 = 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \end{array} \right. \tag{1.1}$$

in the domain $\left\{ x \in R^3, t > 0 \right\}$, where $\overrightarrow{v}(x,t)$ is a velocity field with components v_1, v_2, v_3, the function $p(x,t)$ is the scalar field of the dynamic pressure, $\rho(x,t)$ is the dynamycal density and ρ_*, g, N are positive constants. The equations (1.1) are

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deduced in [1] under the assumption that the function of stationary distribution of density is performed by the function $\rho_* e^{-Nx_3}$.

We consider the Cauchy problem for (1.1):

$$\left\{ \begin{array}{c} \overrightarrow{v}|_{t=0} = \overrightarrow{v}^0(x) \\ \rho|_{t=0} = 0 \end{array} \right. \tag{1.2}$$

For the function $\overrightarrow{v}^0$, we shall add the conditions of the absence of the rotational component in (x_1, x_2)

$$\frac{\partial v_1^0}{\partial x_2} - \frac{\partial v_2^0}{\partial x_1} = 0 \tag{1.3}$$

together with the natural condition

$$div\left(\overrightarrow{v}^0\right) = 0. \tag{1.4}$$

The conditions (1.3) do not restrict the generality and are only necessary in order that the Fourier image of the velocity field be represented as a product of the $\widehat{\overrightarrow{v^0}}(\xi)$ and a function of ξ. If we do not assume (1.3), the Fourier image of the solution will be represented as a linear combination of the Fourier images of the three coordinates of $\widehat{\overrightarrow{v^0}}(\xi)$.

We put $P = \frac{1}{\rho_*}\frac{\partial p}{\partial t}$. We also note that, we can change the scale of the velocity and the density, introducing the modified velocity as $\overrightarrow{v}\rho_*$, and the modified density as $g\rho$. For the qualitative properties of the considered solutions, we also may put $N = 1$. Thus, without loss of generality, we can assume $\rho_* = 1$, $N = 1$, $g = 1$.

2. Construction of solutions

In order to restrict ourselves to convergent integrals, we assume that the initial data have, for example, continuous second derivatives and decrease sufficiently rapidly at infinity together with their derivatives up to the second order. Using the Fourier transform with respect to x , the Laplace transform with respect to t and the conditions (1.3) and (1.4), we obtain the solution of our problem in the form

$$\widehat{\overrightarrow{v}}(\xi,\lambda) = \frac{\lambda|\xi|^2}{\lambda^2|\xi|^2+|\xi'|^2}\widehat{\overrightarrow{v^0}}(\xi), \quad \widehat{\rho}(\xi,\lambda) = \frac{|\xi|^2\widehat{v}_3^0(\xi)}{\left(\lambda^2|\xi|^2+|\xi'|^2\right)}, \quad \widehat{P}(\xi,\lambda) = \frac{i\xi_3\widehat{v}_3^0(\xi)}{\lambda^2|\xi|^2+|\xi'|^2},$$

where $\xi = (\xi_1, \xi_2, \xi_3)$, $|\xi|^2 = \sum\limits_{k=1}^{3}\xi_k^2$, $\left|\xi'\right|^2 = \sum\limits_{k=1}^{2}\xi_k^2$. After an inverse Laplace transform we obtain the solution in the form

$$\begin{aligned} \widehat{\overrightarrow{v}}(\xi,t) &= \widehat{\overrightarrow{v^0}}(\xi)\cos\frac{\left|\xi'\right|}{|\xi|}t, \ \widehat{\rho}(\xi,t) = \widehat{v}_3^0(\xi)\frac{|\xi|}{\left|\xi'\right|}\sin\frac{\left|\xi'\right|}{|\xi|}t, \\ \widehat{P}(\xi,t) &= \widehat{v}_3^0(\xi)\frac{i\xi_3}{|\xi|\left|\xi'\right|}\sin\frac{\left|\xi'\right|}{|\xi|}t\ . \end{aligned} \tag{2.1}$$

We now find the inverse Fourier transform of the required solution. We first obtain the solution in the form of integrals with weak singularities of the kernels. For this we seek a vector $\overrightarrow{v}(x,t)$ expressed in terms of the Laplace operator and a function $P(x,t)$ expressed in terms of the first derivatives of the initial data. As we see from

(2.1), it is sufficient to calculate only two kernels

$$K_1(x-y,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i(\xi,x-y)}\frac{1}{|\xi|^2}\cos\frac{|\xi'|}{|\xi|}t\,d\xi\ , \tag{2.2}$$

$$K_2(x-y,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i(\xi,x-y)}\frac{1}{|\xi|\,|\xi'|}\sin\frac{|\xi'|}{|\xi|}t\,d\xi\ . \tag{2.3}$$

We note that K_2 is the primitive of K_1 with respect to t , and it is therefore sufficient to calculate only one of these integrals.

