On a linearized system arising in the study of Bénard - Marangoni convection

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1 Introduction

Let $\mathbb{T}^2 = \mathbb{R}^2 / \left(\frac{2\pi}{a}\mathbb{Z} \times \frac{2\pi}{b}\mathbb{Z} \right)$, $a\,,\, b\,>0$, be the two dimensional torus for the horizontal coordinates, and let $\Omega = \mathbb{T}^2 \times I$ where I is the interval $\{x_3\,;\; 0 < x_3 < 1\}$. We denote the boundary $\partial\Omega$ by $S_F = \mathbb{T}^2 \times \{\,x_3 = 1\,\}$ and $S_B = \mathbb{T}^2 \times \{\,x_3 = 0\,\}$. In this article we are concerned with the linear nonstationary problem

$$\frac{\partial h}{\partial t} - v_3 = g_0 \qquad \text{on } S_F\ , \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta - v_3 = f_0\ , \tag{1.2}$$

$$\frac{1}{P_r}\frac{\partial \boldsymbol{v}}{\partial t} - \Delta\boldsymbol{v} + \nabla p - R_a\,\theta\boldsymbol{e} = \boldsymbol{f}\ , \quad \operatorname{div}\boldsymbol{v} = 0 \qquad \text{in } \Omega\ , \tag{1.3}$$

$$\theta = 0 \quad , \quad \boldsymbol{v} = \boldsymbol{0} \qquad \text{on } S_B\ , \tag{1.4}$$

$$\frac{\partial \theta}{\partial x_3} + B_i\,(\theta - h) = b_0\ , \tag{1.5}$$

$$\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} + M_a\left(\frac{\partial \theta}{\partial x_j} - \frac{\partial h}{\partial x_j}\right) = b_j\ ,\ j = 1, 2, \tag{1.6}$$

$$p - 2\frac{\partial v_3}{\partial x_3} - (G_a - C_a\varDelta_F)\,h = b_3\ ,\ \text{on } S_F\ . \tag{1.7}$$

Here $\boldsymbol{e} = (0,0,1)^T$ and $\varDelta_F = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$. This problem arises in the linearization around the basic heat conductive state of the Bénard - Marangoni convection with deformable upper surface. P_r, R_a, M_a, G_a, B_i and C_a are the constants appearing in nondimensionalizing the physical model. Except M_a, B_i these are positive, and $B_i \geq 0$. See, for example, [5] or [6] for the physical model. The unknown h on S_F corresponds to an upper free surface in the original full nonlinear problem. The unknowns v and p denote velocity vector and scalar pressure respectively. θ denotes disturbances from the basic linear heat distribution. Most of this article is devoted to study the linear stationary problem with complex parameter λ

$$\lambda h - v_3 = g_0 \qquad \text{on } S_F\ , \tag{1.8}$$

$$\lambda\theta - \Delta\theta - v_3 = f_0 \qquad \text{in } \Omega \tag{1.9}$$

$$\frac{\lambda}{P_r}\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla p - R_a\,\theta\boldsymbol{e} = \boldsymbol{f}\ , \quad \operatorname{div}\boldsymbol{v} = 0 \qquad \text{in } \Omega\ , \tag{1.10}$$

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$$\theta = 0 \quad , \quad \boldsymbol{v} = \mathbf{0} \qquad \text{on } S_B \; , \tag{1.11}$$

$$\frac{\partial\theta}{\partial x_3} + B_i\,(\theta - h) = 0 \; , \tag{1.12}$$

$$\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} + M_a \left(\frac{\partial\theta}{\partial x_j} - \frac{\partial h}{\partial x_j} \right) = 0 \; , \; j = 1, 2, \tag{1.13}$$

$$p - 2\frac{\partial v_3}{\partial x_3} - (G_a - C_a \Delta_F\,)\, h = 0 \; , \text{ on } S_F \; . \tag{1.14}$$

This problem is closely related to construction of the resolvent operator when we rewrite (1.1) – (1.7) in the form of evolution equation. We solve the problem (1.8) – (1.14) for λ with sufficiently large real part, and derive the estimate which shows that the linear operator appearing in the form of evolution equation has compact resolvent and generates the analytic semigroup in some function space. These facts are crucial in bifurcation analysis for the model system of Bénard - Marangoni convection, which we will publish separately. We here apply our result to the initial boundary value problem for full nonlinear system, and show in the final section that global in time solution exists uniquely for sufficiently small initial data.

