# THE NAVIER-STOKES FLOW FOR GLOBALLY LIPSCHITZ CONTINUOUS INITIAL DATA

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Abstract. Consider the Cauchy problem of the incompressible Navier-Stokes equations with initial velocity $U_0$ of the form $U_0(x) := u_0(x) - f(x)$, where $f$ is a Lipschitz function and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. It is shown that under these assumptions the equations of Navier-Stokes admit a unique local in time mild solution.

## 1. Introduction

We consider the flow of an incompressible, viscous fluid in the whole space $\mathbb{R}^n$, $n \geq 2$ described by the Cauchy problem for the system of the Navier-Stokes equations, i.e.,

$$\left\{ \begin{array}{rcll} U_t - \Delta U + (U, \nabla)U + \nabla P &=& F, & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot U &=& 0 & \text{in } \mathbb{R}^n \times (0,T), \\ U|_{t=0} &=& U_0 & (\text{with} \quad \nabla \cdot U_0 = 0) \qquad \text{in } \mathbb{R}^n. \end{array} \right. \tag{1.1}$$

Here, $U = (U^1, \ldots, U^n)$ and $P$ represent the unknown velocity and the unknown pressure of the fluid; $U_0$ is the given initial velocity, and $F$ is a given external force term.

There is a vast literature on existence of solutions of (1.1) in $\mathbb{R}^n$, see e.g. [1, 7, 9, 12, 16, 19, 22]. All these results assume that the initial data decay as $|x| \to \infty$. In particular, when $F = 0$, it is well known that there exists a locally-in-time smooth solution to (1.1) provided the initial velocity $U_0$ belongs to $L^p_\sigma(\mathbb{R}^n)$ and $p \geq n$ (see e.g. [15, 19]).

On the other hand, there is strong interest in equation (1.1) for initial data which do not decay at infinity. For results in this direction, we refer to [5, 6, 13] and [8]. Also, H. Okamoto [24] showed that for certain concrete flow problems there exist exact solutions to (1.1) which have the property that $u$ grows linearly as $|x| \to \infty$.

In this paper, we consider initial data of the form

$$U_0(x) = u_0(x) - f(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $u_0 \in L^p(\mathbb{R}^n)^n$ satisfies $\nabla \cdot u_0 = 0$ and $f$ fulfills the following three conditions:

$$\begin{array}{ll} \text{(H1)} & \nabla \cdot f = 0, \\ \text{(H2)} & \Delta f \in L^p_\sigma, \\ \text{(H3)} & \exists \Pi : \text{ scalar function s.t.} \quad (f, \nabla)f + \nabla \Pi \in L^p_\sigma. \end{array}$$

The particular case where $f(x) = Mx$ was considered in [17]. Here $M$ denotes a real $n \times n$ matrix having $\operatorname{tr} M = 0$. It was shown that this case there exists a unique, local solution to (1.1). It was also shown that this solution is analytic in the spatial variables provided $M$ is skew-symmetric. In this paper, we generalize the result of [17] to the case of Lipschitz continuous functions $f$ satisfying (H1), (H2) and (H3).

For the time being consider again the case where $f(x) = Mx$. Then it is known that (1.1) admits many exact solutions, which are studied e.g. in [10, 21, 25]. In fact, let $f$ be of the form

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$f(x) = Mx+V$, where $M = (m_{ij})_{i,j}$ is an $n \times n$ real-valued constant matrix satisfying that $\operatorname{tr} M = 0$ and such that $M^2$ is symmetric. Moreover, let $V$ be a vector. Then $\Delta f = 0$ and

$$\operatorname{tr} M = 0 \;\Leftrightarrow \text{(H1)} \qquad \text{and} \qquad M^2 \text{ is symmetric } \;\Leftrightarrow \text{(H3)}.$$

In fact, take $\Pi = \frac{1}{2}(M^2 x, x) + (V, M^T x)$. Then $(U, P)$ given by $U = -f$ and $P = -\Pi$ solves (1.1) with $F \equiv 0$ provided $\Delta f = 0$ and (H3) holds.