An integral of type (2.3) is calculated in [2] by means of Sonine´s formulas for Bessel functions and is given by

$$K_2(x-y,t) = \frac{1}{4\pi}\frac{1}{r}\int_0^t J_0(t-\tau)\,J_0\left(\frac{\rho\tau}{r}\right)d\tau\ , \tag{2.4}$$

where $\rho^2 = (x_3-y_3)^2$, $r^2 = \sum\limits_{k=1}^{3}(x_k-y_k)^2$, and J_0 is the Bessel function of order zero. Therefore

$$K_1(x-y,t) = \frac{1}{4\pi}\frac{1}{r}J_0\left(\frac{\rho t}{r}\right) - \frac{1}{4\pi}\frac{1}{r}\int_0^t J_1(t-\tau)\,J_0\left(\frac{\rho\tau}{r}\right)d\tau. \tag{2.5}$$

If we now use (2.4) and (2.5), the solution of the Cauchy problem for system (1) can be written in the form

$$\overrightarrow{v}(x,t) = \iiint_{R^3}\left\{-\Delta\overrightarrow{v}^0(y)\,K_1(x-y,t)\right\}dy\ , \tag{2.6}$$

$$P(x,t) = \iiint_{R^3}\left\{\frac{\partial v_3^0}{\partial y_3}K_2(x-y,t)\right\}dy\ . \tag{2.7}$$

These are basic formulas defining our solution with weak singularities of the kernels. To obtain the exact estimates in L_p-norms it is helpful to rewrite these formulas in another form with strong singularities of the kernels.

In (2.6) and (2.7) we integrate by parts in order that the solution be expressed in terms of the initial functions rather than their derivatives. This is easily done in (2.7) for $P(x,t)$, because after one integration by parts the kernels will still have an integrable singularity. However, in (2.6), after the second integration by parts we shall have a strong (locally non-integrable) singularity.

We remove from our space the ball K_ε of radius ε with boundary S_ε and center at (x_1,x_2,x_3) and denote the rest of the domain by Ω_ε. The component $v_1(x,t)$ of $\overrightarrow{v}(x,t)$ is given by

$$v_1(x,t) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\iiint_{\Omega_\varepsilon}\left(-\Delta v_1^0\right)\left[\frac{1}{r}J_0\left(\frac{\rho t}{r}\right) - \frac{1}{r}\int_0^t J_1(t-\tau)\,J_0\left(\frac{\rho\tau}{r}\right)d\tau\right]dy =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi}\iiint_{\Omega_\varepsilon}\left(-v_1^0\right)\left[\Delta\left(\frac{1}{r}J_0\left(\frac{\rho t}{r}\right)\right) - \int_0^t J_1(t-\tau)\,\Delta\left(\frac{1}{r}J_0\left(\frac{\rho\tau}{r}\right)\right)d\tau\right]dy+$$

$$+\lim_{\varepsilon\to 0}\frac{1}{4\pi}\iint\limits_{S_\varepsilon}\left(-v_1^0\right)\left[\frac{\partial}{\partial n}\left(\frac{1}{r}J_0\left(\frac{\rho t}{r}\right)\right)-\int\limits_0^t J_1\left(t-\tau\right)\frac{\partial}{\partial n}\left(\frac{1}{r}J_0\left(\frac{\rho\tau}{r}\right)\right)d\tau\right]ds,$$

where n is the normal to the surface S_ε which is interior with respect to Ω_ε.

We calculate the principal value of the integral over the surface of the sphere and denote it by I . Also we remark that on the surface of the sphere we have -$\frac{\partial}{\partial n}\left(\frac{1}{r}J_0\left(\frac{\rho t}{r}\right)\right)=\frac{1}{r^2}J_0\left(\frac{\rho t}{r}\right)$. We transform to spherical coordinates on the surface of the sphere of radius ε so that

$$I=\frac{1}{4\pi}\lim_{\varepsilon\to 0}v_1^0\left(x\right)\iint\limits_{S_\varepsilon}\left[\frac{1}{r^2}J_0\left(\frac{\rho t}{r}\right)-\int\limits_0^t J_1\left(t-\tau\right)\frac{1}{r^2}J_0\left(\frac{\rho\tau}{r}\right)d\tau\right]ds=$$

$$=v_1^0\left(x\right)\int\limits_0^{\pi/2}\left[J_0\left(t\cos\theta\right)\sin\theta-\left(\int\limits_0^t J_1\left(t-\tau\right)J_0\left(\tau\cos\theta\right)d\tau\right)\sin\theta\right]d\theta=v_1^0\left(x\right)\Phi\left(t\right),$$

where

$$\Phi\left(t\right)=\frac{1}{t}\int\limits_0^t J_0\left(\eta\right)d\eta-\int\limits_0^t J_1\left(t-\tau\right)\left(\frac{1}{\tau}\int\limits_0^\tau J_0\left(\eta\right)d\eta\right)d\tau. \tag{2.8}$$

On carrying out exactly similar argumants for the other components of $\overrightarrow{v}\left(x,t\right)$ and $P\left(x,t\right)$, we obtain

$$\overrightarrow{v}\left(x,t\right)=\overrightarrow{v}^0\left(x\right)\Phi\left(t\right)+$$