We introduce some function spaces. For $\ell \geq 0$ $H^\ell(\Omega)$ denotes the usual Sobolev space or its generalization based on $L^2(\Omega)$. $H^\ell(\mathbb{T}^2)$ is defined in a similar way, and set

$$H_0^\ell(\mathbb{T}^2) = \left\{ \phi \in H^\ell(\mathbb{T}^2) \; ; \; \int_{\mathbb{T}^2} \phi(x')dx' = 0 \; , \; x' = (x_1, x_2) \right\} \; .$$

We also introduce the spaces

$$K^\ell(\Omega \times (0,T)) = H^0\left(0,T; H^\ell(\Omega)\right) \cap H^{\frac{\ell}{2}}\left(0,T; H^0(\Omega)\right)$$

and

$$K_0^{\ell+\frac{1}{2}}\left(\mathbb{T}^2 \times (0,T)\right) = H^0\left(0,T; H_0^{\ell+\frac{1}{2}}(\mathbb{T}^2)\right) \cap H^{\frac{\ell+1/2}{2}}\left(0,T; H_0^0(\mathbb{T}^2)\right) \; ,$$

where $\ell \geq 1$. We also set $K_{(0)}^\ell(\Omega \times (0,T)) = \{ f \in K^\ell(\Omega \times (0,T)) \; ; \; f(\cdot,0) = 0 \}$ for $\ell \geq 1/2$. We define $K_{0,(0)}^{\ell+\frac{1}{2}}\left(\mathbb{T}^2 \times (0,T)\right)$ similarly. See [8] for these.

2 Formulation of the problem

To formulate our problem in the form of an evolution equation in some function spaces, we use the orthogonal projection introduced by J.T. Beale [3]. Set $\mathcal{G}^0 = \left\{ \nabla\phi \; ; \; \phi \in H^1(\Omega) \; , \; \phi = 0 \text{ on } S_F \right\}$. Denote the projection orthogonal to $\mathcal{G}^0$ by $\mathbb{P}^0$ from $L^2(\Omega)$ to $\left(\mathcal{G}^0\right)^\perp$. We cite Lemma 3.1 in [3] .

Lemma 2.1. *Let $\ell \geq 0$. $\mathbb{P}^0$ is a bounded operator on $H^\ell(\Omega)$ and $K^\ell(\Omega \times (0,T))$. If $\phi \in H^{\ell+1}(\Omega)$, then $\mathbb{P}^0(\nabla\phi) = \nabla\psi$, where ψ satisfies*

$$\Delta\psi = 0 \text{ in } \Omega \; , \quad \psi = \phi \text{ on } S_F \; , \quad \partial_n\psi = 0 \text{ on } S_B \quad . \tag{2.1}$$

By this lemma, if we apply $\mathbb{P}^0$ to (1.3) and recover p on S_F from (1.7), we have

$$\frac{\partial\boldsymbol{v}}{\partial t} - P_r\mathbb{P}^0\Delta\boldsymbol{v} + P_r\left(\nabla p_1 + \nabla p_2\right) - P_r R_a \mathbb{P}^0\theta\boldsymbol{e} = P_r\mathbb{P}^0\boldsymbol{f} - P_r\nabla p_3 \; ,$$

where

$$\Delta p_j = 0 \text{ in } \Omega \; , \; \frac{\partial p_j}{\partial x_3} = 0 \text{ on } S_B \; , \; j = 1, 2, 3,$$

$$p_1 = 2\frac{\partial v_3}{\partial x_3} \; , \; p_2 = (G_a - C_a\Delta_F\,)\, h \; , \; p_3 = b_3 \text{ on } S_F \; .$$

The system (1.1), (1.2) and (1.3) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} h \\ \boldsymbol{v} \\ \theta \end{pmatrix} - G\begin{pmatrix} h \\ \boldsymbol{v} \\ \theta \end{pmatrix} = \begin{pmatrix} g_0 \\ P_r\mathbb{P}^0\boldsymbol{f} - P_r R^* b_3 \\ f_0 \end{pmatrix} \; . \tag{2.2}$$