The particular case, where $M = R$ describes pure rotation, was investigated by Hishida and by Babin, Mahalov and Nicolaenko. Indeed, Hishida constructed in [18] a local solution to the equation (1.3) written below in the $L^2$ context and provided $u_0$ belongs to a certain fractional power space. Babin, Mahalov and Nicolaenko [2, 3] proved the existence of a local solution and even a global solution to (1.1)-(1.2) provided the speed of rotation is fast enough. Further, the case $f(x) = (ax_1, ax_2, -2ax_3)$ with some constant $a \in \mathbb{R}$, was investigated by Giga and Kambe [14]. They studied the axisymmetric irrotational flow and the stability of the vortex.

In [26], the third author proved the existence of a local solution of (1.1)-(1.2), still for $M = R$ provided $u_0$ belongs to the homogeneous Besov space $\dot{B}^0_{\infty,1}$. Although $\dot{B}^0_{\infty,1}$ is strictly smaller than $L^\infty$, this space still contains the nondecaying function $f(x) = \sin x$. He also showed the uniqueness of the solution for general matrices $M$; see [27].

In the following consider the the substitutions $u := U + f$ and $\tilde{P} := P + \Pi$. Then the pair $(U, P)$ satisfies (1.1) in the classical sense, if and only if $(u, \tilde{P})$ satisfies

$$\text{(1.3)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u - (f, \nabla)u - (u, \nabla)f + \nabla \tilde{P} &= \tilde{F} \quad \text{in } \mathbb{R}^n \times (0, T), \\ \nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\ u(0) &= u_0 \quad \text{with } \nabla \cdot u_0 = 0 \quad \text{in } \mathbb{R}^n. \end{cases}$$

Here $\tilde{F} := F + \Delta f - (f, \nabla)f - \nabla \Pi$. Of course, if $(f, \Pi)$ is a stationary solution to (1.1) with $F = F(x)$, then $\tilde{F} \equiv 0$. Our approach to equation (1.1) is based on equation (1.3).

## 2. Main Results

Let $u_0 \in L^p_\sigma(\mathbb{R}^n)$ for some $p$ satisfying $1 < p < \infty$. Moreover, let $f$ be a vector-valued globally Lipschitz continuous function satisfying hypothesis (H1), (H2), (H3).

We then rewrite the first equation of (1.3) as the abstract equation

$$\text{(2.1)} \qquad u' + Au + (u, \nabla)u - 2(u, \nabla)f + \nabla \tilde{P} = \tilde{F}.$$

with $A$ being an operator in $L^p_\sigma(\mathbb{R}^n)$ defined by

$$\text{(2.2)} \qquad Au := -\Delta u - (f, \nabla)u + (u, \nabla)f.$$

Equipped with the domain $D(A) := \{u \in W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n); (f, \nabla)u \in L^p(\mathbb{R}^n)\}$, $-A$ generates a $C_0$-semigroup $\{e^{-tA}\}_{t \geq 0}$ on $L^p_\sigma$ for $1 < p < \infty$. This follows from the results in [20] and standard perturbation theory.

Applying the Helmholtz projection $\mathbb{P}$ to (2.1), we may rewrite (1.3) as

$$\text{(2.3)} \qquad \begin{cases} u' + Au + \mathbb{P}(u, \nabla)u - 2\mathbb{P}(u, \nabla)f &= \tilde{F} \\ u(0) &= u_0. \end{cases}$$

Note that in our case the Helmholtz projection $\mathbb{P}$ can be expressed explicitly by $\mathbb{P} := (\delta_{ij} + R_i R_j)_{i,j}$, where $\delta_{ij}$ stands for Kronecker's delta, and $R_i$ is the Riesz transform defined by $R_i := \partial_i(-\Delta)^{-1/2}$ for $i = 1, \ldots, n$. Observe that $A$ and $\mathbb{P}$ commute in our case, since $\nabla \cdot Au = 0$ if $\nabla \cdot u = 0$. Since $u$, $F$ and $f$ are divergence-free, $\mathbb{P}u = u$ as well as $\mathbb{P}\tilde{F} = \tilde{F}$.

For $T>0$ we call a function $u \in C([0,T); L^p_\sigma(\mathbb{R}^n))$ a *mild solution* of (2.3) if $u$ satisfies the integral equation

$$
\begin{aligned}
u(t) = e^{-tA}u_0 &- \int_0^t e^{-(t-s)A}\mathbb{P}(u(s),\nabla)u(s)ds \\
&+ 2\int_0^t e^{-(t-s)A}\mathbb{P}(u(s),\nabla)f ds + \int_0^t e^{-(t-s)A}\tilde{F}(s)ds
\end{aligned}
\tag{2.4}
$$

for $t \in (0,T)$, and $u(0)=u_0$.