(2.9)

$$+\frac{1}{4\pi}V.p.\iiint\overrightarrow{v}^0\left(y\right)\left(\Delta\left(-\frac{1}{r}J_0\left(\frac{\rho t}{r}\right)\right)+\int\limits_0^t J_1\left(t-\tau\right)\Delta\left(\frac{1}{r}J_0\left(\frac{\rho\tau}{r}\right)\right)d\tau\right)dy,$$

where the integrals are calculated in the principal value sense over the sphere, and the function $\Phi\left(t\right)$ is given by (2.8). For $P\left(x,t\right)$ we obtain

$$P\left(x,t\right)=\frac{1}{4\pi}\iiint v_3^0\left(y\right)\frac{\partial}{\partial y_3}\left(-\frac{1}{r}\int\limits_0^t J_0\left(t-\tau\right)J_0\left(\frac{\rho\tau}{r}\right)d\tau\right)dy, \tag{2.10}$$

where the kernels have integrable singilarities.

In what follows we shall show that the function $\overrightarrow{v}\left(x,t\right)$ defined by (2.9) is a unique solution of the Cauchy problem and that $P\left(x,t\right)$ is defined by (2.10) to within a term depending on t (since the initial Cauchy data for it were not given).

3. L_p-estimates

We shall show that the kernels which are used in writing out the solution and its derivatives satisfy the conditions of the Calderón-Zygmund Theorem [3]. We write (2.9) in the form of convolution:

$$\overrightarrow{v}\left(x,t\right)=\overrightarrow{v}^0\left(x\right)\Phi\left(t\right)+\left(\overrightarrow{v}^0*\Gamma\right)_{R^3}, \tag{3.1}$$

$$\left(\overrightarrow{v}^0 * \Gamma\right)_{R^3} = \iiint\limits_{R^3} \overrightarrow{v}^0 (y)\,\Gamma(x-y,t)\,dy, \tag{3.2}$$

$$\Gamma(x,t) = G(x,t) - \int\limits_0^t J_1(t-\tau)\,G(x,\tau)\,d\tau, \tag{3.3}$$

where the infinite triple integrals are calculated in the sense of principal value, and from (2.9), after the corresponding differentiation, we obtain

$$G(x,t) = \frac{1}{4\pi}\left[\frac{t^2\left(x_1^2+x_2^2\right)}{r^5} J_0\left(\frac{\rho t}{r}\right) + \frac{t}{r^3}\left(\frac{\rho}{r}+\frac{r}{\rho}\right) J_0'\left(\frac{\rho t}{r}\right)\right]. \tag{3.4}$$

It is easy to see that Γ is infinitely differentiable function of t since a singularity in the space x does not increase on differentiation with respect to t. We examine the properties of the kernel Γ for any finite $t : 0 \le t \le T < \infty$. Let us observe that the function Γ satisfies the following three conditions.

(1) Γ is a homogeneous function of x of degree -3 . The proof is obvious from nothing that Bessel functions of the argument $\rho t/r$ are homogeneous functions of degree zero.

(2) Γ may be put in the form $\Gamma(x,t) = \frac{\widetilde{\Omega}(x,t)}{r^3}$, where

$$\widetilde{\Omega}(x,t) = \Omega(x,t) - \int\limits_0^t J_1(t-\tau)\,\Omega(x,\tau)\,d\tau,$$

$$\Omega(x,t) = \frac{1}{4\pi}\left[\frac{t^2\left(x_1^2+x_2^2\right)}{r^2} J_0\left(\frac{\rho t}{r}\right) + \frac{t\left(r^2+\rho^2\right)}{r\rho} J_0'\left(\frac{\rho t}{r}\right)\right].$$

(3) The integrals of $\widetilde{\Omega}(x,t)$ over the unit sphere are zero.

Indeed, transforming to polar coordinates on the unit sphere and using the change of variables $\cos\theta = z$, we obtain

$$\begin{aligned}
\iint\limits_{r=1} \Omega ds &= \int\limits_0^{\pi/2} \left\{t^2 \sin^3\theta J_0(t\cos\theta) + t\sin\theta\left[\cos\theta + \frac{1}{\cos\theta}\right] J_0'(t\cos\theta)\right\} d\theta = \\
&= \int\limits_0^1 \left\{t^2\left(1-z^2\right) J_0(tz) + t\left[z+\frac{1}{z}\right] J_0'(tz)\right\} dz = \\
&= \int\limits_0^1 t^2\left(1-z^2\right) J_0(tz)\,dz - \int\limits_0^1 t^2\left(1+z^2\right)\left(J_0(tz) + J_0''(tz)\right) dz = \\
&= \int\limits_0^1 J_0(tz)\left(-2t^2z^2\right) dz - \int\limits_0^1 t^2\left(1+z^2\right) J_0''(tz)\,dz. \qquad (3.5)
\end{aligned}$$

In (19) we used the Bessel equation $J_0''(y)+\frac{J_0'(y)}{y}+J_0(y)=0$. Integrating by parts and using the Bessel equation for the first integral in (3.5), we obtain

$$-\int_0^1 2t^2z^2J_0(tz)\,dz=2\int_0^1 t^2z^2J_0''(tz)\,dz+2\int_0^1 tzJ_0'(tz)\,dz=2tJ_0'(t)-2\int_0^1 tzJ_0'(tz)\,dz.$$

For the second integral in (3.5), we integrate by parts and obtain

$$\int_0^1 t^2\left(1+z^2\right)J_0''(tz)\,dz=2tJ_0'(t)-2\int_0^1 tzJ_0'(tz)\,dz.$$

Finally, summing up the two last results, we have $\iint\limits_{r=1}\Omega ds=0$.

