# Decay estimates of the Stokes flow around a rotating obstacle

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**Abstract**

We study the motion of a viscous incompressible fluid filling the whole 3-dimensional space exterior to a rigid body that is rotating with constant angular velocity $\omega$. By using a frame attached to the body, the problem is reduced to an equivalent one in a fixed exterior domain. Then the linear part of the reduced equation is

$$\partial_t u = \Delta u + (\omega \times x) \cdot \nabla u - \omega \times u - \nabla p, \qquad \text{div } u = 0.$$

For the exterior problem with the Dirichlet boundary condition, we develop the $L_p$-$L_q$ estimates (and the $L_{p,1}$-$L_{q,1}$ estimates as well) of the generated semigroup, which is no longer analytic due to the drift operator with the coefficient $\omega \times x$. We next apply them to the Navier-Stokes equation to prove the global existence of a unique solution which goes to a stationary flow as $t \to \infty$ with some definite rates when both the stationary flow and the initial disturbance are sufficiently small in $L_{3,\infty}$ (weak-$L_3$ space).

## 1 Introduction

Let us consider the motion of a viscous fluid filling an infinite space exterior to a rigid body, that moves in a prescribed way such as rotation and translation. In order to understand the rotation effect mathematically, this article studies the purely rotating case. Thus, suppose that the body is rotating about $y_3$-axis with constant angular velocity $\omega = (0, 0, a)^T, a \in \mathbb{R}$; here and hereafter, all vectors are column ones. Let $\Omega$ be an exterior domain in $\mathbb{R}^3$

with boundary $\partial\Omega \in C^{1,1}$. Unless the body is axisymmetric, the domain occupied by the fluid varies with time $t$, and it is described as

$$\Omega(t) = \{y = \mathcal{O}(at)x; x \in \Omega\}, \qquad \mathcal{O}(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We consider the Navier-Stokes equation

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla_y \widetilde{u} = \Delta_y \widetilde{u} - \nabla_y \widetilde{p}, \qquad \mathrm{div}_y \widetilde{u} = 0,$$

for $y \in \Omega(t), t > 0$, subject to the boundary and initial conditions

$$\widetilde{u}|_{\partial\Omega(t)} = \omega \times y, \qquad \widetilde{u} \to 0 \text{ as } |y| \to \infty, \qquad \widetilde{u}(y, 0) = u_0(y),$$

where $\widetilde{u}(y,t) = (\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)^T$ and $\widetilde{p}(y,t)$ are respectively unknown velocity and pressure of the fluid; $u_0$ is the given initial velocity; $\omega \times y = a(-y_2, y_1, 0)^T$ is the velocity of the rotating body so that the boundary condition is the usual nonslip one. A reasonable way from both mathematical and physical points of view is to take the frame $x = \mathcal{O}(at)^T y$ attached to the body ([3], [13], [21]). The following change of functions is thus made:

$$u(x,t) = \mathcal{O}(at)^T \widetilde{u}(y,t), \qquad p(x,t) = \widetilde{p}(y,t).$$

The problem is then reduced to

$$\partial_t u + u \cdot \nabla u = \Delta u + M_a u - \nabla p, \qquad \mathrm{div}\ u = 0 \tag{1.1}$$

in the fixed domain $\Omega \times (0, \infty)$ subject to

$$u|_{\partial\Omega} = \omega \times x, \qquad u \to 0 \text{ as } |x| \to \infty, \qquad u(x,0) = u_0(x), \tag{1.2}$$

where

$$M_a = (\omega \times x) \cdot \nabla - \omega \times, \qquad \omega = (0, 0, a)^T. \tag{1.3}$$

Our goal is to prove that the problem (1.1)–(1.2) possesses a unique global solution $u(t)$ which goes to a stationary flow $u_s$ as $t \to \infty$ when $\omega$ and $u_0 - u_s$ are small in a sense. Thus the first step is to find a solution $u_s$ of the stationary problem

$$-\Delta u_s - M_a u_s + \nabla p_s + u_s \cdot \nabla u_s = 0, \qquad \mathrm{div}\ u_s = 0$$

in $\Omega$ subject to

$$u_s|_{\partial\Omega} = \omega \times x, \qquad u_s \to 0 \text{ as } |x| \to \infty.$$

Look at the linear part of the first equation of (1.1). The crucial drift operator $(\omega \times x)\cdot\nabla$ has a variable coefficient growing at infinity and causes some difficulties, which indicate that the term $(\omega \times x)\cdot\nabla u$ is never subordinate to the viscous term $\Delta u$ even if $|\omega|$ is small. In fact, the semigroup generated by the operator $\Delta + M_a$ is not an analytic one in, say, $L_2$ ([21], [22], [10]). And also, the pointwise estimate of the fundamental solution of the operator $\Delta + M_a$ is slightly worse than $1/|x-y|$ for large $(x,y)$ ([9], [25]).

Up to now, particularly in the last decade, a lot of efforts have been made on the problems above or some related ones; see [3], [6], [15], [17], [18], [21] for the nonstationary flow, [7], [8], [9], [13], [14], [16], [25], [38] for the stationary one. Among them, the stationary solutions of [14] and [8] can be taken as the basic flow around which a global solution exists since their solutions enjoy so good asymptotic behavior at infinity that one can expect the stability. In fact, Galdi [14] derived pointwise estimates

$$|u_s(x)| \leqq c/|x|, \qquad |\nabla u_s(x)| + |p_s(x)| \leqq c/|x|^2$$

of a unique stationary solution provided that $\omega$ is small enough and that, in case the external force $f = \mathrm{div}\, F$ is present, it has some decay properties and is also small in a sense. Another outlook on the pointwise estimates above in a different framework by use of function spaces has been recently provided by Farwig and Hishida [8] when the external force $f = \mathrm{div}\, F$ is taken from a larger class $F \in L_{3/2,\infty}(\Omega)$, where $L_{q,\infty}(\Omega)$ is the weak-$L_q$ space, one of the Lorentz spaces introduced below. To be more precise, a stationary solution of class

$$u_s \in L_{3,\infty}(\Omega), \qquad (\nabla u_s, p_s) \in L_{3/2,\infty}(\Omega) \tag{1.4}$$

has been uniquely constructed for small $\omega$ and $\|F\|_{L_{3/2,\infty}(\Omega)}$, subject to

$$\|u_s\|_{L_{3,\infty}(\Omega)} + \|(\nabla u_s, p_s)\|_{L_{3/2,\infty}(\Omega)} \leqq C\Big(|\omega| + \|F\|_{L_{3/2,\infty}(\Omega)}\Big). \tag{1.5}$$

This result can be ragarded as a generalization of [29] and [37] to the rotating body problem.

