Uniform local solvability for the Navier-Stokes equations with the Coriolis force

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Abstract

This is a supplementary note of the paper: Y. Giga, K. Inui, A. Mahalov and S. Matsui (2005, *Methods and Applications of Analysis*), where local-in-time existence and uniqueness of mild solution for the 3-dimensional Navier-Stokes equations with the Coriolis force were established with its uniform existence time in the Coriolis parameter. The crucial part of the proof is to seek an appropriate class for initial data which allows uniform boundedness in $t \in \mathbb{R}$ of the Riesz semigroup whose symbol is $\exp(t(i\xi_j/|\xi|))$ $(\xi = (\xi_1, \xi_2, \xi_3),\ i = \sqrt{-1})$ for $j = 1, 2, 3$. For this purpose we found a new space denoted by FM_0, Fourier preimage of finite Radon measures having no-point mass at the origin. In Appendix we give an observation on the Mikhlin theorem in the Besov-type space $\dot{B}^0_{Z,1}$ for a Banach space Z which is included in the space of temperd distributions $\mathcal{S}'$.

1 Introduction

In this note we discuss local-in-time existence and uniqueness for the Cauchy problem of the Navier-Stokes equations with the Coriolis term;

$$u_t + (u \cdot \nabla)u + \Omega e_3 \times u - \Delta u = -\nabla p, \quad \nabla \cdot u = 0, \quad u|_{t=0} = u_0, \tag{1.1}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ is the unknown velocity vector field and $p = p(x,t)$ is the unknown scalar pressure at the point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ in space and time $t > 0$, while $u_0 = u_0(x)$ is the given initial velocity vector field. Here, the real constant Ω represents the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$ and it is called the Coriolis parameter. By $\times$ we denote the exterior product, hence, the Coriolis term is represented by $e_3 \times u \equiv \mathbf{J}u$ with the corresponding skew-symmetric 3×3 matrix $\mathbf{J}$. Clearly, (1.1) can be regarded as generalized equations of the Navier-Stokes equations ($\Omega = 0$).

Research of the equations (1.1) (for $\Omega \neq 0$) originated in the paper [2] by Babin-Mahalov-Nicolaenko (see the survey [11] for full list of their papers on this problem), and in [3] they showed that weak solutions of (1.1) are global in time strong solutions for sufficiently large Ω independent of the size of initial data in periodic lattice and cylindrical domains. The main methods are singular limits of oscillating operator as $\Omega \to \infty$, nonlinear averaging, and a lemma on interaction of the nonlinear oscillations of the vorticity field.

On the other hand, solubility for non-periodic and non-decaying initial data has been investigated in Hieber-Sawada [9], Sawada [12], and Giga-Inui-Mahalov-Matsui [5], where they proved existence and uniqueness of local-in-time solution of the Cauchy problem for initial data in the homogeneous Besov space $\dot{B}^0_{\infty,1}$ and each fixed $\Omega \in \mathbb{R}$. We have choice of how to deal with the additional Coriolis term $\Omega e_3 \times u$ in (1.1). In [9] and [12] the Coriolis term was regarded as perturbation of the Laplacian, while in [5] as the principal term by considering the Poincaré-Sobolev equations, the linearized equations of (1.1);

$$u_t + \Omega e_3 \times u - \Delta u = -\nabla p, \quad \nabla \cdot u = 0, \quad u|_{t=0} = u_0.$$

After multiplying the Helmholtz projection operator $\mathbf{P} = (\delta_{ij} + R_i R_j)_{1 \le i,j \le 3}$ to these equations and transforming into time variable evolution equation, by virtue of skew-symmetry of the Coriolis term, it turns out (as explained in Section 4) that solubility of (1.1) reduces the boundedness of the Riesz semigroup $\exp(tR_j)$ defined by

$$\exp(tR_j)f = F^{-1}(\exp(ti\frac{\xi_j}{|\xi|})Ff), \qquad t \in \mathbb{R},\ j = 1,2,3, \tag{1.2}$$

where $i = \sqrt{-1}$ and R_j denotes the Riesz operator with its symbol $i\frac{\xi_j}{|\xi|}$. Here, by F and F^{-1} we denote Fourier transform and inverse Fourier transform.