Thus the three conditions of the Calderón-Zygmund Theorem [3] are satisfied, and we therefore have the following estimate for the vector $\overrightarrow{v}(x,t)$ in the L_p-norm, $1<p<\infty$, in the layer $E_4^T=\{-\infty<x_i<+\infty, 0\le t\le T\}$ (and also on each cross-section t=const):

$$\|\overrightarrow{v}\|_{L_p\left(E_4^T\right)}\le C(p,T)\left\|\overrightarrow{v}^0\right\|_{L_p(R^3)}, \tag{3.6}$$

where C depends only on p and T. We observe that, in differentiation with respect to t, the properties 1), 2) and 3) for the kernel Γ in (3.1)-(3.4) are preserved. In this way, for any derivative with respect to t we shall have

$$\left\|D_t^k\overrightarrow{v}\right\|_{L_p\left(E_4^T\right)}\le C(p,T)\left\|\overrightarrow{v}^0\right\|_{L_p(R^3)}. \tag{3.7}$$

If we denote the lth order derivative with respect to x_1,x_2,x_3 by D_x^l (that is, $D_x^l=\frac{\partial^l}{\partial x_1^{l_1}\partial x_2^{l_2}\partial x_3^{l_3}}$), then we can see that the convolution (3.2) possesses the property

$$D_x^l\left(\overrightarrow{v}^0*\Gamma\right)_{R^3}=\left(D_x^l\overrightarrow{v}^0*\Gamma\right)_{R^3}. \tag{3.8}$$

Thus, on accout of (3.7) and (3.8) we have the following expression for the derivatives of $\overrightarrow{v}(x,t)$:

$$D_t^kD_x^l\overrightarrow{v}(x,t)=D_x^l\overrightarrow{v}^0(x)\,D_t^k\Phi(t)+\left(D_x^l\overrightarrow{v}^0*D_t^k\Gamma\right)_{R^3}. \tag{3.9}$$

Bearing in mind (3.9) and the property of the kernel (3.8), we have the following estimate for the derivatives of $\overrightarrow{v}(x,t)$:

$$\left\|D_t^kD_x^l\overrightarrow{v}\right\|_{L_p\left(E_4^T\right)}\le C(p,T)\left\|D_x^l\overrightarrow{v}^0\right\|_{L_p(R^3)}$$

Making similar calculations for ∇P , we obtain the estimates

$$\|\nabla P\|_{L_p\left(E_4^T\right)}\le C(p,T)\left\|\overrightarrow{v}^0\right\|_{L_p(R^3)}, \tag{3.10}$$

$$\left\|D_t^kD_x^l\nabla P\right\|_{L_p\left(E_4^T\right)}\le C(p,T)\left\|D_x^l\overrightarrow{v}^0\right\|_{L_p(R^3)}. \tag{3.11}$$

REMARK 3.1. A solution of the Cauchy problem for system (1.1) has the following property: as it is seen from (3.10) and (3.12), the smoothness of the solution with respect to t does not depend on the smoothness of the initial conditions.

If we denote by $W_{p,t,x}^{k,l}\left(E_4^T\right)$ the Sobolev space having k derivatives with respect to t and l derivatives with respect to x which are pth power summable, then we have proved the following theorem.

THEOREM 3.2. *If the initial data satisfy* $\overrightarrow{v}^0(x) \in W_p^l(R^3)$ *and if* $\overrightarrow{v}(x,t)$, $P(x,t)$ *is a solution of the problem (1.1)-(1.4) for which the norms given below are finite, then the following estimates will hold:*

$$\|\overrightarrow{v}\|_{W_{p,t,x}^{k,l}(E_4^T)} \le C_1(p,T)\left\|\overrightarrow{v}^0\right\|_{W_p^l(R^3)}, \tag{3.12}$$

$$\|\nabla P\|_{W_{p,t,x}^{k,l}(E_4^T)} \le C_2(p,T)\left\|\overrightarrow{v}^0\right\|_{W_p^l(R^3)}, \tag{3.13}$$

where the constants C_i *depend only on* p *and, in general, on* T $(0 \le t \le T < \infty)$, k *and* l.

Basing ourselves on Theorem 3.1 and the explicit formulas obtained for the solution and using the closure in the corresponding spaces, we obtain the following existence theorem.