The solvability of the initial value problem (1.1)–(1.2) was studied in [3], [15], [18] and [21]. Borchers [3] constructed weak solutions for $u_0$ in $L_2(\Omega)$. As usual, we do not know the uniqueness of weak solutions. Later on, in [21] the existence of a unique solution locally in time was proved when, roughly speaking, $u_0$ possesses the regularity $W_2^{1/2}(\Omega)$. This local existence result has been recently extended to the general $L_q$-theory by Geissert, Heck and Hieber [18] to replace $W_2^{1/2}(\Omega)$ by $L_3(\Omega)$. Galdi and Silvestre [15] showed the unique existence of local and global strong solutions by the Galerkin method. Their global solution was constructed around a stationary solution $u_s$ of Galdi [14] and the stability of the solution $u_s$ was also proved. To be more precise, if $\omega$ is small and if $u_0 - u_s$ is taken from $W_2^2(\Omega)$ with small $W_2^1$-norm, together with $u_0|_{\partial\Omega} = \omega \times x$ and $(\omega \times x) \cdot \nabla(u_0 - u_s) \in L_2(\Omega)$, then there is a global solution $u(t)$ which satisfies $\|\nabla(u(t) - u_s)\|_{L_2(\Omega)} \to 0$ as $t \to \infty$.

In this article we prove the stability of the stationary solution $u_s$ of [8], [14] for small $\omega$ and $u_0 - u_s \in L_{3,\infty}(\Omega)$. Although the global solution of [15] is more regular than ours, new contribution of our global existence theorem is to deduce the definite decay rates of the disturbance $u(t) - u_s$, see (9.4), which seem to be optimal.

For the proof, the most difficult step is to derive some decay estimates of the solution to the following Stokes equation with rotation effect, which is of own interest:

$$\partial_t u = \Delta u + M_a u - \nabla \pi, \qquad \text{div } u = 0 \tag{1.6}$$

in $\Omega \times (0, \infty)$ subject to $u|_{\partial\Omega} = 0$ and $u(x, 0) = u_0(x)$. The strategy based on some cut-off techniques together with spectral analysis is traced back to Shibata [34] and is similar to that of Iwashita [27] and also Kobayashi and Shibata [28], however, we need several new ideas because the semigroup generated by the problem above is never analytic unlike [27], [28]. In particular, it is important to derive the behavior of the associated resolvent for large $\lambda$ along the imaginary axis in the complex plane as well as its regularity for small $\lambda$.

We give a remark on the case of time-dependent angular velocity $\omega = \omega(t)$, which was discussed by [3], [6] on weak solutions, by [23] on a local unique solution and by [17] on time-periodic solutions. For the global existence and large time behavior of a unique solution as in this article, we need two difficult steps. One is to find a suitable basic flow instead of stationary solutions, the other is not only the asymptotic behavior of evolution operator generated by (1.6) with $M_a = M_a(t)$ but also the analysis of full linearization around the basic flow above. Although the method developed in this article cannot be directly employed, some ideas could be applied and the problem will be an interesting object in the future.

Finally, we would like to mention a physical example which has relation to our theory, that is, the particle sedimentation in a viscous fluid. This is of practical interest and the problem is to find a falling motion of a rigid body under its own weight in an infinite fluid, see Weinberger [39] and Galdi

[13] for details. The body undergoes a rotation and a translation which are to be determined from equilibrium conditions on the boundary; that is, a fluid-body interaction system has to be solved. Indeed the present article is devoted to the fluid motion around a body which moves in a prescribed way, but our study is certainly a step toward an analysis of that problem.

This article is organized as follows. The next section provides the main results on some decay estimates of the solution to (1.6), namely the Stokes semigroup with rotation effect. In sections 3 and 4 the resolvent problems in the whole space $\mathbb{R}^3$ and in a bounded domain are respectively studied. We construct, in section 5, a parametrix of the resolvent in exterior domains. Section 6 provides the reconstruction and some estimates near $t = 0$ of the semigroup when the initial velocity has a bounded support. In section 7 the local energy decay estimate of the semigroup is deduced. In section 8 we derive $L_p$-$L_q$ estimates of the semigroup. The final section is devoted to an application of decay estimates of the semigroup to the Navier-Stokes flow. In each section only some ideas and key points are explained. For details, see [26].

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## 2 Main results on the Stokes flow

To state our results, we introduce some function spaces. We adopt the same symbols for vector and scalar function spaces. Let $C_0^\infty(\Omega)$ consist of all $C^\infty$-functions with compact supports in $\Omega$. For $1 \leqq q \leqq \infty$ and $0 \leqq k \in \mathbb{Z}$, we denote by $W_q^k(\Omega)$, with $W_q^0(\Omega) = L_q(\Omega)$, the usual $L_q$-Sobolev space of order $k$. Let $1 < q < \infty$ and $1 \leqq r \leqq \infty$. Then the Lorentz spaces are defined by

$$L_{q,r}(\Omega) = \Big(L_1(\Omega), L_\infty(\Omega)\Big)_{1-1/q,r},$$

where $(\cdot,\cdot)$ is the real interpolation functor, see [1]. It is well known that $f$ is in $L_{q,\infty}(\Omega)$ if and only if

$$\sup_{\sigma>0} \sigma \left|\{x \in \Omega; |f(x)| > \sigma\}\right|^{1/q} < \infty$$

and that $L_{q,\infty}(\Omega)$ is the dual space of $L_{q/(q-1),1}(\Omega)$. Note that $C_0^\infty(\Omega)$ is not dense in $L_{q,\infty}(\Omega)$. We next introduce some solenoidal function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ be the class of all $C_0^\infty$-vector fields $f$ which satisfy $\operatorname{div} f = 0$ in $\Omega$. For $1 < q < \infty$ we denote by $J_q(\Omega)$ the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L_q(\Omega)$. Then the Helmholtz decomposition of $L_q$-vector fields holds, see Miyakawa [32]:

$$L_q(\Omega) = J_q(\Omega) \oplus \{\nabla\pi \in L_q(\Omega); \pi \in L_{q,loc}(\overline{\Omega})\}.$$

Let $P$ denote the projection operator from $L_q(\Omega)$ onto $J_q(\Omega)$ associated with the decomposition. Then the operator $\mathcal{L}_a$ is defined by

$$\begin{cases} D(\mathcal{L}_a) = \{u \in J_q(\Omega) \cap W_q^2(\Omega);\ u|_{\partial\Omega} = 0,\ (\omega \times x) \cdot \nabla u \in L_q(\Omega)\}, \\ \mathcal{L}_a u = -P[\Delta u + M_a u], \end{cases}$$

see (1.3). It is proved by Geissert, Heck and Hieber [18] that the operator $-\mathcal{L}_a$ generates a $C_0$-semigroup $\{T_a(t)\}_{t \geqq 0}$ on the space $J_q(\Omega), 1 < q < \infty$ (see also [20] for the case $q = 2$). Indeed the semigroup $T_a(t)$ enjoys a certain smoothing properties, but it is not an analytic one, see Hishida [21]. Concerning the essential spectrum of the generator $-\mathcal{L}_a$ on $J_2(\Omega)$, we refer to Farwig and Neustupa [10].