Here

$$G \;=\; \begin{pmatrix} 0 & R & 0 \\ -P_r R^* \left(G_a - C_a \Delta_F\right) & -A & P_r R_a \mathbb{P}^0(\,\cdot\,\boldsymbol{e}) \\ 0 & (\,\cdot\;,\;\boldsymbol{e}) & \Delta \end{pmatrix}$$

where $R\,:\,\boldsymbol{v} \to v_3\,|_{S_F}$ is the restriction of the third component of $\boldsymbol{v}$ to the upper boundary S_F. R^* is the formal adjoint to R defined by $R^*\phi = \nabla q$, where

$$\Delta q = 0 \quad \text{in } \Omega, \quad \frac{\partial q}{\partial x_3} = 0 \text{ on } S_B \quad , \quad q = \phi \quad \text{on } S_F \; ,$$

and

$$-A\boldsymbol{v} \;=\; P_r\left(\mathbb{P}^0\Delta\boldsymbol{v} - R^* 2\frac{\partial v_3}{\partial x_3}\right) \; .$$

We need the solution to the the following boundary value problem with the homogeneous boundary data and inhomogeneous right hand sides.

$$\frac{\lambda}{P_r}\boldsymbol{v}^{(0)} \;-\; \Delta\boldsymbol{v}^{(0)} \;+\; \nabla q^{(0)} \;=\; \boldsymbol{f} \qquad \text{in } \Omega \; , \tag{2.3}$$

$$\text{div}\;\boldsymbol{v}^{(0)} \;=\; 0 \qquad \text{in } \Omega \; , \tag{2.4}$$

$$\boldsymbol{v}^{(0)} \;=\; \boldsymbol{0} \qquad \text{on } S_B \; , \tag{2.5}$$

$$\frac{\partial v_j^{(0)}}{\partial x_3} \;+\; \frac{\partial v_3^{(0)}}{\partial x_j} \;=\; 0\;, \qquad j = 1\;,\;2\;, \qquad \text{on } S_F \; , \tag{2.6}$$

$$q^{(0)} \;-\; 2\,\frac{\partial v_3^{(0)}}{\partial x_3} \;=\; 0 \qquad \text{on } S_F \; , \tag{2.7}$$

where $\boldsymbol{f}$ is given in $\mathbb{P}^0 H^{\ell-2}\left(\Omega\right)$. To discuss this, we use the following integral identity: Suppose $\boldsymbol{v}, \boldsymbol{u} \in H^2\left(\Omega\right)$, $q \in H^1\left(\Omega\right)$ and div $\boldsymbol{v} = 0$. Then integration by parts gives

$$\int_\Omega \left(-\,\Delta\boldsymbol{v} + \nabla q\right)\;\boldsymbol{u}^*\,dx = \langle\;\boldsymbol{v}\;,\;\boldsymbol{u}\;\rangle + \int_{\partial\Omega} n_j S_{jk}(v,q) u_k^*\,dS - \int_\Omega q\,\text{div}\,\boldsymbol{u}^*\,dx\;, \tag{2.8}$$

where

$$\langle\;\boldsymbol{v}\;,\;\boldsymbol{u}\;\rangle = \sum_{j,k}\;\int_\Omega \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\right)\;\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)^*\,dx$$

and $S_{jk}(v,q) = q\delta_{jk} - \left(\dfrac{\partial v_j}{\partial x_k} + \dfrac{\partial v_k}{\partial x_j}\right)$. Here and hereafter $\{\cdot\}^*$ denotes the complex conjugate of $\{\cdot\}$. For the solvability of the linear problem with homogeneous boundary conditions we use the lemma below.

Lemma 2.2. *For $\boldsymbol{u} \in H^1\left(\Omega\right)$ with $\boldsymbol{u} = \boldsymbol{0}$ on S_B, we have*

$$|\boldsymbol{u}|^2_{H^1(\Omega)} \;\leq\; C\,\langle\boldsymbol{u}\;,\;\boldsymbol{u}\rangle$$

with $C > 0$ independent of $\boldsymbol{u}$.