THEOREM 3.3. *If the initial data* $\overrightarrow{v}^0(x)$ *belong to* $L_2(R^3)$, *then there exists a solution of system (1.1) in the class*

$$\overrightarrow{v}, \frac{\partial \overrightarrow{v}}{\partial t}, \frac{\partial^2 \overrightarrow{v}}{\partial t^2}, \nabla P \in L_2\left(E_4^T\right),$$

whereby the first four equations of (1) are satisfied almost everywhere and the last equation, in the generalized sense

$$\iiint\limits_{R^3} (\overrightarrow{v}, \nabla\varphi)\, dx = 0$$

for any infinitely differentiable function φ. *If, however,* $\overrightarrow{v}^0(x) \in W_p^l(R^3)$, *the solution obtained will belong to the space* $W_{p,t,x}^{k,l}\left(E_4^T\right)$.

The following uniqueness theorem is also valid.

THEOREM 3.4. *The solution* $\overrightarrow{v}(x,t)$ *of problem (1.1)-(1.4) defined by (2.9) is unique in* L_2, *and* $P(x,t)$ *is defined to within a function depending on* t. *In this case* ∇P *is also determined uniquely in* L_2.

Theorem 3.3 follows from the usual power inequalities obtained by multiplying the first three equations of (1.1) by $\overrightarrow{v}$, $\frac{\partial \overrightarrow{v}}{\partial t}$, $\frac{\partial^2 \overrightarrow{v}}{\partial t^2}$ or ∇P respectively, and integrating by parts. The inequalities are of the form

$$\|\overrightarrow{v}\|_{L_2(E_4^T)} + \left\|\frac{\partial \overrightarrow{v}}{\partial t}\right\|_{L_2(E_4^T)} + \left\|\frac{\partial^2 \overrightarrow{v}}{\partial t^2}\right\|_{L_2(E_4^T)} + \|\nabla P\|_{L_2(E_4^T)} \le C(T)\left\|\overrightarrow{v}^0\right\|_{L_2(R^3)}.$$

4. Asymptotic behavior as $t \to \infty$

THEOREM 4.1. *Let the initial data be finite, and let* $\overrightarrow{v}_0(x) \in C^\infty(R^3)$. *Then the solution* $\overrightarrow{v}(x,t)$ *of the Cauchy problem (1.1)-(1.4) decreases as* $\frac{1}{\sqrt{t}}$ *for* $t \to \infty$.

REMARK 4.2. In fact, in the proof of Theorem 4.1 we require only that $\overrightarrow{v}_0(x) \in C^3(R^3)$, and all the derivatives up to 4th order decrease sufficiently rapidly as $|x| \to \infty$; it is sufficient to require that when multiplied by $|x|^2$ they are integrable over R^3.

Proof.

We write the solution of the Cauchy problem in the form

$$\overrightarrow{v}(x,t)=\iiint\limits_{R^3}\left\{-\Delta\overrightarrow{v}^0(y)\,K_1(x-y,t)\,dy\right\}.$$

It is easy to see that it is sufficient to study the asymptotics of the kernel K_1. We express the kernel K_1 as

$$K_1(x,t)=\frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty}e^{ix_3\xi_3}\iint\limits_{R^2}e^{i(x',\xi')}\frac{\cos\left(\frac{|\xi'|t}{\sqrt{|\xi'|^2+\xi_3^2}}\right)}{|\xi'|^2+\xi_3^2}d\xi' d\xi_3=$$

$$=\frac{1}{2\pi^2}\int\limits_0^{\infty}\int\limits_0^{\infty}\frac{R}{R^2+\xi_3^2}J_0\left(|x'|\,R\right)\cos\left(\frac{Rt}{\sqrt{R^2+\xi_3^2}}\right)\cos\left(x_3\xi_3\right)d\xi_3 dR=$$

$$=\frac{1}{2\pi^2\,|x|}\int\limits_{\lambda}^{1}\frac{w}{\sqrt{1-w^2}}\frac{\cos\left(tw\right)}{\sqrt{w^2-\lambda^2}}dw,\qquad \lambda=\frac{|x_3|}{|x|}.$$

The asymptotics of the above integral as $t\to\infty$ is well-known and can be obtained by the stationary phase method. As it can be seen in [10], the main term of the asymptotic expansion has the form

$$K_1(x,t)=\sqrt{\frac{\pi}{2\lambda\left(1-\lambda^2\right)}}\frac{\cos\left(t\lambda-\frac{\pi}{4}\right)}{2\pi^2\,|x|\,\sqrt{t}}+O\left(t^{-1}\right). \tag{4.1}$$

Thus, the theorem is proved.

Remark 4.3. For $\lambda=0$ we have $K_1(x,t)=\frac{1}{2\pi^2|x|}\int\limits_0^1\frac{\cos(tw)}{\sqrt{1-w^2}}dw$. From the known Lebesgue theorem on the decay of the Fourier coefficients of an integrable function (see, for example, [11]), we can easily obtain the extension of Theorem 4.1 for this case.