The norm of the space $\dot{B}^0_{\infty,1}$ measures low and high frequency separately using the Littlewood-Paley dyadic decomposition in the phase space except the origin;

$$\{\widehat{\phi}_j(\xi)\}_{j\in\mathbb{Z}} \qquad \text{satisfying} \quad \sum_{j=-\infty}^{\infty} \widehat{\phi}_j(\xi) = 1 \quad \text{for } \xi \neq 0. \tag{1.3}$$

The space $\dot{B}^0_{\infty,1}$ is slightly smaller than the spaces L^∞ or BUC, the space of all bounded uniformly continuous functions, however, it can be regarded as a substitute of the spaces L^∞ or BUC, in fact, $\dot{B}^0_{\infty,1}$ is closed on dilation, moreover, it contains the functions of the form

$$f(x) = \sum_{j=1}^{\infty} \alpha_j e^{i\lambda_j \cdot x} \quad \lambda_j \in \mathbb{R}^n \backslash \{0\}, \quad \alpha_j \in \mathbb{C}, \quad \sum_{j=1}^{\infty} |\alpha_j| < \infty, \quad \lambda_j \neq \lambda_k \text{ if } j \neq k, \tag{1.4}$$

which can be an almost periodic function not necessarily periodic by choosing $\{\lambda_j\}_{j=1}^\infty$ appropriately. The advantage of the space $\dot{B}^0_{\infty,1}$ is the boundedness of the Riesz operator R_j, however, for the Riesz semigroup $\exp(tR_j)$ $(t \in \mathbb{R},\ j = 1, 2, 3)$ only estimates depending on t have been obtained so far, and it is still open whether the uniform estimate in $\dot{B}^0_{\infty,1}$ holds or not, being crucial to show global-in-time solubility and regularity. On estimates for the Riesz semigroup, readers can refer [10], which also gives a short review of recent results on non-decaying rotating Navier-Stokes flow both the cases $\Omega = 0$ and $\Omega \neq 0$.

The aim of this note is to give an overview of the paper Giga-Inui-Mahalov-Matsui [6], where we found the new space denoted by FM_0 in which the Riesz semigroup is bounded with uniform bound in $t \in \mathbb{R}$. The space FM_0 is the Fourier preimage of the space M_0, which is the space of finite Radon measures having no-point mass at the origin. In the definition of FM_0 we do not use the Littlewood-Paley decomposition (1.3). Instead, we treat only homogeneous finite Radon measures, elements of M_0, a subspace of the space of all finite Radon measures that we denote by M, when we operate the Riesz semigroup $\exp(tR_j)$ whose symbol $\exp(t(i\xi_j/|\xi|))$ is not continuous at the origin.

The inclusion of the spaces introduced above is as follows;

$$FM_0 \subset \dot{B}^0_{\infty,1} \subset BUC.$$

The both inclusions are continuous and strict, in fact, nonzero constant functions belong to $BUC \setminus \dot{B}^0_{\infty,1}$, while $f = h * e^{ix} i \text{ sgn } x \in L^\infty(\mathbb{R})$ belongs to $\dot{B}^0_{\infty,1} \setminus FM_0$ in the case $n = 1$ when h is a smooth bounded function whose Fourier transform $\hat{h}$ is supported in $\{\xi \in \mathbb{R}\ ;\ |\xi - 1| \leq 1/2\}$ and $\text{sgn}(s)$ denotes the signature function (see [6, Appendix]).

The space FM_0 still contains the functions of the form (1.4) and is closed on dilation like L^∞. In FM_0 we could prove uniform estimate for the Riesz semigroup with the bound 1, the supremum norm of the symbol $\exp(t(i\xi_j/|\xi|))$ (Proposition 4.1, Corollary 4.1). The estimate makes key role in proof of main result, local-in-time existence and uniqueness with its uniform existence time.