We now state the our existence and uniqueness results for mild solutions of (2.3) in $L^p$ spaces.

2.1. **Theorem.** *Let $n \geq 2$, $T>0$, $p \in [n,\infty)$ and $q \in [p,\infty)$. Let $f$ be a vector-valued globally Lipschitz continuous function satisfying* (H1), (H2) *and* (H3). *Assume that $u_0 \in L^p_\sigma(\mathbb{R}^n)$, and that $F \in C(0,T; L^p_\sigma(\mathbb{R}^n))$. Then there exist $T_0 \in (0,T)$ and a unique mild solution $u$ of* (2.3) *such that*

$$
[t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}u(t)] \in C([0,T_0); L^q_\sigma(\mathbb{R}^n))
\tag{2.5}
$$

$$
[t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}\nabla u(t)] \in C([0,T_0); L^q(\mathbb{R}^n)).
\tag{2.6}
$$

2.2. **Remark.** (i) The semigroup $\{e^{-tA}\}_{t\geq 0}$ is not analytic.
(ii) Consider the case $p=\infty$ and $u_0 \in L^\infty_\sigma(\mathbb{R}^n)$ or $u_0 \in BUC_\sigma$, i.e., $u_0$ do not decay at space infinity. In this case, one might expect to obtain the existence result for the mild solutions $u \in C([0,T_0); \dot{B}^0_{\infty,1})$ satisfying (2.3) provided that $u_0 \in \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ and $\nabla \cdot u_0 = 0$. In [27], this is discussed for the case $f(x)=Mx$.

The proof of Theorem 2.1 is based on Kato's iteration procedure. The key is to derive appropriate smoothing estimates for the semigroup and its gradient; see Proposition 3.3. Uniqueness follows the by Gronwall's inequality.

## 3. Estimates for the semigroup

In this section we prepare the linear estimates needed for the iteration scheme. Let $f$ be a vector-valued globally Lipschitz continuous function satisfying (H1), (H2) and (H3).

We the define the realization of the operator

$$
\mathcal{L}u := -\Delta u - (f,\nabla)u, \qquad x \in \mathbb{R}^n,
\tag{3.1}
$$

in $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$ as follows. Set

$$
\begin{aligned}
Lu &:= \mathcal{L}u \\
D(L) &:= \{u \in W^{2,p}(\mathbb{R}^n); (f,\nabla)u \in L^p(\mathbb{R}^n)\}.
\end{aligned}
$$

Then the following result was proved by Lunardi and Metafune [20].

3.1. **Proposition.** *Let $1<p<\infty$. Then the operator $-L$ generates a $C_0$-semigroup $\{e^{-tL}\}_{t\geq 0}$ on $L^p(\mathbb{R}^n)$.*

3.2. **Remark.** (i) The semigroup $\{e^{-tL}\}_{t\geq 0}$ is not analytic; see [20].
(ii) The family $\{e^{-tL}\}_{t\geq 0}$ is also a semigroup on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$, which in the latter case is not strongly continuous.
(iii) If $f(x)=Mx$ where $M$ is a constant matrix, the semigroup $\{e^{-tL}\}_{t\geq 0}$ has an explicit representation given by

$$
e^{-tL}\varphi(x) := \frac{1}{(4\pi)^{n/2}(detQ_t)^{1/2}}\int_{\mathbb{R}^n}\varphi(e^{tM}x-y)e^{-\frac{1}{4}(Q_t^{-1}y,y)}dy, \quad x \in \mathbb{R}^n,\ t>0,
$$

where $Q_t$ for $t>0$ is given by $Q_t := \int_0^t e^{sM}e^{sM^T}ds$.