Remark 4.4. For $\lambda=1$ we use (2.4) and the relations $K_1=\frac{\partial K_2}{\partial t}$ and $\int\limits_0^t J_0(t-\tau)\,J_0(\tau)\,d\tau=\sin t$, which allows us to represent the kernel K_1 in terms of elementary functions: $K_1(x,t)=\frac{\cos t}{4\pi|x|}$, $\lambda=1$. The last relation means that on the vertical axis $(\lambda=1)$, the solution acts as a stationary wave with no limit for $t\to\infty$.

Remark 4.5. Summing up the results obtained for the solution of the Cauchy problem as $t\to\infty$, we may conclude that the solution reveals its irregular, non-uniform character: it tends to zero as a stationary wave with vanishing amplitude for $\lambda=0$, it is a stationary wave which has no limit for $\lambda=1$, and it represents a remarkable wave process for $0<\lambda<1$, as it can be seen from (4.1): the equiphase surfaces of the wave (wave peaks), are described by the relation $\lambda=\frac{|x_3|}{|x|}=\frac{Const}{t}$ and are represented by conic surfaces with the vertex in the origin and the vertical axis which increase their opening with the growth of t, approaching the plane $x_3 = 0$. This geometric situation explains the lack of limit of the solution as t$\to\infty$ for $\lambda = 1$ (on the vertical axis).

REMARK 4.6. The solution of the Cauchy problem is closely related to the function

$$V = \frac{1}{r} J_0 \left(\frac{\rho}{r} t \right) = \frac{1}{r} J_0 \left(t\cos\theta \right).$$

Let us consider the behaviour of the function V as a function of t . We consider a sphere of a constant radius, on which, for every t, the function V depends only on the polar angle θ. The argument of the Bessel function on the sphere changes from 0 to t. With t growing, we will have more and more waves generated by maxima and minima of the Bessel function, all of them situated between the pole and the equator of the sphere. The waves will appear on the pole and then will move towards the equator, accumulating but not disappearing. Thus large waves will generate more and more short ones.

5. Spectral properties

Let us assume $g = 1$ and $\rho_* = 1$, and write the system (1.1) in the following form:

$$\left\{ \begin{array}{c} \frac{\partial^2 \vec{v}}{\partial t^2} + \nabla P + \vec{e}_3 N^2 v_3 = 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \end{array} \right. . \tag{5.1}$$

First, let us analyze certain mathematical similarity of the system (5.1) and the system which describes rotational motions of incompressible fluid over the vertical axis ($\vec{\omega} = (0, 0, \omega)$):

$$\left\{ \begin{array}{c} \frac{\partial^2 \vec{v}}{\partial t^2} + \vec{\omega} \times \vec{v} + \nabla p = 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \end{array} \right. .$$

Particularly, we would like to compare the scalar form of the two systems

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} \right) + N^2 \left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} \right) &= 0, \\ \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} \right) + \omega^2 \frac{\partial^2 \Phi}{\partial x_3^2} &= 0, \end{aligned}$$

and their corresponding singular solutions ([12]):

$$\mathcal{E}(x,t) = \frac{1}{4\pi \left| x_3 \right|} \int_0^{\frac{Nt \left| x_3 \right|}{\left| x \right|}} J_0 \left(\alpha \right) d\alpha \ ,$$

$$\mathcal{E}(x,t) = \frac{1}{4\pi \left| \overline{x} \right|} \int_0^{\frac{\omega t \left| \overline{x} \right|}{\left| x \right|}} J_0 \left(\alpha \right) d\alpha \ .$$

This mathematical analogy between gravitational and rotational waves, may lead to the corresponding analogy in spectral properties.

In [13] we proved that the essential spectrum of normal vibrations generated by rotational inner waves, is the interval of the real axis $[-\omega,\omega]$, so, it seems appropriate to express the conjecture that the operators generated by (5.1) should possess spectral properties, analogous to the rotational system, namely, the essential spectrum of such operators should be the interval $[-N,N]$. Here we prove that this conjecture is true.

Let Ω be a bounded domain in R^3 and let us consider the boundary conditions $\overrightarrow{v}\cdot\overrightarrow{n}|_{\partial\Omega}=0$ for the system (5.1).

It is proved in [14] that there is valid the decomposition $L_2(\Omega) = J_2(\Omega)\oplus G_2(\Omega)$, so we can reduce (5.1) to the problem

$$\left\{\begin{array}{c} \frac{\partial^2\overrightarrow{v}}{\partial t^2}+N^2B\overrightarrow{u}=0 \\ \overrightarrow{v}\in J_2(\Omega) \end{array}\right. , \tag{5.2}$$

where

$$B\overrightarrow{v}=P\left\{v_3\overrightarrow{e_3}\right\},$$

P is the operator of the orthogonal projection of $L_2(\Omega)$ onto $J_2(\Omega)$, $D(B)=J_2(\Omega)$, and $J_2(\Omega)$ as a closure of $J_0(\Omega)$ in the norm of $L_2(\Omega)$, being $J_0(\Omega)$ be the following space of solenoidal fields :

$$J_0(\Omega)=\left\{\overrightarrow{v}(x)\ :\ \overrightarrow{v}(x)\in C^1(\Omega)\ ,\ div\,\overrightarrow{v}=0,\ \overrightarrow{v}\cdot\overrightarrow{n}|_{\partial\Omega}=0\ \right\},$$

and $G_2(\Omega)$ being the space of potential fields in $L_2(\Omega)$:

$$G_2(\Omega)=\left\{\overrightarrow{v}(x)\in L_2(\Omega)\ :\ \overrightarrow{v}(x)=\nabla\psi,\ \psi\in W_2^1(\Omega)\right\}.$$