The notion of the spaces FM_0 or FM is implicitly hinted by a lemma in proof of Mikhlin's theorem in the Besov spaces. To prove the boundedness of multiplier $F^{-1}mF$ in $\dot{B}^0_{\infty,1}$, one needs to estimate of FL^1-norm of $m\widehat{\phi_j}$, multiplication of the symbol m and the decomposition $\widehat{\phi_j}$. When one formally replaces L^1 of the letters FL^1 by the space M which can be regarded as the dual space of $C_\infty(\mathbb{R}^n)$, the space of all continuous functions decaying at space infinity equipped with L^∞-norm, we have FM. In appendix we give a short observation on Mikhlin's theorem in the Besov spaces.

To show global solubility Giga-Inui-Mahalov-Saal [7] recently utilized the exponential decay estimate in the space FM for the heat semigroup $e^{t\Delta}$ such that

$$||e^{t\Delta}f||_{FM} \leq e^{-t\delta^2}||f||_{FM} \qquad \text{for } t>0,\ f\in FM,$$

where the constant $\delta > 0$ denotes distance of support of $\widehat{f} = (Ff)(\xi)$ from the origin in the phase space. In the contraction mapping argument handling the nonlinear term it is required that distance of support from the origin doesn't shrink after multiplication, that is, there must be a set $F_\delta \subseteq \mathbb{R}^3$ whose distance from the origin is $\delta > 0$ satisfying that

$$\text{supp } \widehat{f} \subseteq F_\delta,\ \text{supp } \widehat{g} \subseteq F_\delta \qquad \text{imply} \qquad \text{supp } \widehat{fg} \subseteq F_\delta \cup \{0\}.$$

This property is fulfilled by a "sum-closed set", which is a closed set $F \subseteq \mathbb{R}^3 \setminus \{0\}$ satisfying $F+F := \{x+y;\ x,y\in F\} \subseteq F \cup \{0\}$. We showed existence of global-in-time regular solutions for initial data $u_0 \in FM_\delta$, that is, for $u_0 \in FM$ whose support is in a sum-closed set F_δ provided the distance $\delta > 0$ of F_δ from the origin is sufficiently large. One can see another application of the space FM_δ to general parabolic systems in Giga-Mahalov-Saal [8].

This note is organised as follows. In Section 2 we define the spaces for measures M, M_0, and their Fourier preimage spaces FM and FM_0. The main results are given in Section 3. In Section 4 we explain how the Riesz semigroup arises in the analysis of the linearized equations of (1.1) and give the key uniform estimate. Since the sections above are summary or extracts of [6] or [10], we do not repeat proofs nor details. In Appendix A, we give an observation on Mikhlin's theorem in the Besov spaces.

2 Key Function Spaces

In this section we define four spaces for measures, which make key role in the main results. The main results (in Section 3) on the Navier-Stokes equations with the Coriolis term are valid only for 3-dimensional case, however, definition is given in general space dimension $n = 1, 2, 3, \ldots$.

Before defining spaces, we introduce Fourier transform and inverse Fourier transform by

$$F\,f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx,$$
$$F^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi,$$

where, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi \cdot x$ denotes the standard inner product in $\mathbb{R}^n$.

Then we define

$$M = \{\mu;\ \mathbb{R}^n \to \mathbb{C};\ \text{finite Radon measure}\}$$

with the total variation norm $||\mu||_M$. It is known by the Riesz representation theorem that M is characterized by the dual space of $C_\infty(\mathbb{R}^n)$, where $C_\infty(\mathbb{R}^n)$ is the space of all continuous functions $\psi(x)$ satisfying $\lim_{|x|\to\infty}\psi(x) = 0$ equipped with the supremum norm $||\cdot||_\infty$. Hence, we also have $M \subset \mathcal{S}'$ because the inclusion $\mathcal{S} \subset C_\infty$ is dense. Here, by $\mathcal{S}$ we denote the class of rapidly decreasing functions and by $\mathcal{S}'$ its topological dual space, the space of tempered distributions.