We need also the solenoidal Lorentz spaces, which are defined by

$$J_{q,r}(\Omega) = \Big(J_{q_0}(\Omega), J_{q_1}(\Omega)\Big)_{\theta,r}$$

where $1 < q_0 < q < q_1 < \infty, 1 \leqq r \leqq \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Then $\{T_a(t)\}_{t \geqq 0}$ is extended to the semigroup on the space $J_{q,r}(\Omega)$.

We now fix $R > 0$ such that $\mathbb{R}^3 \setminus \Omega \subset B_R = \{x \in \mathbb{R}^3; |x| < R\}$, and set $\Omega_R = \Omega \cap B_R$. By $W_q^{-1}(\Omega_R)$ we denote the dual space of $\dot{W}^1_{q/(q-1)}(\Omega_R) = \{v \in W^1_{q/(q-1)}(\Omega_R); v|_{\partial\Omega_R} = 0\}$. The space $L_{q,[R+2]}(\Omega)$ or $L_{q,[R+2]}(\mathbb{R}^3)$ consists of $L_q$-vector fields $v$ satisfying $v(x) = 0$ almost everywhere for $|x| \geqq R + 2$.

In the linear theory, one does not need any smallness condition on the angular velocity $\omega = (0, 0, a)^T$. But most of the estimates are not uniform for large $\omega$, however, they are uniform for $\omega$ with $|\omega| = |a| \leqq a_0$, where $a_0 > 0$ is arbitrary. In what follows, the constant $C$ which may change from line to line depends on $a_0 > 0$ and increases as $a_0$ grows even if this will not be specified.

The main theorems on some decay properties of the semigroup $T_a(t)$ read as follows.

**Theorem 2.1 (Local energy decay).** *Let $1 < q < \infty$. For arbitrary $a_0 > 0$, there is a constant $C = C(q, R, a_0) > 0$ such that*

$$\|T_a(t)Pf\|_{W_q^1(\Omega_{R+3})} \leqq C\,\ell_0(t)\|f\|_{L_q(\Omega)} \tag{2.1}$$

$$\|\partial_t T_a(t)Pf\|_{W_q^{-1}(\Omega_{R+3})} + \|\pi(t)\|_{L_q(\Omega_{R+3})} \leqq C\,\ell_1(t)\|f\|_{L_q(\Omega)} \tag{2.2}$$

*for all $t > 0$, $f \in L_{q,[R+2]}(\Omega)$ and $\omega$ with $|\omega| = |a| \leqq a_0$. Here, $\pi(x,t)$ is the associated pressure that satisfies $\int_{\Omega_{R+3}} \pi(x,t)\,dx = 0$, see (1.6), and*

$$\ell_0(t) = \begin{cases} t^{-1/2}, & 0 < t \leqq 1, \\ t^{-3/2}, & t > 1, \end{cases} \qquad \ell_1(t) = \begin{cases} t^{-\frac{1}{2}\left(1+\frac{1}{q}\right)}, & 0 < t \leqq 1, \\ t^{-3/2}, & t > 1. \end{cases}$$

**Theorem 2.2** (**$L_p$-$L_q$ estimate**). *Suppose that*

$$\begin{cases} 1 < p \leqq q < \infty & \text{for } j = 0, \\ 1 < p \leqq q \leqq 3 & \text{for } j = 1, \end{cases} \tag{2.3}$$

*and let $a_0 > 0$ be arbitrary. Set*

$$\kappa = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

*Then there is a constant $C = C(p, q, a_0) > 0$ such that*

$$\|\nabla^j T_a(t) f\|_{L_q(\Omega)} \leqq C t^{-j/2-\kappa} \|f\|_{L_p(\Omega)} \tag{2.4}$$

*for all $t > 0$, $f \in J_p(\Omega)$ and $\omega$ with $|\omega| = |a| \leqq a_0$. For $q = \infty$ and $j = 0$ as well, estimate (2.4) holds.*

**Theorem 2.3** (**$L_{p,r}$-$L_{q,r}$ estimate**). *Suppose (2.3) and let $a_0 > 0$ be arbitrary. Let $1 \leqq r < \infty$. Then there is a constant $C = C(p, q, r, a_0) > 0$ such that*

$$\|\nabla^j T_a(t) f\|_{L_{q,r}(\Omega)} \leqq C t^{-j/2-\kappa} \|f\|_{L_{p,r}(\Omega)} \tag{2.5}$$

*for all $t > 0$, $f \in J_{p,r}(\Omega)$ and $\omega$ with $|\omega| = |a| \leqq a_0$, where $\kappa$ is the same as in Theorem 2.2.*

Estimate (2.4) with the case $p = q$ tells us the uniform boundedness of the semigroup in $t$ on $J_q(\Omega)$, which was not shown in [18], while the semigroup is contractive on $J_2(\Omega)$, see [20]. The restriction $q \leqq 3$ for the gradient estimate, which was first proved by Iwashita [27] for the case of the usual Stokes semigroup ($\omega = 0$), is caused by the fact that the effect from the solution to the whole space problem remains near the boundary. In fact, Maremonti and Solonnikov [31] pointed out that one cannot avoid that restriction even when $\omega = 0$. In view of their proof, we find that this is also related to the decay structure of stationary solutions. As was mentioned, the decay of our fundamental solution is slightly worse than that of the usual Stokes one and thus it is hopeless to improve the restriction $q \leqq 3$ for the gradient estimate in our problem as well.

In an application to the Navier-Stokes equation, we will employ (2.5) with $r = 1$ rather than (2.4). The reason why we need the estimate in the Lorentz space will be clarified in the final section.

# 3 Resolvent for the whole space problem

The large time behavior of the semigroup $T_a(t)$ is closely related to the regularity for small $\lambda$ of the resolvent $(\lambda I + \mathcal{L}_a)^{-1}$. In the proof of (2.1), the first step is thus the analysis of the resolvent problem

$$\lambda u - \Delta u - M_a u + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } \mathbb{R}^3. \tag{3.1}$$