For the system (5.2) we consider the problem of normal vibrations

$$\begin{array}{rcl} \overrightarrow{v}(x,t) & = & \overrightarrow{u}(x)\,e^{i\lambda t}\ ,u\in J_2(\Omega) \\ P(x,t) & = & q(x)e^{i\lambda t},q\in W_2^1(\Omega). \end{array}$$

Thus, the system (5.2) can be written in spectral form

$$\left\{\begin{array}{c} \lambda^2\overrightarrow{u}-N^2B\overrightarrow{u}=0 \\ \overrightarrow{u}\in J_2(\Omega) \end{array}\right. .$$

Let us note that, if q is the solution of the system

$$\left\{\begin{array}{c} -\lambda^2u_1+\frac{\partial q}{\partial x_1}=0 \\ -\lambda^2u_2+\frac{\partial q}{\partial x_2}=0 \\ \left(-\lambda^2+N^2\right)u_3+\frac{\partial q}{\partial x_3}=0 \\ \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0 \end{array}\right. , \tag{5.3}$$

then q satisfies the equation

$$\frac{\partial^2 q}{\partial x_1^2}+\frac{\partial^2 q}{\partial x_2^2}+\frac{\partial^2 q}{\partial x_3^2}=-div\left(N^2u_3\overrightarrow{e}_3\right),$$

which implies

$$div\left(N^2u_3\overrightarrow{e}_3+grad\ q\right)=0.$$

Thus, the projection operator B obtains its explicit form as

$$N^2B\overrightarrow{u}=N^2u_3\overrightarrow{e}_3+grad\ q.$$

Our aim now is to investigate the spectrum of the operator B.

We recall that the *essential* spectrum is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity ([15], [16]). We shall use the following criterion which is attributed to Weyl ([15], [16]) : A necessary and sufficient condition that a real finite value μ be a point of the essential spectrum of a self-adjoint operator B is that there exist a sequence of elements $x_n \in D\,(B)$ such that

$$\|x_n\| = 1 \quad , \quad x_n \rightharpoonup 0 \quad , \quad \|(B - \mu I)\, x_n\| \to 0\ . \tag{5.4}$$

LEMMA 5.1. *B is a positive self-adjoint operator in $J_2\,(\Omega)$.*

LEMMA 5.2. *$\lambda = 0$ is an eigenvalue of infinite multiplicity for B.*

The proof of Lemma 5.1 is based on the fact that for bounded operators the property of self-adjointness follows from the symmetry, and the symmetry for B is a consequence of the permutability of the operators of projection and inner product in $L_2\,(\Omega)$.

The proof of Lemma 5.2 follows from the property that the kernel of B is the subspace of $J_2\,(\Omega)$ with trivial third component.

THEOREM 5.3. *The essential spectrum of the operator N^2B is the interval of the real axis $[-N, N]$. Moreover, the points 0, $\pm N$ are eigenvalues of infinite multiplicity.*

We shall draw the general idea of the proof.

Let us denote $\lambda^2 = \mu$, $\mu \neq 0$. Then, the system (5.3) can be written in the matrix form:

$$\begin{pmatrix} -\mu & 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & -\mu & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & N^2 - \mu & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.5}$$

One can easily see that the main symbol of the differential operator in (5.5) is:

$$L(\xi) = \begin{pmatrix} -\mu & 0 & 0 & \xi_1 \\ 0 & -\mu & 0 & \xi_2 \\ 0 & 0 & N^2 - \mu & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix}$$

As $\det(L(\xi)) = \mu \left(-\mu\,|\xi| + N^2 \sqrt{\xi_1^2 + \xi_2^2} \right)$, we may conclude that the operator N^2B is not elliptic in the sense of Douglis-Nieremberg if and only if $\mu \in [0, N^2]$ ([17]).

Now, let us consider $\mu_0 \in (0, N^2)$ and choose a vector $\xi \neq 0$ such that

$$-\mu_0\,|\xi| + N^2 \sqrt{\xi_1^2 + \xi_2^2} = 0.$$

Therefore, there exists $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ such that $L(\xi)\eta = 0$:

$$\left\{ \begin{array}{c} -\mu_0\eta_1 + \xi_1\eta_4 = 0 \\ -\mu_0\eta_2 + \xi_2\eta_4 = 0 \\ \left(-\mu_0 + N^2\right)\eta_3 + \xi_3\eta_4 = 0 \\ \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 = 0 \end{array} \right. \tag{5.6}$$

Solving (5.6) with respect to η , we obtain one of possible solutions:

$$\begin{cases} \eta_1 = \frac{\xi_1}{\mu_0} \\ \eta_2 = \frac{\xi_2}{\mu_0} \\ \eta_3 = \frac{\xi_3}{\mu_0 - N^2} \\ \eta_4 = 1 \end{cases} .$$

We observe that $\eta_i \neq 0$, $i = 1, 2, 3, 4$.