On the other hand, we also define the subspace M_0 of M by

$$M_0 = \{\mu \in M; \mu\lfloor\{0\} = 0\}. \tag{2.1}$$

Here, a Radon measure $\mu\lfloor\psi$ for a bounded Borel measurable function ψ is defined by

$$(\mu\lfloor\psi)(O) = \int_O \psi(\xi)\mu(d\xi)$$

for an open set O. For a Borel set B we simply write $\mu\lfloor\chi_B$ by $\mu\lfloor B$ if ψ is a characteristic function χ_B of a Borel set B. Hence, the condition $\mu\lfloor\{0\} = 0$ in (2.1) implies that the Radon measure μ has no point mass at the origin.

Next we define two spaces for measures which belong to M and M_0 in the phase space. The first one is

$$FM = \{f \in \mathcal{S}'; (F^{-1}f)(x) \in M\}.$$

The space FM is the Fourier image of M, and is the same as the preimage;

$$FM = \{f \in \mathcal{S}'; (Ff)(\xi) \in M\}$$

since $(F^{-1}f)(x) = F(f(-\xi)(x))$. Similarly, we define the preimage of M_0 by

$$FM_0 = \{f \in \mathcal{S}'; (Ff)(\xi) \in M_0\}.$$

In the spaces FM and FM_0 we employ the norm $||f||_{FM} = (2\pi)^{-n/2}||Ff||_M$.

The space FM contains constant functions since $||1||_{FM} = ||F(1)||_M = ||\delta(\xi)||_M = 1$, where δ denotes the Dirac delta measure. However, FM_0 does not contain nonzero constant functions. In fact, $\delta(\xi) \notin M_0$. Moreover, FM has a topological direct sum decomposition $FM = FM_0 \oplus \mathbb{C}$.

On the other hand, FM_0 contains the functions in the form (1.4), which does not decay at space infinity. In fact, we have

$$||\sum_{j=1}^{\infty} \alpha_j e^{i\lambda_j \cdot x}||_{FM} = ||\sum_{j=1}^{\infty} \alpha_j \delta(\xi - \lambda_j)||_M \leq \sum_{j=1}^{\infty} |\alpha_j| ||\delta(\xi - \lambda_j)||_M = \sum_{j=1}^{\infty} |\alpha_j| < \infty$$

since $\delta(\xi - \lambda_j)$ for $\lambda_j \neq 0$ has no point mass at the origin, hence belongs to M_0 and its total variation norm is 1. We also remark that FM_0 is closed on dilation, that is, if $f \in FM_0$, then $f(\lambda\cdot)$ for $\lambda \neq 0$ also belongs to FM_0. It is because $F(f(\lambda\cdot))(\xi) = \frac{1}{\lambda^n}(Ff)(\frac{\xi}{\lambda})$.

3 Main Results

Theorem 3.1. [6, Theorem 1.1] *Assume that $u_0 \in FM_0$ with div $u_0 = 0$. Then there exist $T_0(\geq c/||u_0||^2_{FM}) > 0$ independent of the Coriolis parameter Ω and a unique mild solution $u = u(t) \in C([0, T_0]; FM_0)$ of (1.1), where the numerical constant c equals to $e/576$.*

To state the second theorem we denote by BMO the space of functions of bounded mean oscillation.

Theorem 3.2. **[6, Theorem 1.2]** *Assume that $u_0 \in FM_0$ with div $u_0 = 0$.*
(1) *Let $u = u(t)$ be the mild solution obtained in Theorem 3.1. If we set the pressure $p = p(x,t)$ as*

$$\partial_i p(t) = \partial_i \sum_{j,k=1}^{3} R_j R_k u^j u^k(t) + \Omega R_i \ (R_2 u^1 - R_1 u^2)(t) \tag{3.1}$$

for $t > 0$ and $i = 1, 2, 3$, then the pair $(u, \nabla p)$ is a classical solution of (1.1).
(2) *Let $u \in L^\infty((0,T)\times\mathbb{R}^3)$ and $p \in L^1_{loc}((0,T); BMO)$ be a solution of (1.1) in the distributional sense for some $T > 0$. Then the pair $(u, \nabla p)$ is unique. Moreover, the relation (3.1) holds.*