See Lemma 2.7 in [3] and [7] for of this lemma. Suppose $\boldsymbol{v}^{(0)}$ and $q^{(0)}$ satisfy (2.3),(2.4),(2.5),(2.6) and (2.7). Then, by the integral identity (2.8) it holds

$$\frac{\lambda}{P_r}\left(\boldsymbol{v}^{(0)}\;,\;\boldsymbol{v}\right)_{L^2} + \langle\boldsymbol{v}^{(0)}\;,\;\boldsymbol{v}\;\rangle \;=\; \left(\boldsymbol{f}\;,\;\boldsymbol{v}\right)_{L^2} \tag{2.9}$$

for any $\boldsymbol{v} \in H^1(\Omega)$ satisfying div $\boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} = \boldsymbol{0}$ on S_B. By Lemma 2.2 we see that the real part of the left hand side of (2.9) with $\boldsymbol{v}$ replaced by $\boldsymbol{v}^{(0)}$ is positive definite for any $\text{Re}\,\lambda \geq 0$. By the Lax – Milgram's lemma we first obtain a unique weak solution to the above problem. Since the boundary conditions (2.5) and (2.6), (2.7) satisfy the complementary condition of [2], we obtain the higher regularity of the weak solution. Thus we have

Proposition 2.3. *Suppose $\ell \geq 2$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. For a given $\boldsymbol{f} \in \mathbb{P}^0 H^{\ell-2}(\Omega)$ there is a unique solution $\boldsymbol{v}^{(0)}$, $q^{(0)}$ of (2.3) – (2.7), which satisfies*

$$\left|\boldsymbol{v}^{(0)}\right|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}} \left|\boldsymbol{v}^{(0)}\right|_{H^0(\Omega)} \leq C\left(|\boldsymbol{f}|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}} |\boldsymbol{f}|_{H^0(\Omega)}\right) ,$$

$$\left|\nabla q^{(0)}\right|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}} \left|\nabla q^{(0)}\right|_{H^0(\Omega)} + \left|q^{(0)}{}_{|S_F}\right|_{H^{\ell-3/2}(\mathbb{T}^2)}$$
$$+ |\lambda|^{\frac{\ell-3/2}{2}} \left|q^{(0)}{}_{|S_F}\right|_{H^0(\mathbb{T}^2)} \leq C\left(|\boldsymbol{f}|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}} |\boldsymbol{f}|_{H^0(\Omega)}\right) .$$

For the details of Proposition2.3, see Lemma 3.3 of [4]. To see how to recover $q^{(0)}$ we refer to Section 3 of [3] and [11].

3 Auxiliary linear problem

We first consider an auxiliary linear problem for given $a_0 \in H_0^{3/2}(\mathbb{T}^2)$ and $a_j \in H_0^{1/2}(\mathbb{T}^2)$, $j = 1, 2, 3$

$$\lambda h - v_3 = a_0 \qquad \text{on } S_F , \tag{3.1}$$

$$\frac{\lambda}{P_r}\boldsymbol{v} - \Delta \boldsymbol{v} + \nabla p = \boldsymbol{0} , \tag{3.2}$$

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } \Omega , \tag{3.3}$$

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } S_B , \tag{3.4}$$

$$\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} = a_j \ , \ j = 1, 2, \tag{3.5}$$

$$p - 2\frac{\partial v_3}{\partial x_3} + C_a \mathit{\Delta}_F h = a_3 , \quad \text{on } S_F . \tag{3.6}$$

To solve this we expand $h(x')$, $v(x', x_3)$ and $p(x', x_3)$ in the Fourier series in $x' \in \mathbb{T}^2$:

$$h(x') = \sum_{\mathbb{Z}^2} h^{(\ell,m)} e^{i(a\ell x_1 + bm x_2)} \quad , \quad v(x', y) = \sum_{\mathbb{Z}^2} v^{(\ell,m)}(y)\, e^{i(a\ell x_1 + bm x_2)} \quad ,$$
$$p(x', y) = \sum_{\mathbb{Z}^2} p^{(\ell,m)}(y)\, e^{i(a\ell x_1 + bm x_2)} .$$