Now, let $C_0^\infty(\Omega)$ be a space of smooth functions with compact support in Ω and let us choose a function

$$\psi_0(x) \in C_0^\infty(\Omega), \int\limits_{|x| \leqslant 1} \psi_0^2(x)\, dx = 1 .$$

We fix $x_0 \in \Omega$ and define

$$\psi_k(x) = k^{\frac{3}{2}} \psi_0 \left(k(x - x_0) \right) \quad , \quad k = 1, 2, ...$$

We define the Weyl sequence

$$\widetilde{v}^k = (v_1^k, v_2^k, v_3^k, q^k)$$

as follows:

$$\begin{cases} v_j^k(x) = \eta_j e^{ik^3 < x, \xi >} \left(\psi_k - \frac{1}{ik^3 \xi_j} \frac{\partial \psi_k}{\partial x_j} \right) \quad , \quad j = 1, 2, 3 \\ q^k = -\frac{i}{k^3} \psi_k e^{ik^3 < x, \xi >} \\ < x, \xi > = x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3 \quad , \quad k = 1, 2, ... \end{cases} \tag{5.7}$$

It can be shown that the sequence (5.7) actually satisfies all the conditions (5.4).

We have seen that $\lambda = 0$ is an eigenvalue of infinite multiplicity. The same statement holds for the points $\lambda = \pm N$.

Indeed, for $\lambda = \pm N$ the system (5.3) transforms into

$$\begin{cases} -N^2 v_1 + \frac{\partial q}{\partial x_1} = 0 \\ -N^2 v_2 + \frac{\partial q}{\partial x_2} = 0 \\ \frac{\partial q}{\partial x_3} = 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \end{cases} .$$

It can be easily seen that any function of the type $(0, 0, \varphi(x_1, x_2), 0)$, $\varphi \in C_0^\infty$, satisfies the last system.

Thus, the Theorem is proved.

References

[1] A. Brekhovskih and V. Goncharov. *Introduction to the mechanics of continuous media*, Nauka, Moscow, (1982) (in Russian).

[2] S. Szekers-Zenkovic, *Construction of the fundamental solution for the operator of inner waves*, Dokl.Akad.Nauk **246** (1979), 286–288 (in Russian).

[3] A. Calderón and A. Zygmund, *On singular integrals*, Amer.J.Math, **78** (1956), 289–309.

[4] A. Giniatoulline, *On the essential spectrum of the operators generated by PDE systems of stratified fluids*, Internat.J.Computer Research, **12** (2003), 63-72.

[5] A. Giniatoulline and C.Rincón, *On the spectrum of normal vibrations for stratified fluids*, Computational Fluid Dynamics J. **13** (2004), 273-281.

[6] S. Sobolev, *On a new problem of Mathematical Physics,* Izv.Akad.Nauk,Ser.Mat, **18** (1954), 7-50 (in Russian).

[7] V. Maslennikova, *About asymptotic decay of the solutions of Sobolev viscous system,* Mat.Sbornik **134** (1973), 49–60.

[8] V. Maslennikova and A. Giniatoulline, *On the intrusion problem in a viscous stratified fluid for three space variables,* Math.Notes, **51** (1992), 374-379.

[9] A. Giniatoulline, *On the uniqueness of solutions in the class of increasing functions for a system describing the dynamics of a viscous weakly stratified fluid in three dimensional space,* Rev.Colombiana Mat., **31** (1997), 71-76.

[10] E.T.Copson, *Asymptotic Expansions,* CUP (2004), 120p.

[11] A.Kolmogorov and S.Fomin, *Eléments de la théorie des fonctions et de l'analise fonctionelle,* MIR (1977).

[12] S.Gabov and A.Sveshnikov, *Dynamic problems for the stratified fluids,* Nauka, Moscow, (1986), 288p. (in Russian).

[13] V.Maslennikova, A.Giniatoulline, *Spectral properties of operators for systems of hydrodynamics,* Siberian Math.J., **29** (1988), 812–824.

[14] M.Bogovskii, *Decomposition of* L_2, Dokl.Akad.Nauk, **286** (1986), 781-786 (in Russian).

[15] T.Kato, *Perturbation theory for linear operators,* Springer, Berlin (1966).

[16] F.Riesz, B.Sz.-Nagy, *Functional analysis,* Fr.Ungar, N.Y., (1972).

[17] A.Giniatoulline, *Sobre los sistemas elípticos en el sentido de Petrovski y en el sentido de Douglis-Nirenberg,* Lect.Mat., **17** (1996), 37-47.

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