4 The Poincaré-Riesz semigroup and the Riesz semigroup

In this section we see that the Riesz semigroup arises in the analysis of the Poincaré-Sobolev equations, the linearised equations of (1.1), which have the form;

$$\text{(PS)} \qquad u_t + \Omega e_3 \times u - \Delta u = -\nabla p, \quad \nabla \cdot u = 0, \quad u|_{t=0} = u_0.$$

Multiplying the Helmholtz operator $\mathbf{P} = (\delta_{ij} + R_i R_j)_{1\le i,j\le 3}$, the equations (PS) are transformed into the abstract ordinary equation

$$u_t - \Delta u + \Omega \mathbf{PJ} u = 0 \qquad \text{for } t > 0, \qquad u|_{t=0} = u_0. \tag{4.1}$$

Because of $\mathbf{P}u = u$, Instead of (4.1), we can deal rather

$$u_t - \Delta u + \Omega \mathbf{PJP} u = 0 \qquad \text{for } t > 0, \qquad u|_{t=0} = u_0, \tag{4.2}$$

whose the solution operator is given by

$$\exp(-\mathbf{A}(\Omega)t) = \exp(t\Delta)\exp(-\Omega \mathbf{S} t),$$

where $\mathbf{A}(\Omega) = -\Delta + \Omega\mathbf{S}$ and $\mathbf{S} := \mathbf{PJP}$ is the Poincaré-Riesz operator. The advantage of the operator $\mathbf{PJP}$ than $\mathbf{PJ}$ is its skew-symmetry;

$$\mathbf{S}(\xi) \equiv \mathbf{P}(\xi)\mathbf{J}\mathbf{P}(\xi) = \left(\frac{\xi_3}{|\xi|}\right)\mathbf{R}(\xi),$$

where the operator $\mathbf{R}$ is given by;

$$\sigma(\mathbf{R}) \equiv \mathbf{R}(\xi) = \begin{pmatrix} 0 & -\frac{\xi_3}{|\xi|} & \frac{\xi_2}{|\xi|} \\ \frac{\xi_3}{|\xi|} & 0 & -\frac{\xi_1}{|\xi|} \\ -\frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & 0 \end{pmatrix}. \qquad \text{(skew-symmetry)}$$

Here, $\sigma(T)$ denotes the symbol of an operator T. Also it is known ([2], [3]) that $\exp(\Omega\mathbf{S}t)$ (the Poincaré-Riesz semigroup) is given by the symbol

$$\exp(\Omega\mathbf{S}(\xi)t) = \cos(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{I} + \sin(\frac{\xi_3}{|\xi|}\Omega t)\mathbf{R}(\xi), \tag{4.3}$$

where $\mathbf{I}$ is the 3×3 identity matrix.

When one discusses the existence time of solution, we remark that dependence on the parameter Ω of the symbol (4.3) appears only in scalar coefficients $\cos(\frac{\xi_3}{|\xi|}\Omega t)$ and $\sin(\frac{\xi_3}{|\xi|}\Omega t)$, which are functions of the Riesz operator R_3. Therefore, noticing that the functions $\cos(x)$ and $\sin(x)$ can be represented by $\exp(ix)$ and $\exp(-ix)$, we thus essentially need the estimate in the following form to solve the ordinary differential equation (4.2) in time variable t in some function space X.

$$||F^{-1}(\exp(\Omega ti\frac{\xi_3}{|\xi|})Ff)||_X \leq C(t)||f||_X, \qquad t \in \mathbb{R},\ f \in X$$

for some $C(t) > 0$. Replacing Ωt by t for simplicity, we reduce the above estimate to the following estimate for our target operator (1.2);

$$||\exp(tR_3)f||_X \leq C(t)||f||_X \qquad \text{for } t \in \mathbb{R},\ f \in X. \tag{4.4}$$

In this note we discuss the case of $f \in FM_0$, where the dependency of $C(t)$ can be taken 1, uniform in t. To deal with the operator $\exp(tR_3)$ whose symbol is $\exp(t(i\xi_3/|\xi|))$, we need define for $f \in FM_0$ the operators whose symbol may not be continuous at the origin.