Then we obtain the system of ordinary differential equations in the interval $I = \{0 < y < 1\}$

$$\frac{\lambda}{P_r} v_1^{(\ell,m)} - \left(\left(\frac{d}{dy}\right)^2 - |\xi|^2\right) v_1^{(\ell,m)} + ia\ell p^{(\ell,m)} = 0 , \tag{3.7}$$

$$\frac{\lambda}{P_r} v_2^{(\ell,m)} - \left(\left(\frac{d}{dy}\right)^2 - |\xi|^2\right) v_2^{(\ell,m)} + ibm p^{(\ell,m)} = 0 , \tag{3.8}$$

$$\frac{\lambda}{P_r} v_3^{(\ell,m)} - \left(\left(\frac{d}{dy}\right)^2 - |\xi|^2\right) v_3^{(\ell,m)} + \frac{d}{dy} p^{(\ell,m)} = 0 , \tag{3.9}$$

$$ia\ell\, v_1^{(\ell,m)} + ibm\, v_2^{(\ell,m)} + \frac{d}{dy} v_3^{(\ell,m)} = 0 , \qquad \text{in} \quad 0 < y < 1 . \tag{3.10}$$

supplemented with the boundary conditions

$$\lambda h^{(\ell,m)} - v_3^{(\ell,m)} = a_0^{(\ell,m)} , \tag{3.11}$$

$$ia\ell v_3^{(\ell,m)} + \frac{d}{dy} v_1^{(\ell,m)} = a_1^{(\ell,m)} , \tag{3.12}$$

$$ibm v_3^{(\ell,m)} + \frac{d}{dy} v_2^{(\ell,m)} = a_2^{(\ell,m)} , \tag{3.13}$$

$$p^{(\ell,m)} - 2\frac{d}{dy} v_3^{(\ell,m)} - C_a |\xi|^2 h^{(\ell,m)} = a_3^{(\ell,m)} \quad \text{on } y = 1 , \tag{3.14}$$

$$v_1^{(\ell,m)} = v_2^{(\ell,m)} = v_3^{(\ell,m)} = 0 \quad \text{on } y = 0 , \tag{3.15}$$

where $|\xi| = \sqrt{(a\ell)^2 + (bm)^2}$ and $a_j^{(\ell,m)}$ $(j = 0, 3)$ denotes the corresponding Fourier coefficient of the boundary data. For a while we set $(a\ell, bm) = (\xi_1, \xi_2)$. We see easily that $\boldsymbol{v}^{(0,0)} = \mathbf{0}$, $p^{(0,0)} = 0$, $h^{(0,0)} = 0$, so we assume $(\xi_1, \xi_2) \neq (0,0)$. The solution to the system (3.7) – (3.10) is written as follows

$$v_j^{(\ell,m)}(y) = C_1^j e^{-r(1-y)} - \frac{C_2^j}{r} e^{-ry} + \frac{i\xi_j}{|\xi|} C_1 e^{-|\xi|(1-y)} - \frac{i\xi_j}{|\xi|} C_2 e^{-|\xi| y} \ , \ j = 1, 2, \tag{3.16}$$

$$\begin{aligned} v_3^{(\ell,m)}(y) &= C_1 e^{-|\xi|(1-y)} + C_2 e^{-|\xi| y} \\ &\quad - \left(\frac{i\xi_1}{r} C_1^1 + \frac{i\xi_2}{r} C_1^2 \right) e^{-r(1-y)} - \left(\frac{i\xi_1}{r^2} C_2^1 + \frac{i\xi_2}{r^2} C_2^2 \right) e^{-ry} \ , \end{aligned} \tag{3.17}$$

$$p^{(\ell,m)}(y) = -\frac{\lambda}{P_r} \frac{1}{|\xi|} \left(C_1 e^{-|\xi|(1-y)} - C_2 e^{-|\xi| y} \right) \tag{3.18}$$

where $r = \sqrt{\dfrac{\lambda}{P_r} + |\xi|^2}$ with $|\arg r| < \dfrac{\pi}{4}$. $h^{(\ell,m)}$ in (3.14) is replaced by $h^{(\ell,m)} = \dfrac{1}{\lambda}\left(v_3^{(\ell,m)} + a_0^{(\ell,m)} \right)$ which comes from (3.11). Substituting (3.16), (3.17) and (3.18) we obtain the algebraic system for the coefficients $C_1, \ldots, C_2^2$. From (3.12), (3.13) and (3.14) we obtain