For the iteration scheme described in the next section it is essential that the associated semigroup maps an $L^p$-function $u$ with $\nabla \cdot u = 0$ into the space of $L^p$-functions which are divergence free. We therefore introduce the operator $\mathcal{A}$ by

$$\mathcal{A}u := \mathcal{L}u + (u, \nabla)f,$$

where $u = (u^1, \ldots, u^n)$. Thus $\mathcal{A}$ is an $n \times n$ operator matrix given by

$$\mathcal{A} = \mathcal{L}\,\mathrm{Id} + (\nabla f)$$

where Id denotes the identity matrix. Observe that

$$\nabla \cdot \{(f, \nabla)u - (u, \nabla)f\} = 0, \qquad \text{provided} \quad \nabla \cdot u = 0 \ \text{ and } \ \nabla \cdot f = 0.$$

Hence, we define the realization $A$ of $\mathcal{A}$ in $L^p_\sigma(\mathbb{R}^n)$ as

$$\begin{aligned} Au &:= \mathcal{A}u, \\ D(A) &:= D(L)^n \cap L^p_\sigma(\mathbb{R}^n). \end{aligned} \tag{3.2}$$

By standard perturbation theory, $-A$ generates a $C_0$-semigroup $\{e^{-tA}\}_{t\geq 0}$ on $L^p_\sigma$ for all $p \in (1, \infty)$.

In the case where $f(x) = Mx$, the semigroup $\{e^{-tA}\}_{t\geq 0}$ is given by

$$(e^{-tA}u)(x) := \frac{1}{(4\pi)^{n/2}(detQ_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^n} u(e^{tM}x - y)e^{-\frac{1}{4}(Q_t^{-1}y,y)}dy. \tag{3.3}$$

We cannot expect to have such a formula for the semigroup $\{e^{-tA}\}_{t\geq 0}$, in general.

We are now state $L^p - L^q$ smoothing properties for the semigroup $e^{-tA}$ as well as gradient estimates for $e^{-tA}$. Note that due to the non-analyticity of $\{e^{-tA}\}_{t\geq 0}$, gradient estimates for $e^{-tA}$ do not follow from the general theory of semigroups (like the Stokes semigroup). Notice also that in the special case where $f(x) = x$, $L^p - L^q$ smoothing estimates as well as gradient estimates for $e^{-tA}$ were obtain by Gallay and Wayne [11]. For $f(x) = Mx$, these estimates were obtained in [17]. For the general case, we rely on the recent results of Lunardi and Metafune [20] and Bertholdi and Lorenzi [4].

3.3. **Proposition.** [[20], Prop. 5.4], [[4], Thm. 4.7, Cor, 4.8]. *Let $n \geq 2$, $1 < p < \infty$ and $p \leq q \leq \infty$.*

a) *Then there exist constants $C > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|e^{-tA}\varphi\|_q \leq Ce^{\omega t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_p, \quad t \geq 0, \ \varphi \in L^p(\mathbb{R}^n), \tag{3.4}$$

$$\|\nabla e^{-tA}\varphi\|_p \leq Ce^{\omega t}t^{-\frac{1}{2}}\|\varphi\|_p, \quad t \geq 0, \ \varphi \in L^p(\mathbb{R}^n). \tag{3.5}$$

b) *There exist constants $C' > 0$ and $\omega' \in \mathbb{R}$ such that*

$$\|\nabla^2 e^{-tA}\varphi\|_p \leq C'e^{\omega' t}t^{-1}\|\varphi\|_p, \quad t \geq 0, \ \varphi \in L^p(\mathbb{R}^n). \tag{3.6}$$

c) *Moreover, let $1 < p < q \leq \infty$ and $\varphi \in L^p(\mathbb{R}^n)$. Then*

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{-tA}\varphi\|_q \to 0 \quad \text{as } t \to 0, \tag{3.7}$$

$$t^{\frac{1}{2}}\|\nabla e^{-tA}\varphi\|_p \to 0 \quad \text{as } t \to 0, \tag{3.8}$$

$$t\|\nabla^2 e^{-tA}\varphi\|_p \to 0 \quad \text{as } t \to 0. \tag{3.9}$$

## 4. Proof of the Main Result

For a given globally Lipschitz continuous function $f$ satisfying (H1), (H2), (H3), consider the substitution $u(x,t) := U(x,t) + f(x)$ and $\tilde{P}(x,t) := P(x,t) + \Pi(x)$. Then $(U, P)$ is a solution of (1.1) in the classical sense if and only if $(u, \tilde{P})$ satisfies (1.3). We thus consider in the following (1.3) and its abstract formulation in (2.3), or (2.4). We only show the proof for the case $p = n$; the case $p > n$ is similar.