The solution, which we denote by $A_{\mathbb{R}^3}(\lambda) f$, is described as the Laplace transform of the semigroup

$$\left(S_a(t) f\right)(x) = \mathcal{O}(at)^T \left(e^{t\Delta} f\right)(\mathcal{O}(at)x) \tag{3.2}$$

in the whole space, whose Fourier transform is

$$\left(\widehat{S_a(t) f}\right)(\xi) = \mathcal{O}(at)^T e^{-|\xi|^2 t} \widehat{f}\left(\mathcal{O}(at)\xi\right),$$

and thus we have

$$\begin{aligned} u(x) &= \left(A_{\mathbb{R}^3}(\lambda) f\right)(x) \\ &= \frac{1}{(2\pi)^3} \int_0^\infty \int_{\mathbb{R}^3} e^{-(\lambda + |\xi|^2)t} e^{i(\mathcal{O}(at)x)\cdot\xi} \mathcal{O}(at)^T P(\xi) \widehat{f}(\xi) d\xi dt \end{aligned} \tag{3.3}$$

for Re $\lambda \geqq 0$ and $f \in L_q(\mathbb{R}^3)$, where $P(\xi) = I - \xi \otimes \xi / |\xi|^2$. The associated pressure is given by

$$p(x) = \left(Q_{\mathbb{R}^3} f\right)(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \frac{\xi \cdot \widehat{f}(\xi)}{i|\xi|^2} d\xi + c_0(f) \tag{3.4}$$

where the constant $c_0(f)$ is determined so that $\int_{\Omega_{R+3}} \left(Q_{\mathbb{R}^3} f\right)(x) dx = 0$. If in particular $f \in L_{q,[R+2]}(\mathbb{R}^3)$, its Fourier image $\widehat{f}$ is a smooth function, from which we find that $A_{\mathbb{R}^3}(\lambda) f$ possesses a certain regularity for small $\lambda$ in the localized space $W_q^2(B_{R+3})$. In fact, we find

$$\partial_\lambda A_{\mathbb{R}^3}(\lambda) f \sim |\lambda - kia|^{-1/2}, \qquad k = 0, \pm 1$$

near $\lambda = 0, \pm ia$ and $A_{\mathbb{R}^3}(\cdot) f$ is of class $C^1$ on $\overline{\mathbb{C}_+} \setminus \{0, \pm ia\}$ with values in $W_q^2(B_{R+3})$. Furthermore, we observe

$$\partial_\lambda^2 A_{\mathbb{R}^3}(\lambda) f \sim |\lambda - kia|^{-3/2}, \qquad k = 0, \pm 1$$

$$\partial_\lambda^2 A_{\mathbb{R}^3}(\lambda) f \sim |\lambda - kia|^{-1/2}, \qquad k = \pm 2, \pm 3$$

near $\Lambda = \{0, \pm ia, \pm 2ia, \pm 3ia\}$ and $A_{\mathbb{R}^3}(\cdot)f$ is of class $C^2$ on $\overline{\mathbb{C}_+} \setminus \Lambda$ with values in $W^2_q(B_{R+3})$. The observations above provide a justification of the regularity $C^{3/2}$ of the resolvent $A_{\mathbb{R}^3}(\lambda)$ for small $\lambda$ in the sense that

$$\int_{-K}^{K} \|(\partial_\lambda A_{\mathbb{R}^3})(i(\tau+h)) - (\partial_\lambda A_{\mathbb{R}^3})(i\tau)\|_{W^2_q(B_{R+3})} d\tau \leqq C|h|^{1/2}\|f\|_{L_q(\mathbb{R}^3)} \tag{3.5}$$

for $|h| \leqq 1$ and $f \in L_{q,[R+2]}(\mathbb{R}^3)$, where $K > 0$ is a fixed large number.

The analysis of the resolvent for large $\lambda$ is also quite important in our problem because of lack of analyticity of the semigroup. The main point is that, by integration by parts with respect to $t$, the resolvent $A_{\mathbb{R}^3}(\lambda)$ can be divided into two parts as below: the first term arising from $t = 0$ is something like parabolic part, that is, its analytic continuation into a sectorial subset of the left half complex plane is possible, while the second term decays rapidly as $|\lambda| \to \infty$ in $\overline{\mathbb{C}_+}$, even along the imaginary axis. To be precise, given arbitrary $N \in \mathbb{N}$, we have the representation

$$\begin{aligned}(A_{\mathbb{R}^3}(\lambda)f)(x) = &\sum_{k=0}^{N-1} M_a^k(\lambda - \Delta_{\mathbb{R}^3})^{-(k+1)}P_{\mathbb{R}^3}f(x)\\ &+ \frac{1}{(2\pi)^3}\int_0^\infty \int_{\mathbb{R}^3} \frac{e^{-(\lambda+|\xi|^2)t}e^{ix\cdot\xi}}{(\lambda+|\xi|^2)^N}\\ &\qquad \mathcal{O}(at)^T\left[\widetilde{M}_{\xi,a}^N\left(P(\cdot)\widehat{f}\right)\right](\mathcal{O}(at)\xi)\,d\xi dt\end{aligned} \tag{3.6}$$

in $W^2_q(B_{R+3})$ for all $f \in L_{q,[R+2]}(\mathbb{R}^3)$, where $M_a$ is as in (1.3) and

$$\widetilde{M}_{\xi,a} = (\omega \times \xi)\cdot\nabla_\xi - \omega\times, \qquad \omega = (0,0,a)^T.$$

# 4 Resolvent for the interior problem

Given $f \in L_q(\Omega_{R+3})$, let us consider the resolvent problem

$$\lambda u - \Delta u - M_a u + \nabla p = f, \qquad \mathrm{div}\ u = 0 \qquad \text{in } \Omega_{R+3} \tag{4.1}$$

subject to $u|_{\partial\Omega_{R+3}} = 0$ and $\int_{\Omega_{R+3}} p(x)dx = 0$. It is worth while emphasizing that the structure of the pressure, see (4.3) with (4.2) below, plays an important role later.

Using the Helmholtz decomposition (Fujiwara and Morimoto [11])

$$f = P_{\Omega_{R+3}}f + \nabla Q_{\Omega_{R+3}}f, \qquad \int_{\Omega_{R+3}} \left(Q_{\Omega_{R+3}}f\right)(x)dx = 0,$$

one can rewrite the equation

$$(\lambda - \Delta - M_a)u + \nabla\left(p - Q_{\Omega_{R+3}} f\right) = P_{\Omega_{R+3}} f, \qquad \mathrm{div}\ u = 0$$

which can be treated as a perturbation from the usual Stokes equation ($\omega = 0$). In fact, by compactness argument with use of the Fredholm theorem, we find that the operator $I - M_a \widetilde{A}_0(\lambda)$ has a bounded inverse on $J_q(\Omega_{R+3})$ for $\lambda \in \overline{\mathbb{C}_+}$, where $\left(\widetilde{A}_0(\lambda), \widetilde{\Pi}_0(\lambda)\right)$ denotes the solution operator $J_q(\Omega_{R+3}) \ni g \mapsto (u, \pi)$ for the problem

$$\lambda u - \Delta u + \nabla \pi = g, \qquad \mathrm{div}\ u = 0$$

in $\Omega_{R+3}$ subject to $u|_{\partial\Omega_{R+3}} = 0$ and $\int_{\Omega_{R+3}} \pi(x)dx = 0$. Furthermore, the operator norm of the inverse $\left(I - M_a \widetilde{A}_0(\lambda)\right)^{-1}$ is uniformly bounded in $\lambda \in \overline{\mathbb{C}_+}$ because this inverse can be also described as the Neumann series for large $\lambda$. Now, given $g \in J_q(\Omega_{R+3})$, it is easily verified that the pair of