$$A_1 \begin{pmatrix} C_1 \\ C_1^1 \\ C_1^2 \end{pmatrix} + A_2 \begin{pmatrix} C_2 \\ C_2^1 \\ C_2^2 \end{pmatrix} = \begin{pmatrix} a_1^{(\ell,m)} \\ a_2^{(\ell,m)} \\ a_3^{(\ell,m)} + C_a \frac{|\xi|^2}{\lambda} a_0^{(\ell,m)} \end{pmatrix} \ , \tag{3.19}$$

where

$$A_1 = \begin{pmatrix} 2i\xi_1 & \dfrac{r^2 + \xi_1^2}{r} & \dfrac{\xi_1 \xi_2}{r} \\ 2i\xi_2 & \dfrac{\xi_1 \xi_2}{r} & \dfrac{r^2 + \xi_2^2}{r} \\ -\dfrac{r^2 + |\xi|^2}{|\xi|} - C_a \dfrac{|\xi|^2}{\lambda} & 2i\xi_1 + C_a \dfrac{|\xi|^2}{\lambda} \dfrac{i\xi_1}{r} & 2i\xi_2 + C_a \dfrac{|\xi|^2}{\lambda} \dfrac{i\xi_2}{r} \end{pmatrix} \tag{3.20}$$

and

$$A_2 = \begin{pmatrix} 2i\xi_1 e^{-|\xi|} & e^{-r} \dfrac{r^2 + \xi_1^2}{r} & e^{-r} \dfrac{\xi_1 \xi_2}{r} \\ 2i\xi_2 e^{-|\xi|} & e^{-r} \dfrac{\xi_1 \xi_2}{r} & e^{-r} \dfrac{r^2 + \xi_2^2}{r} \\ \left(\dfrac{r^2 + |\xi|^2}{|\xi|} - C_a \dfrac{|\xi|^2}{\lambda} \right) e^{-|\xi|} & e^{-r} \left(-2\dfrac{i\xi_1}{r} + C_a \dfrac{|\xi|^2}{\lambda} \dfrac{i\xi_1}{r^2} \right) & e^{-r} \left(-2\dfrac{i\xi_2}{r} + C_a \dfrac{|\xi|^2}{\lambda} \dfrac{i\xi_2}{r^2} \right) \end{pmatrix}. \tag{3.21}$$

The next lemma is based on Lemma 2.5 in [1].

Lemma 3.1. $\det \left(\begin{array}{c|c} A_1 & O \\ \hline O & A_2 \end{array} \right) \neq 0.$

Proof. Suppose that this vanishes. Then the boundary value problem (3.7) - (3.10), (3.11) - (3.15) with $a_j = 0$, $j = 0, \ldots, 3$ has a nontrivial solution. Multiply (3.7), (3.8) and (3.9) by v_j , $j = 1, 2, 3$, integrate these on $0 \le y \le 1$, then take sum. Integrate by parts and take the homogeneous boundary conditions into account. Then it holds that

$$\begin{aligned} 0 = \frac{\lambda}{P_r} \int_0^1 |\boldsymbol{v}|^2 \, dy + \frac{1}{2} \Bigg(4 \int_0^1 \xi_1^2 |v_1|^2 dy + 2 \int_0^1 |i\xi_1 v_2 + i\xi_2 v_1|^2 \, dy + 4 \int_0^1 \xi_2^2 |v_2|^2 dy \\ +2 \int_0^1 \left| i\xi_1 v_3 + \frac{dv_1}{dy} \right|^2 dy + 2 \int_0^1 \left| i\xi_2 v_3 + \frac{dv_2}{dy} \right|^2 dy + 4 \int_0^1 \left| \frac{dv_3}{dy} \right|^2 dy \Bigg) + C_a \frac{|\xi|^2}{\lambda} |v_3(1)|^2 \ . \end{aligned} \tag{3.22}$$

From this it follows that $\boldsymbol{v} = \mathbf{0}$ since $\mathrm{Re}\, \lambda > 0$. Hence it also follows that $p = 0$ and $h = 0$. This contradicts that we have a nontrivial solution. $\square$