*Proof of Theorem 2.1.* Let $n \geq 2$ and $u_0 \in L^n_\sigma(\mathbb{R}^n)$. Assume that $F \in C(0,\infty; L^n_\sigma(\mathbb{R}^n))$. Recall that $\tilde{F} = F + \Delta f - (f,\nabla)f - \nabla\Pi$ and $\nabla \cdot \tilde{F} = 0$. For $j \geq 1$ and $t > 0$ we define functions $u_j$ successively by

$$u_1(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\tilde{F}(s)ds, \tag{4.1}$$

$$u_{j+1}(t) := u_1(t) - \int_0^t e^{-(t-s)A}\mathbb{P}(u_j(s),\nabla)u_j(s)ds + 2\int_0^t e^{-(t-s)A}\mathbb{P}(u_j(s),\nabla)f ds. \tag{4.2}$$

Since $\{e^{-tA}\}_{t\geq 0}$ acts on $L^p_\sigma(\mathbb{R}^n)$ for $p \in (1,\infty)$, it follows from the definition of the Helmholtz projection that the functions $u_j$ are divergence-free for all $t > 0$ and all $j$.

For $T \in (0,1]$ and $\delta \in (0,1)$ we define

$$K_0 := \sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\|e^{-tA}u_0\|_{n/\delta} \quad \text{and} \quad K_0' := \sup_{0<t\leq T} t^{1/2}\|\nabla e^{-tA}u_0\|_n.$$

By (3.7) and (3.8) in Proposition 3.3-(c), $K_0 \to 0$ and $K_0' \to 0$ as $T \to 0$. Similarly, we define $K_j := K_j(T)$ and $K_j' := K_j'(T)$ for $j \geq 1$ by

$$K_j(T) := \sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\|u_j(t)\|_{n/\delta} \quad \text{and} \quad K_j'(T) := \sup_{0<t\leq T} t^{1/2}\|\nabla u_j(t)\|_n.$$

Let us estimate $K_1$ and $K_1'$: by definition and the $L^p - L^q$ smoothing property (3.4), we have

$$\begin{aligned}
K_1 &= \sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\|u_1(t)\|_{n/\delta} \\
&\leq K_0 + C\sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\int_0^t \|e^{-(t-s)A}\tilde{F}(s)\|_{n/\delta}ds \\
&\leq K_0 + C\sup_{0<t\leq T} t^{\frac{1-\delta}{2}}\int_0^t (t-s)^{-\frac{1-\delta}{2}}\|\tilde{F}(s)\|_n ds \\
&\leq K_0 + CT\big(\|-\Delta f + (f,\nabla)f + \Pi\|_n + \|F\|_{L^\infty(0,T;L^n(\mathbb{R}^n))}\big).
\end{aligned}$$

Similarly,

$$K_1' \leq K_0' + CT\big(\|-\Delta f + (f,\nabla)f + \Pi\|_n + \|F\|_{L^\infty(0,T;L^n(\mathbb{R}^n))}\big).$$

We thus have

$$K_1, K_1' \to 0 \quad \text{as} \quad T \to 0. \tag{4.3}$$

Next, it follows from (4.2), the $L^p - L^q$ smoothing of the semigroup and from the boundedness of $\mathbb{P}$ from $L^p(\mathbb{R}^n)$ into $L^p_\sigma(\mathbb{R}^n)$ that

$$\begin{aligned}
&\|u_{j+1}(t)\|_{n/\delta} \\
&\leq \|u_1\|_{n/\delta} + \int_0^t \|e^{-(t-s)A}\mathbb{P}(u_j(s),\nabla)u_j(s)\|_{n/\delta}ds + 2\int_0^t \|e^{-(t-s)A}\mathbb{P}(u_j(s),\nabla)f\|_{n/\delta}ds \\
&\leq t^{-\frac{1-\delta}{2}}K_1 + C\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{\delta}{n})}\|(u_j(s),\nabla)u_j(s)\|_r ds + C\int_0^t \|u_j(s)\|_{n/\delta}ds,
\end{aligned}$$

where $r = \frac{n}{1+\delta}$. In order to estimate the second term on the right hand side of last inequality, we apply Hölder's inequality to conclude that

$$\|(u_j(s),\nabla)u_j(s)\|_r \leq \|u_j(s)\|_{n/\delta}\|\nabla u_j(s)\|_n \leq K_jK_j's^{-\frac{1-\delta}{2}-\frac{1}{2}}.$$

This implies

$$\|u_{j+1}(t)\|_{n/\delta} \leq t^{-\frac{1-\delta}{2}}K_1 + CK_jK_j'\int_0^t (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}ds + CK_j\int_0^t s^{-\frac{1-\delta}{2}}ds.$$

Multiplying with $t^{\frac{1-\delta}{2}}$ and taking $\sup_{0<t\leq T}$ on both sides, we obtain

$$K_{j+1} \leq K_1 + C_1K_jK_j' + C_2TK_j$$

with some constants $C_1, C_2$, independent of $j$ and $T$.