Let $\sigma \in C(\mathbb{R}^n \setminus \{0\})$. Define the operator Σ by

$$\Sigma f := F^{-1}((Ff)\lfloor\sigma). \tag{4.5}$$

Then, we have

Proposition 4.1. **[6, Lemma 2.2]** *Assume that $\sigma \in C(\mathbb{R}^n\backslash\{0\})$ is bounded in $\mathbb{R}^n\backslash\{0\}$. Then Σ defined in (4.5) is a bounded linear operator in FM_0 and*

$$\| \Sigma f \|_{FM} \leq \| \sigma \|_\infty \| f \|_{FM} \tag{4.6}$$

for $f \in FM_0$. If, furthermore, σ is continuous at the origin, then Σ is a bounded linear operator in FM and (4.6) holds for all $f \in FM$.

This proposition is proved as follows.

$$\begin{aligned} ||\Sigma f||_{FM} &\leq ||F^{-1}((Ff)(\xi)\lfloor\sigma(\xi))(x)||_{FM} \\ &= ||(Ff)(\xi)\lfloor\sigma(\xi)||_M \\ &\leq ||\sigma||_\infty||Ff||_M = ||\sigma||_\infty||f||_{FM}. \end{aligned}$$

Then, the Riesz operator and the Riesz semigroup, which are defined for $f \in FM_0$ by

$$\begin{aligned} R_j f &= F^{-1}((Ff)\lfloor\sigma(R_j)), \qquad \sigma(R_j) = i\frac{\xi_j}{|\xi|} \\ \exp(tR_j)f &= F^{-1}((Ff)\lfloor\sigma(\exp(tR_j))), \qquad \sigma(\exp(tR_j)) = \exp(it\frac{\xi_j}{|\xi|}) \end{aligned}$$

for $j = 1, 2, \cdots, n$, can be estimated uniformly in $t \in \mathbb{R}$ as follows.

Corollary 4.1. *The Riesz operator R_j and the Riesz semigroup e^{tR_j} are bounded in FM_0 for $j = 1, 2, \cdots, n$. Moreover,*

$$\text{(i)} \parallel R_j f \parallel_{FM} \leq \parallel f \parallel_{FM}, \qquad \text{(ii)} \parallel e^{tR_j} f \parallel_{FM} \leq \parallel f \parallel_{FM} \quad t \in \mathbb{R}$$

for all $f \in FM_0$.

This is trivial since the symbols of the operators are bounded with the bound 1.

Remark 4.1. The operators in the corollary R_j and e^{tR_j} are unbounded in FM since the symbols have singularity at the origin. We also note that they are unbounded in the inhomogeneous Besov spaces $B^s_{p,q}$ for all indices $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.

The above corollary cannot apply directly the Coriolis solution operator $\exp(\Omega \mathbf{S} t)$ or the Helmholtz operator $\mathbf{P}$ which are matrix operators. Although we do not mention the operator for vector measure whose symbol is matrix, here, it is shown in [6, Lemma 2.9] that the operators $\exp(\Omega \mathbf{S} t)$, $\mathbf{P}$ are also uniform bounded in t with the constant 1 as well as definition of the space for vector measures, $FM(\mathbb{R}^n)^d$, $FM(\mathbb{R}^n)^d$ for $d \in \mathbb{N}$. The theorems in Section 3 are proved by standard iteration scheme using the above estimates.

A Appendix: The Mikhlin theorem in the Besov-type spaces

In this appendix we review the proof of the Mikhlin theorem in the Besov spaces from Amann [1], in particular, the case of the homogeneous Besov space $\dot{B}^0_{\infty,1}$. We extract one of steps of the proof as one lemma which invoked the notion of the spaces FM and FM_0 discussed in the preceding sections.

Let X and Y be Banach spaces and assume that the topological dual of X is Y. We also assume that both of the pair X and Y are included in $\mathcal{S}'$ so that we can define their Fourier preimage spaces FX and FY. Here, similarly as in Section 2, for a Banach space $Z \subset \mathcal{S}'$ we define

$$FZ = \{f \in \mathcal{S}'; (F^{-1}f)(x) \in Z\},$$

with the norm $||f||_{FZ} = (2\pi)^{-n/2} ||Ff||_Z$.