$$\begin{aligned} u &= \widetilde{A}_a(\lambda)g := \widetilde{A}_0(\lambda)\left(I - M_a \widetilde{A}_0(\lambda)\right)^{-1} g, \\ \pi &= \widetilde{\Pi}_a(\lambda)g := \widetilde{\Pi}_0(\lambda)\left(I - M_a \widetilde{A}_0(\lambda)\right)^{-1} g, \end{aligned}$$

satisfies

$$(\lambda - \Delta - M_a)u + \nabla \pi = g, \qquad \mathrm{div}\ u = 0$$

in $\Omega_{R+3}$ subject to $u|_{\partial\Omega_{R+3}} = 0$ and $\int_{\Omega_{R+3}} \pi(x)dx = 0$. Obviously, $\widetilde{A}_a(\lambda)$ enjoys the same decay properties for large $\lambda$ as the parabolic resolvent $\widetilde{A}_0(\lambda)$ does. The following decay of the pressure, which follows from that of $\widetilde{\Pi}_0(\lambda)$, is also important:

$$\|\partial_\lambda^k \widetilde{\Pi}_a(\lambda)g\|_{L_q(\Omega_{R+3})} \leqq C(1 + |\lambda|)^{-(1-1/q)/2-k} \|g\|_{L_q(\Omega_{R+3})} \tag{4.2}$$

for $k \in \mathbb{N} \cup \{0\}, 1 < q < \infty$. For the case $\omega = 0$, this rate of decay is a refinement of a related result shown by Noll and Saal [33, Lemma 3.11] and it is proved by means of duality argument together with an interpolation inequality. Set

$$A_{\Omega_{R+3}}(\lambda)f = \widetilde{A}_a(\lambda)P_{\Omega_{R+3}} f, \quad \Pi_{\Omega_{R+3}}(\lambda)f = \widetilde{\Pi}_a(\lambda)P_{\Omega_{R+3}} f + Q_{\Omega_{R+3}} f \tag{4.3}$$

for $f \in L_q(\Omega_{R+3})$. Then they actually provide a solution to (4.1).

# 5 Resolvent for the exterior problem

In this section we derive a representation of the resolvent $(\lambda I + \mathcal{L}_a)^{-1}$ in exterior domains. We employ a cut-off technique to construct a parametrix by use of resolvents in the whole space $\mathbb{R}^3$ and in the bounded domain $\Omega_{R+3}$ together with the Bogovskii operator ([2]) to recover the solenoidal condition.

Given $f \in L_{q,[R+2]}(\Omega)$, we denote by $f_0$ the zero extension of $f$ to the whole space $\mathbb{R}^3$ and by $r_{\Omega_{R+3}}f$ the restriction of $f$ on $\Omega_{R+3}$. Using (3.3), (3.4) and (4.3), we set

$$
\begin{aligned}
u = A(\lambda)f &:= (1-\varphi)A_{\mathbb{R}^3}(\lambda)f_0 + \varphi A_{\Omega_{R+3}}(\lambda)r_{\Omega_{R+3}}f + B[(\mathcal{C}(\lambda)f)\cdot\nabla\varphi], \\
p = \Pi(\lambda)f &:= (1-\varphi)Q_{\mathbb{R}^3}f_0 + \varphi\Pi_{\Omega_{R+3}}(\lambda)r_{\Omega_{R+3}}f,
\end{aligned}
$$

with

$$
\mathcal{C}(\lambda)f = A_{\mathbb{R}^3}(\lambda)f_0 - A_{\Omega_{R+3}}(\lambda)r_{\Omega_{R+3}}f,
$$

where $\varphi \in C_0^\infty(\mathbb{R}^3)$ is fixed cut-off function so that $\varphi(x) = 1$ for $|x| \leqq R+1$ and $\varphi(x) = 0$ for $|x| \geqq R+2$, and $B$ denotes the Bogovskii operator on $A_{R+1,R+2} = \{x \in \mathbb{R}^3; R+1 < |x| < R+2\}$. Then the pair $(u,p)$ should obey

$$
(\lambda - \Delta - M_a)u + \nabla p = f + R(\lambda)f, \qquad \operatorname{div} u = 0
$$

in $\Omega$ subject to $u|_{\partial\Omega} = 0$, where $R(\lambda)f$ is the remainder term arising from the cut-off procedure. The operator $R(\lambda)$ is divided into two or three parts

$$
R(\lambda) = R_1 + R_2(\lambda) = R_1 + R_{21}(\lambda) + R_{22}(\lambda), \tag{5.1}
$$

where $R_1$ is independent of $\lambda$ and consists of $Q_{\mathbb{R}^3}f_0$ and $Q_{\Omega_{R+3}}r_{\Omega_{R+3}}f$, while $R_2(\lambda) \equiv R_{21}(\lambda) + R_{22}(\lambda)$ depends on $\lambda$. To be precise, $R_{21}(\lambda)$ is extended into a sectorial subset $S$ of the left half complex plane, as analytic continuation, with estimate

$$
\|R_{21}(\lambda)\|_{\mathcal{L}(L_{q,[R+2]}(\Omega))} \leqq C(1+|\lambda|)^{-(1-1/q)/2}, \qquad \lambda \in S \cup \overline{\mathbb{C}_+} \tag{5.2}
$$

where the rate of decay comes from (4.2) with $k=0$, while $R_{22}(\lambda)$ consists of some terms arising from the second term of (3.6) and, therefore, cannot be extended into the left half complex plane at all, but it decays rapidly, say,

$$\|R_{22}(\lambda)\|_{\mathcal{L}(L_{q,[R+2]}(\Omega))} \leqq C(1+|\lambda|)^{-3}, \qquad \lambda \in \overline{\mathbb{C}_+} \tag{5.3}$$

since we can take $N \in \mathbb{N}$ as large as we want in the representation formula (3.6).

As usual, a compactness argument implies the existence of the bounded inverse $(I+R(\lambda))^{-1}$; however, the behavior of $(I+R(\lambda))^{-1}$ for large $\lambda$ is not clear. We thus reconstruct this inverse of the form

$$\begin{aligned} (I+R(\lambda))^{-1} &= \left[I+(I+R_1)^{-1}R_2(\lambda)\right]^{-1}(I+R_1)^{-1} \\ &= \sum_{k=0}^{\infty}\left\{-(I+R_1)^{-1}R_2(\lambda)\right\}^k (I+R_1)^{-1} \end{aligned} \tag{5.4}$$

for large $\lambda$ by using $R_2(\lambda) \to 0$ as $|\lambda| \to \infty$ as given in (5.2) and (5.3). The important step is to show the invertibility of $I+R_1$ by the uniqueness of the Helmholtz decomposition. As a consequence, the operator norm of the inverse $(I+R(\lambda))^{-1}$ is uniformly bounded in $\lambda$, and therefore, both the behavior for large $\lambda$ and the regularity for small $\lambda$ of the resolvent

$$(\lambda I+\mathcal{L}_a)^{-1}Pf = A(\lambda)(I+R(\lambda))^{-1}f$$

are governed by those of resolvents in the whole space $\mathbb{R}^3$ and in $\Omega_{R+3}$, see (7.2) and (7.3) below.