By this lemma the algebraic system (3.19) is always solvable uniquely. From (3.15) we obtain

$$\begin{pmatrix} C_2 \\ \\ C_2^1 \\ \\ C_2^2 \end{pmatrix} = B \begin{pmatrix} C_1 \\ \\ C_1^1 \\ \\ C_1^2 \end{pmatrix} , \tag{3.23}$$

where

$$B = \frac{r^3}{r-|\xi|} \begin{pmatrix} e^{-|\xi|}\left(-\frac{1}{r^2} - \frac{|\xi|}{r^3}\right) & e^{-r}\frac{2i\xi_1}{r^3} & e^{-r}\frac{2i\xi_2}{r^3} \\ \frac{2i\xi_1}{r|\xi|}e^{-|\xi|} & e^{-r}\left(\frac{1}{r} + \frac{\xi_1^2 - \xi_2^2}{r^2|\xi|}\right) & e^{-r}\frac{2\xi_1\xi_2}{r^2|\xi|} \\ \frac{2i\xi_2}{r|\xi|}e^{-|\xi|} & e^{-r}\frac{2\xi_1\xi_2}{r^2|\xi|} & e^{-r}\left(\frac{1}{r} - \frac{\xi_1^2 - \xi_2^2}{r^2|\xi|}\right) \end{pmatrix} .$$

We now have the algebraic system for C_1, C_1^1, C_1^2

$$(A_1 + A_2 B) \begin{pmatrix} C_1 \\ \\ C_1^1 \\ \\ C_1^2 \end{pmatrix} = \begin{pmatrix} a_1^{(\ell,m)} \\ a_2^{(\ell,m)} \\ a_3^{(\ell,m)} + C_a\frac{|\xi|^2}{\lambda}a_0^{(\ell,m)} \end{pmatrix} . \tag{3.24}$$

The determinant of A_1 is $-\frac{1}{|\xi|}\mathcal{D}$, where $\mathcal{D} = \left(r^2 + |\xi|^2\right)^2 - 4r|\xi|^3 + \frac{C_a}{P_r}|\xi|^3$.

Lemma 3.2. *For $\mathcal{D}$ it holds*

$$|\mathcal{D}| \geq \frac{2}{P_r}|\lambda||\xi|^2 , \quad \left(\frac{\sqrt{P_r}}{2} + \frac{P_r}{C_a}\frac{7}{2}\right)|\mathcal{D}| \geq |\xi|^3, \quad \left(3 + \frac{C_a\sqrt{P_r}}{2}\right)|\mathcal{D}| \geq \frac{|\lambda|^2}{P_r^2} .$$

For the proof of this lemma see Lemma 2.5 in [12]. The system (3.24) can be rewritten as follows

$$\left(I + A_2 B A_1^{-1}\right) A_1 \begin{pmatrix} C_1 \\ \\ C_1^1 \\ \\ C_1^2 \end{pmatrix} = \begin{pmatrix} a_1^{(\ell,m)} \\ a_2^{(\ell,m)} \\ a_3^{(\ell,m)} + C_a\frac{|\xi|^2}{\lambda}a_0^{(\ell,m)} \end{pmatrix} . \tag{3.25}$$

The inverse of A_1 is given by

$$A_1^{-1} = \frac{|\xi|}{\mathcal{D}} \times \begin{pmatrix} i\left(2\xi_1 r + C_a\frac{\xi_1|\xi|^2}{\lambda}\right) & i\left(2\xi_2 r + C_a\frac{\xi_2|\xi|^2}{\lambda}\right) & -\left(r^2+|\xi|^2\right) \\ \frac{\left(r^2+|\xi|^2\right)\left(r^2+\xi_2^2\right)}{r|\xi|} - 4\xi_2^2 + C_a\frac{|\xi|^2}{\lambda}\frac{r^2-\xi_2^2}{r} & -\frac{r^2+|\xi|^2-4r|\xi|}{r|\xi|}\xi_1\xi_2 + C_a\frac{|\xi|^2}{\lambda}\frac{\xi_1\xi_2}{r} & 2i\xi_1 r \\ -\frac{r^2+|\xi|^2-4r|\xi|}{r|\xi|}\xi_1\xi_2 + C_a\frac{|\xi|^2}{\lambda}\frac{\xi_1\xi_2}{r} & \frac{\left(r^2+|\xi|^2\right)\left(r^2+\xi_1^2\right)}{r|\xi|} - 4\xi_1^2 + C_a\frac{|\xi|^2}{\lambda}\frac{r^2-\xi_1^2}{r} & 2i\xi_2 r \end{pmatrix} . \tag{3.26}$$