Typical examples of the pair (X, Y) are

$$(L^p, L^q) \text{ with } 1 < p, q < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \qquad (L^1, L^\infty), \qquad (C_\infty, M), \qquad (\mathcal{H}^1, BMO).$$

Here, $\mathcal{H}^1$ and BMO denote the Hardy space and the space of functions of bounded mean oscillation, respectively. Although elements in BMO can not be viewed as in $\mathcal{S}'$, we regard the elements modulo constants to take their Fourier transform.

Before describing the lemma we give the definition of $\dot{B}^0_{Z,1}$, the Besov-type space based on a Banach space $Z \subset \mathcal{S}'$ by

$$\dot{B}^0_{Z,1} := \{f \in \mathcal{Z}'; ||f; \dot{B}^0_{Z,1}|| < \infty\},$$

Here, by $\mathcal{Z}'$ we denote the topological dual space of the space $\mathcal{Z}$, which is defined by $\mathcal{Z} := \big\{ f \in \mathcal{S};\ D^\alpha \hat{f}(0) = 0$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)\big\}$ and the norm is defined by

$$||f; \dot{B}^0_{Z,1}|| := \sum_{j=-\infty}^{\infty} ||\phi_j * f||_Z.$$

Here, $\{\phi_j\}_{j=-\infty}^{\infty}$ is the Littlewood-Paley dyadic decomposition satisfying $\widehat{\phi_j}(\xi) = \widehat{\phi_0}(2^{-j}\xi) \in C_c^\infty(\mathbb{R}^n)$ with

$$\mathrm{supp}\widehat{\phi_0} \subset \{1/2 < |\xi| < 2\}, \qquad \sum_{j=-\infty}^{\infty} \widehat{\phi_j}(\xi) = 1 \quad \text{for } \xi \neq 0.$$

In particular, the case $Z = L^p \quad (1 \le p \le \infty)$, the Besov-type space $\dot{B}^0_{Z,1}$ becomes the usual Besov space $\dot{B}^0_{p,1}$, which we denote by $\dot{B}^0_{L^p,1}$ in this note.

Remark A.1. Let $X \subset Y$, that is X is continuously embedded in Y. Then we have
(i) $FX \subset FY$, and
(ii) $\dot{B}^0_{X,1} \subset \dot{B}^0_{Y,1}$.

This is trivial since $|| \cdot ||_Y \le || \cdot ||_X$ implies $||Ff||_Y \le ||Ff||_X$ and

$$\sum_{j=-\infty}^{\infty} ||\phi_j * f||_Y \le \sum_{j=-\infty}^{\infty} ||\phi_j * f||_X.$$

Similarly, we also have $B^0_{X,1} \subset B^0_{Y,1}$ if $X \subset Y$. (See [4] for definition of the inhomogeneous Besov spaces $B^0_{X,1}$. The argument in this section can be repeated for $B^0_{X,1}$.)

Then we extract one of steps of proof of the Mikhlin theorem as the following lemma.

Lemma A.1. *Assume $m \in \mathcal{S}'$ and that the dual space of X is Y. Let $m\widehat{\phi_j} \in FX$ and*

$$\sup_{j \in \mathbb{Z}} ||m\widehat{\phi_j}||_{FX} \le C \qquad \textit{for some } \ C > 0. \tag{A.1}$$

Then the operator $T = F^{-1}mF$ is bounded from $\dot{B}^0_{Y,1}$ to $\dot{B}^0_{L^\infty,1}$.

Proof. Since $\sum_{j=-\infty}^{\infty} \widehat{\phi_j} = 1$ (except at 0) and $\mathrm{supp}\widehat{\phi_j} \cap \mathrm{supp}\widehat{\phi_k} = \emptyset$ for $|j-k| \ge 3$, we calculate

$$\begin{aligned} ||F^{-1}mFf||_{\dot{B}^0_{L^\infty,1}} &= \sum_{k=-\infty}^{\infty} ||\phi_k * (F^{-1}mFf)||_{L^\infty} \\ &\le \sum_{k=-\infty}^{\infty} \sum_{j=k-2}^{k+2} ||\phi_j * (F^{-1}mFf) * \phi_k||_{L^\infty}. \end{aligned}$$