# 6 Reconstruction of the semigroup

We reconstruct the semigroup $T_a(t)$ and derive some estimates near $t=0$ when the initial velocity has a bounded support. Thus, given $f \in L_{q,[R+2]}(\Omega)$, let us consider the nonstationary problem (1.6) subject to $u|_{\partial\Omega}=0$ and $u(x,0)=(Pf)(x)$. Although $T_a(t)$ is not an analytic semigroup, thanks to the boundedness of the support of $f$, we get a unique strong solution

$$u \in C^1\left((0,\infty);J_q(\Omega)\right) \cap C\left((0,\infty);D(\mathcal{L}_a)\right), \quad \nabla p \in C\left((0,\infty);L_q(\Omega)\right) \tag{6.1}$$

with estimates

$$\begin{aligned} &\|u(t)\|_{L_q(\Omega)} + t^{1/2}\|\nabla u(t)\|_{L_q(\Omega)} \\ &\qquad + t\left\|(\partial_t u(t),\nabla^2 u(t),\nabla p(t))\right\|_{L_q(\Omega)} \leqq C_\gamma e^{\gamma t}\|f\|_{L_q(\Omega)}, \\ &t^{(1+1/q)/2}\left(\|\partial_t u(t)\|_{W_q^{-1}(\Omega_b)} + \|p(t)\|_{L_q(\Omega_b)}\right) \leqq C_\gamma e^{\gamma t}\|f\|_{L_q(\Omega)}, \end{aligned} \tag{6.2}$$

for any $t > 0, \gamma > 0$ and $b > R+3$. Indeed, the proof does not rely upon the existence of the semigroup due to [18] at all. We make full use of the structure of the resolvent $(\lambda I + \mathcal{L}_a)^{-1}$ studied in the previous section to construct the solution $(u,p)$ concretely by

$$\begin{aligned} u(t) &= \lim_{\ell\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\ell}^{\gamma+i\ell} e^{\lambda t}A(\lambda)(I+R(\lambda))^{-1}f\,d\lambda \qquad \text{in } W^1_q(\Omega), \\ p(t) &= \lim_{\ell\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\ell}^{\gamma+i\ell} e^{\lambda t}\Pi(\lambda)(I+R(\lambda))^{-1}f\,d\lambda \qquad \text{in } L_q(\Omega_b), \end{aligned} \tag{6.3}$$

where the convergence is uniform in any compact interval of $(0,\infty)$. And then, we know $u(t) = T_a(t)Pf$; as a consequence, our argument provides a justification of representation of the semigroup $T_a(t)Pf$ in terms of the inverse Laplace transform of $(\lambda I + \mathcal{L}_a)^{-1}Pf$ when $f \in L_{q,[R+2]}(\Omega)$. Since we will prove a local energy decay property of the semigroup for $f \in L_{q,[R+2]}(\Omega)$ in the next section, it is sufficient for us to justify that representation for such $f$. Note, however, that this does not follow from the usual semigroup theory, in which the initial data are assumed to be rather smooth. It is also remarkable that the formula of the associated pressure is available.

In the proof of the regularity (6.1) of the solution $u(t)$ given by (6.3), the following decomposition of $(I+R(\lambda))^{-1}$, see (5.4) with (5.1), plays a crucial role:

$$(I+R(\lambda))^{-1} = (I+R_1)^{-1} + U_1(\lambda) + U_2(\lambda)$$

with

$$\|U_2(\lambda)\|_{\mathcal{L}(L_{q,[R+2]}(\Omega))} \leqq C(1+|\lambda|)^{-3}, \qquad \lambda \in \overline{\mathbb{C}_+}.$$

This is actually possible when we take $m \in \mathbb{N}$ so large that $(m+1)(1-1/q)/2 \geqq 3$ and set

$$U_1(\lambda) = \sum_{k=1}^{m} \left\{-(I+R_1)^{-1}R_{21}(\lambda)\right\}^k (I+R_1)^{-1}.$$

Using the decomposition above combined with equations of the resolvent problems (3.1) and (4.1), we obtain also the following decomposition from which we find the time-differentiability of $u(t)$:

$$\begin{aligned} \lambda(\lambda+\mathcal{L}_a)^{-1}Pf &= \lambda A(\lambda)\left[(I+R_1)^{-1} + U_1(\lambda) + U_2(\lambda)\right] f \\ &= Kf + Y(\lambda)f + V_1(\lambda)f + V_2(\lambda)f, \end{aligned}$$

where $Kf$ does not depend on $\lambda$; $Y(\lambda)f$ depends on $\lambda$ but does not decay for large $\lambda$; $V_1(\lambda)f$ is extended into a sectorial subset $S$ of the left half complex plane, as analytic continuation, with

$$\|V_1(\lambda)f\|_{L_q(\Omega)} \leqq C(1+|\lambda|)^{-(1-1/q)/2}\|f\|_{L_q(\Omega)}, \qquad \lambda \in S \cup \overline{\mathbb{C}_+};$$

and $V_2(\lambda)f$ cannot be extended into the left half complex plane but it decays rapidly, in fact,

$$\|V_2(\lambda)f\|_{W_q^2(\Omega)} \leqq \frac{C}{\gamma}|\lambda|^{-3}\|f\|_{L_q(\Omega)}, \qquad \mathrm{Re}\,\lambda \geqq \gamma > 0.$$

# 7 Local energy decay of the semigroup

As in Iwashita [27] and also in Kobayashi and Shibata [28] for the Oseen equation, the crucial step in the proof of $L_p$-$L_q$ estimate (2.4) is to derive the local energy decay estimate (2.1) for $t > 1$. The strategy is traced back to Shibata [34]. Estimates (2.1) and (2.2) for $0 < t \leqq 1$ have been already shown in (6.2).