Similarly, applying $\nabla$ to (4.2) and estimating it with respect to the $L^n$-norm, it follows from (3.4) and (3.5) that

$$K'_{j+1} \le K'_1 + C_3 K_j K'_j + C_4 T K_j$$

for some constants $C_3$ and $C_4$. By (4.3), for any $\lambda > 0$ there exists $\tilde{T}_0 > 0$ such that $K_1, K'_1 \le \lambda$ for all $T \le \tilde{T}_0$. So, we fix $\tilde{T}_0 \le \min(1, \frac{1}{3C_2}, \frac{1}{3C_4})$ provided $\lambda \le \min(\frac{1}{9C_1}, \frac{1}{9C_3})$. We thus obtain bounds for $K_j(T)$ and $K'_j(T)$ for any $T \le \tilde{T}_0$ uniformly in $j$ provided that $\tilde{T}_0$ is small enough. Indeed, $\sup_j K_j, K'_j \le 3\lambda$ for $T \le \tilde{T}_0$.

The uniform bounds of $K_j$ and $K'_j$ imply that $t^{\frac{1}{2}-\frac{n}{2q}}\|u_j(t)\|_q$ as well as $t^{1-\frac{n}{2q}}\|\nabla u_j(t)\|_q$ are bounded for $q \in [n,\infty)$, $t \le \tilde{T}_0$ and all $j \in \mathbb{N}$. The continuity of the above functions follows from similar calculations and (3.7).

We finally derive estimates for the differences $u_{j+1} - u_j$. Indeed, for all $j \ge 1$ put

$$L_j(T) := \sup_{0<t\le T} t^{\frac{1-\delta}{2}}\|u_{j+1}(t) - u_j(t)\|_{n/\delta} \quad \text{and} \quad L'_j(T) := \sup_{0<t\le T} t^{1/2}\|\nabla u_{j+1}(t) - \nabla u_j(t)\|_n.$$

Similarly as before, we have for all $j \ge 1$

$$\begin{aligned} L_j(T) &\le C_5\lambda(L_{j-1} + L'_{j-1}) + C_6 T L_{j-1}, \\ L'_j(T) &\le C_7\lambda(L_{j-1} + L'_{j-1}) + C_8 T L_{j-1} \end{aligned}$$

with some positive constants $C_5$, $C_6$, $C_7$ and $C_8$. We now choose $T_0 \le \tilde{T}_0$ small enough so that $T_0 \le \min(\frac{1}{8C_6}, \frac{1}{8C_8})$ provided $4(C_5 + C_7)\lambda \le 1$. Hence we have $(L_{j+1} + L'_{j+1})/(L_j + L'_j) \le 1/2$ for all $j \ge 1$ and $T \le T_0$. This implies that $L_j$ and $L'_j$ tend to zero as $j \to \infty$. It thus follows that the above sequences are Cauchy sequences and we conclude that there are unique limit functions

$$[t \mapsto t^{\frac{1}{2}-\frac{n}{2q}} u(t)] \in C([0,T_0]; L^q_\sigma), \qquad [t \mapsto t^{1-\frac{n}{2q}} v(t)] \in C([0,T_0]; L^q)$$

of the sequences $\{t^{\frac{1}{2}-\frac{n}{2q}} u_j(t)\}_{j\ge1}$ (if necessary, we shall take its subsequence) and $\{t^{1-\frac{n}{2q}} \nabla u_j(t)\}_{j\ge1}$. Finally, note that $v(t) = t^{1/2}\nabla u(t)$ and that $u$ is a mild solution of (2.3) on $[0,T_0]$.

Uniqueness of mild solutions follows from standard Gronwall's inequality. This completes the proof of the first assertion of Theorem 2.1. □

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