It follows from $\phi_j * (F^{-1}mFf) = \phi_j * (F^{-1}m) * f = (F^{-1}(m\widehat{\phi_j})) * f$ that

$$||F^{-1}mFf||_{\dot{B}^0_{L^\infty,1}} \le \sum_{k=-\infty}^{\infty} \sum_{j=k-2}^{k+2} ||(F^{-1}(m\widehat{\phi_j})) * f * \phi_k||_{L^\infty}.$$

Then the duality $(X)' = Y$ implies that

$$\begin{aligned} ||F^{-1}mFf||_{\dot{B}^0_{L^\infty,1}} &\le \sum_{k=-\infty}^{\infty} \sum_{j=k-2}^{k+2} ||F^{-1}(m\widehat{\phi_j})||_X ||f * \phi_k||_Y \\ &= \sum_{k=-\infty}^{\infty} \sum_{j=k-2}^{k+2} ||m\widehat{\phi_j}||_{FX} ||f * \phi_k||_Y \\ &\le 5 \sup_{j\in\mathbb{Z}} ||m\widehat{\phi_j}||_{FX} \sum_{k=-\infty}^{\infty} ||f * \phi_k||_Y \le 5C||f||_{\dot{B}^0_{Y,1}}. \end{aligned}$$

We have proved Lemma A.1. □

Remark A.2. Obviously, the above lemma holds even if we exchange the roles of X and Y, that is, if $\sup_{j\in\mathbb{Z}} ||m\widehat{\phi_j}||_{FY} \le C$, then the operator $T = FmF^{-1}$ is bounded from $\dot{B}^0_{X,1}$ to $\dot{B}^0_{L^\infty,1}$.

In the case $X = L^1$, the quantity $||m\widehat{\phi_j}||_{FL^1}$ appearing in the condition (A.1) can be controlled by the following quantity μ_j determined only by the symbol m without the decomposition $\{\widehat{\phi_j}\}$;

Proposition A.1. [1, Lemma 4.2] *Let k be an integer satisfying $k > n/2$. Assume that*

$$\mu_j := \max_{|\alpha|\le k} \sup_{2^{j-1}\le|\xi|\le 2^{j+1}} |\xi|^\alpha |D^\alpha m(\xi)| < \infty \qquad \textit{for some } j \in \mathbb{Z}.$$

Then, $m\widehat{\phi_j} \in FL^1$ and

$$||m\widehat{\phi_j}||_{FL^1} \le C\mu_j. \tag{A.2}$$

Here, $C = C(k) > 0$ is independent of m and j.

The above proposition is the key for the proof of the Mikhlin theorem in the Besov spaces.

Corollary A.1. [1, Theorem 6.2] (Mikhlin's Theorem) *Let k be an integer satisfying $k > n/2$. Assume $m \in C^k(\mathbb{R}^n \setminus \{0\})$ satisfies*

$$K_m := \max_{|\alpha|\le k} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{|\alpha|} |D^\alpha m(\xi)| < \infty. \tag{A.3}$$

Then $F^{-1}mF$ is a bounded operator in $\dot{B}^0_{L^\infty,1}$ and moreover

$$||F^{-1}mFf||_{\dot{B}^0_{L^\infty,1}} \le CK_m ||f||_{\dot{B}^0_{L^\infty,1}},$$

where $C = C(k) > 0$ is independent of m.

Proof. Since $\{\xi \in \mathbb{R}^n;\ \xi \ne 0\} = \cup_{j\in\mathbb{Z}}\{\xi \in \mathbb{R}^n;\ 2^{j-1} \le |\xi| \le 2^{j+1}\}$, the assumption (A.3) implies $\sup_{j\in\mathbb{Z}} \mu_j < K_m < \infty$. Hence, by (A.2) we have

$$\sup_{j\in\mathbb{Z}} ||m\widehat{\phi_j}||_{FL^1} \le C \sup_{j\in\mathbb{Z}} \mu_j < CK_m < \infty.$$

Then, applying Lemma A.1 to the case $X = L^1$, we have proved the conclusion. $\square$

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