Let $f \in L_{q,[R+2]}(\Omega)$. To complete the proof of (2.1), the representation formula (6.3) of the semigroup $T_a(t)$ by the inverse Laplace transform of the resolvent $(\lambda I + \mathcal{L}_a)^{-1}$ is now useful. After integration by parts with respect to $\lambda$, we shift the path of integration to the imaginary axis ($\lambda = i\tau$) to obtain

$$T_a(t)Pf = \frac{-1}{2\pi i t}\int_{-\infty}^{\infty} e^{i\tau t}\partial_\tau(i\tau I + \mathcal{L}_a)^{-1}Pf\,d\tau \tag{7.1}$$

in the localized space $W_q^1(\Omega_{R+3})$. This procedure and the resulting formula (7.1) can be justified because the analysis done in sections 3, 4 and 5 implies the following behavior of

$$\begin{aligned}&\partial_\lambda(\lambda I + \mathcal{L}_a)^{-1}P\\ &= \partial_\lambda A(\lambda)(I + R(\lambda))^{-1} - A(\lambda)(I + R(\lambda))^{-1}\partial_\lambda R_2(\lambda)(I + R(\lambda))^{-1}\end{aligned}$$

as well as that of $(\lambda I + \mathcal{L}_a)^{-1}P$ itself:

$$\|\partial_\lambda^j(\lambda I + \mathcal{L}_a)^{-1}Pf\|_{W_q^1(\Omega_{R+3})} \leqq C\,|\lambda|^{-j-1/2}\|f\|_{L_q(\Omega)}, \quad j = 0, 1 \tag{7.2}$$

in $\{\lambda \in \overline{\mathbb{C}_+}; |\lambda| \geqq K\}$, where $K > 0$ is a fixed large number, and

$$\|\partial_\lambda(\lambda I + \mathcal{L}_a)^{-1}Pf\|_{W_q^1(\Omega_{R+3})} \leqq C\,|\lambda - kia|^{-1/2}\|f\|_{L_q(\Omega)}, \quad k = 0, \pm 1 \tag{7.3}$$

in $\{\lambda \in \overline{\mathbb{C}_+}; |\lambda - kia| \leqq |a|/4\}$. Moreover, we have (7.2) for $j = 2$ as well as the $C^{3/2}$-regularity of $(\lambda I + \mathcal{L}_a)^{-1}P$ in the same sense as in (3.5).

We now consider

$$G(t) = \int_{-\infty}^{\infty} e^{i\tau t}\partial_\tau(i\tau I + \mathcal{L}_a)^{-1}Pf\,d\tau$$

in $W^1_q(\Omega_{R+3})$ and split it into some integrals near $\tau = ka$ ($k = 0, \pm 1, \pm 2, \pm 3$) and for large $\tau$. Then the former decays like $t^{-1/2}$ that follows from the relation between the $C^{3/2}$-regularity of a function and the decay property of its inverse Fourier image ([34], [28], [36], [24]), while the latter decays like $t^{-1}$ that follows from (7.2) for $j = 1, 2$ by integration by parts once more.

# 8 $L_p$-$L_q$ estimates of the semigroup

We proceed to the local energy decay estimates of the semigroup $T_a(t)$ for general data $f \in J_q(\Omega)$.

**Proposition 8.1 (Local energy decay).** *Let $1 < q < \infty$. For arbitrary $a_0 > 0$, there is a constant $C = C(q, R, a_0) > 0$ such that*

$$\|T_a(t)f\|_{W^1_q(\Omega_{R+3})} \leqq C\,\widetilde{\ell}_0(t)\|f\|_{L_q(\Omega)} \tag{8.1}$$

$$\|\partial_t T_a(t)f\|_{W^{-1}_q(\Omega_{R+3})} + \|\pi(t)\|_{L_q(\Omega_{R+3})} \leqq C\,\widetilde{\ell}_1(t)\|f\|_{L_q(\Omega)} \tag{8.2}$$

*for all $t > 0$, $f \in J_q(\Omega)$ and $\omega$ with $|\omega| = |a| \leqq a_0$. Here, $\pi(x, t)$ is the associated pressure that satisfies $\int_{\Omega_{R+3}} \pi(x, t)\, dx = 0$, see (1.6), and*

$$\widetilde{\ell}_0(t) = \begin{cases} t^{-1/2}, & 0 < t \leqq 1, \\ t^{-3/2q}, & t > 1, \end{cases} \qquad \widetilde{\ell}_1(t) = \begin{cases} t^{-\frac{1}{2}\left(1+\frac{1}{q}\right)}, & 0 < t \leqq 1, \\ t^{-3/2q}, & t > 1. \end{cases}$$

This is proved by use of a cut-off technique together with $L_p$-$L_q$ estimate of the related semigroup $S_a(t)$, see (3.2), in the whole space $\mathbb{R}^3$. In fact, as the first approximation of the solution $u(t) = T_a(t)f$, one can take

$$v(t) = (1 - \varphi)S_a(t)f + B[(S_a(t)f) \cdot \nabla\varphi],$$

where $\varphi \in C^\infty_0(\mathbb{R}^3)$ is a fixed cut-off function so that $\varphi(x) = 1$ for $|x| \leqq R+1$ and $\varphi(x) = 0$ for $|x| \geqq R + 2$, and $B$ denotes the Bogovskii operator on $A_{R+1,R+2}$. Here, $f$ may be assumed to be in $C^\infty_{0,\sigma}(\Omega)$, therefore in $C^\infty_{0,\sigma}(\mathbb{R}^3)$ as well, since it is dense in $J_q(\Omega)$. Then $w(t) = u(t) - v(t)$ should obey

$$w(t) = T_a(t)\Big(\varphi f - B[f \cdot \nabla\varphi]\Big) + \int_0^t T_a(t - s)F(s)ds,$$

which can be estimated in $W^1_q(\Omega_R)$ by using Theorem 2.1 because the support of the (solenoidal) ramainder term $F(t)$ as well as $w(0)$ is contained in $A_{R+1,R+2}$. We thus obtain Proposition 8.1. The argument above may provide another proof of generation of $C_0$-semigroup $\{T_a(t)\}_{t\geqq 0}$.

And then, combining (8.1), (8.2) with estimate for $|x| \geqq R+3$, we prove (2.4). In the proof of estimate for $|x| \geqq R+3$ by a cut-off technique again and by using $L_p$-$L_q$ estimate for (3.2), unlike the case of the usual Stokes semigroup ($\omega = 0$), we need (8.2) near $t = 0$ for the pressure $\pi(t)$ which is contained in the remainder term arising from the cut-off procedure. This would not be necessary if $T_a(t)$ were an analytic semigroup.

Since both the estimate of $T_a(t)f$ in (8.1) and that of $\pi(t)$ in (8.2) hold for all $q \in (1,\infty)$, one can employ the real interpolation so that the similar estimates are obtained in the Lorentz spaces $L_{q,r}(\Omega_{R+3})$. Along the same line by use of those estimates as in the proof of (2.4), we obtain (2.5).

# 9 Application to the Navier-Stokes flow

Let $(u_s, p_s)$ be the stationary solution of class (1.4) subject to (1.5). Set

$$v(x,t) = u(x,t) - u_s(x), \qquad \pi(x,t) = p(x,t) - p_s(x)$$

and $v_0(x) = u_0(x) - u_s(x)$. Then our stability problem is reduced to the global existence and decay of solutions to

$$\partial_t v + v\cdot\nabla v + u_s\cdot\nabla v + v\cdot\nabla u_s = \Delta v - M_a v - \nabla\pi, \quad \mathrm{div}\ v = 0 \tag{9.1}$$

in $\Omega\times(0,\infty)$ subject to

$$v|_{\partial\Omega} = 0, \qquad v \to 0 \text{ as } |x|\to\infty, \qquad v(x,0) = v_0(x). \tag{9.2}$$

By use of the semigroup $T_a(t)$, the problem (9.1) is converted into the integral equation

$$v(t) = T_a(t)v_0 - \int_0^t T_a(t-\tau)P\mathrm{div}\ G(u_s, v(\tau))d\tau,$$

where

$$G(u_s, v(t)) = v(t)\otimes v(t) + u_s\otimes v(t) + v(t)\otimes u_s.$$

In view of the class (1.4) of the stationary solution $u_s$, however, it is difficult to treat the additional linear terms $\mathrm{div}(u_s\otimes v + v\otimes u_s)$ directly. We thus consider the weak formulation

$$\langle v(t),\varphi\rangle = \langle v_0, T_{-a}(t)\varphi\rangle + \int_0^t \langle G(u_s, v(\tau)), \nabla T_{-a}(t-\tau)\varphi\rangle\, d\tau \qquad \text{for } \forall\varphi\in C_{0,\sigma}^\infty(\Omega), \tag{9.3}$$

in terms of the semigroup $T_{-a}(t)$, where $T_{-a}(t)^* = T_a(t)$ for $a \in \mathbb{R}$. We use the $L_{p,1}$-$L_{q,1}$ estimates (2.5) rather than (2.4). Let $1 < p < q \leqq 3$ and $1/q = 1/p - 1/3$. Then the interpolation technique developed by Yamazaki [40] combined with (2.5) for $r = 1$ implies

$$\int_0^\infty \|\nabla T_a(t) f\|_{L_{q,1}(\Omega)} dt \leqq C\|f\|_{L_{p,1}(\Omega)}$$

for $f \in J_{p,1}(\Omega)$. This enables us to deal with the additional linear terms in $G(u_s, v)$ as a perturbation from the semigroup $T_a(t)$. As a result, by the contraction mapping principle, we obtain the following global existence theorem.

**Theorem 9.1.** *Let $u_s \in L_{3,\infty}(\Omega)$ and $v_0 \in J_{3,\infty}(\Omega)$.*

1. *There is a constant $\delta > 0$ such that if $\|u_s\|_{L_{3,\infty}(\Omega)} + \|v_0\|_{L_{3,\infty}(\Omega)} \leqq \delta$, then the problem (9.3) possesses a unique global solution*

$$v \in BC\left((0,\infty); J_{3,\infty}(\Omega)\right) \quad \text{with} \quad w^*\text{-}\lim_{t\to 0} v(t) = v_0 \text{ in } J_{3,\infty}(\Omega).$$

2. *Let $3 < q < \infty$. Then there is a constant $\widetilde{\delta}(q) \in (0, \delta]$ such that if $\|u_s\|_{L_{3,\infty}(\Omega)} + \|v_0\|_{L_{3,\infty}(\Omega)} \leq \widetilde{\delta}(q)$, then the solution $v(t)$ obtained above enjoys*

$$\|v(t)\|_{L_r(\Omega)} = O\left(t^{-1/2+3/2r}\right) \quad \textit{as } t \to \infty \tag{9.4}$$

*for every $r \in (3, q)$.*

Note that the problem is well-posed only in $BC\left((0,\infty); J_{3,\infty}(\Omega)\right)$ without any additional norm. But $\|v(t)\|_{L_{3,\infty}(\Omega)}$ does not decay in general as $t \to \infty$ since $C_0^\infty(\Omega)$ is not dense in $L_{3,\infty}(\Omega)$. As in the second assertion of Theorem 9.1, the decay property of $v(t)$ is obtained in $L_r(\Omega)$, $r > 3$, when the data are still smaller. By (1.5) the assumptions on the stationary solution $u_s$ are satisfied for small $\omega$ and $f = \operatorname{div} F$.

We finally remark on the boundary condition of the solution obtained in Theorem 9.1. This is verified in the sense of trace when the stationary solution $u_s$ possesses a slight regularity such as

$$u_s \in L_{q_*}(\Omega), \qquad \nabla u_s \in L_q(\Omega) \tag{9.5}$$

for some $q \in (3/2, 3)$ and $1/q_* = 1/q - 1/3$, in addition to (1.4). For the discussion on the boundary condition, it is convenient to introduce the fractional powers of the usual Stokes operator $A = -P\Delta$ with $D(A) = \{u \in J_r(\Omega) \cap W_r^2(\Omega);\, u|_{\partial\Omega} = 0\}$ for $1 < r < \infty$. Especially, one needs

$$\|A^{1/2}T_a(t)f\|_{L_r(\Omega)} \leqq Ct^{-1/2-(3/q-3/r)/2}\|f\|_{L_q(\Omega)} \tag{9.6}$$

for $0 < t \leqq 2$ (resp. $t > 0$), $1 < q \leqq r < \infty$ (resp. $1 < q \leqq r \leqq 3$) and $f \in J_q(\Omega)$. This is a simple consequence of Theorem 2.2 combined with $\|A^{1/2}u\|_{L_r(\Omega)} \leqq C\|\nabla u\|_{L_r(\Omega)}$ for $u \in D(A), 1 < r < \infty$ (Borchers and Miyakawa [4, Theorem 4.4]). Given $t_0 > 0$, by use of (9.5) and (9.6), it is easy to show that the integral equation

$$v(t) = T_a(t-t_0)v(t_0) - \int_{t_0}^{t} T_a(t-\tau)P(v\cdot\nabla v + u_s\cdot\nabla v + v\cdot\nabla u_s)(\tau)d\tau$$

has a unique local solution $v \in C((t_0, t_0+T_*]; D(A^{1/2}))\cap C([t_0, t_0+T_*]; J_r(\Omega))$ when $v(t_0) \in J_r(\Omega)$ with some $r \in (3, q_*)$, and that the length $T_* > 0$ of the existence interval can be estimated. Now, suppose that $v(t)$ is the global solution obtained in Theorem 9.1 (the second part). Then, $v(t) \in J_r(\Omega) \cap J_{3,\infty}(\Omega)$ for some $r > 3$ and every $t > 0$. For each $\tau > 0$ one can find an interval $I_\tau = [t_0, t_0 + T_*]$ with $0 < t_0 < \tau < t_0 + T_*$ so that $v(t)$ coincides with a solution mentioned above for all $t \in I_\tau$ by uniqueness of solutions in $C(I_\tau; J_r(\Omega))$ (this kind of argument was developed in [30]). Hence, $v(t)|_{\partial\Omega} = 0$ for every $t > 0$.

The stationary flow that satisfies (9.5) as well as (1.4) can be actually constructed for the external force $f = \mathrm{div}\, F$ with $F \in L_{r,\infty}(\Omega) \cap L_{3/2,\infty}(\Omega)$ for some $r \in (3/2, 3)$ when $|\omega|$ and $\|F\|_{L_{3/2,\infty}(\Omega)}$ are small enough, see Farwig and Hishida [8].

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