Since $|\arg r| < \dfrac{\pi}{4}$ it holds that $\mathrm{Re}\, r > \dfrac{|r|}{\sqrt{2}}$. Hence $|e^{-r}| \le e^{-|r|/\sqrt{2}}$. Each component of the matrices A_2 and B has the factor $e^{-|\xi|}$ or e^{-r}. Taking this fact and Lemma 3.2 into account, we can show by direct computation that each component of the product $A_2BA_1^{-1}$ tends to 0 as $|\xi|$ goes to infinity uniformly for $\mathrm{Re}\,\lambda \ge 1$. Hence for sufficiently large $|\xi|$ the inverse of $\left(I \;+\; A_2BA_1^{-1}\right)$ can be given by the Neumann series and be written as

$$\left(I \;+\; A_2BA_1^{-1}\right)^{-1} = \sum_{n=0}^{\infty}(-1)^n \left(A_2BA_1^{-1}\right)^n = I + e^{-|\xi|}(c_{jk})$$

where the components of the 3×3 matrix (c_{jk}) are bounded uniformly for $\mathrm{Re}\,\lambda \ge 1$. Set

$$\widetilde{a_k^{(\ell,m)}} = a_k^{(\ell,m)}\;,\; k=1,2\;,\quad \widetilde{a_3^{(\ell,m)}} = a_3^{(\ell,m)} + C_a\frac{|\xi|^2}{\lambda}a_0^{(\ell,m)}\;.$$

Solving (3.25) we have the explicit forms of the principal terms of $v_j^{(\ell,m)}$, $j=1,2,3$, $p^{(\ell,m)}$, $h^{(\ell,m)}$

$$\begin{aligned}
v_1^{(\ell,m)} &= -\frac{|\xi|(r-|\xi|)}{\mathcal{D}}\frac{e^{-r(1-y)}-e^{|\xi|(1-y)}}{r-|\xi|}\frac{i\xi_1}{|\xi|}\Bigg\{\left(2i\xi_1 r + C_a\frac{i\xi_1|\xi|^2}{\lambda}\right)\left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|}c_{1k}\widetilde{a_k^{(\ell,m)}}\right) \\
&+\left(2i\xi_2 r + C_a\frac{i\xi_2|\xi|^2}{\lambda}\right)\left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|}c_{2k}\widetilde{a_k^{(\ell,m)}}\right) - \left(r^2+|\xi|^2\right)\left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|}c_{3k}\widetilde{a_k^{(\ell,m)}}\right)\Bigg\} \\
&+e^{-r(1-y)}\frac{|\xi|}{\mathcal{D}}\Bigg\{\left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|}c_{1k}\widetilde{a_k^{(\ell,m)}}\right)\Bigg(\frac{(r-|\xi|)\left(r^3+|\xi|r^2+3\xi_2^2 r-\xi_2^2|\xi|\right)}{r|\xi|} \\
&\qquad +C_a\frac{|\xi|(r-|\xi|)\left(|\xi|r+\xi_2^2\right)}{r\lambda}\Bigg) \\
&+\left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|}c_{2k}\widetilde{a_k^{(\ell,m)}}\right)\left(-\frac{\xi_1\xi_2(r-|\xi|)(3r+|\xi|)}{r|\xi|} - C_a\frac{\xi_1\xi_2|\xi|(r-|\xi|)}{\lambda r}\right) \\
&-\left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|}c_{3k}\widetilde{a_k^{(\ell,m)}}\right)\frac{i\xi_1\left(r-|\xi|\right)^2}{|\xi|}\Bigg\} \\
&+\frac{i\xi_1}{|\xi|}\frac{e^{-ry}-e^{-|\xi|y}}{r-|\xi|}\left(-(r+|\xi|)e^{-|\xi|}C_1 + 2i\xi_1 e^{-r}C_1^1 + 2i\xi_2 e^{-r}C_1^2\right) \\
&-\left(\frac{i\xi_1}{|\xi|}e^{-|\xi|}C_1 + e^{-r}C_1^1\right)e^{-ry}\;,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
v_2^{(\ell,m)} &= -\frac{|\xi|(r-|\xi|)}{\mathcal{D}}\frac{e^{-r(1-y)}-e^{|\xi|(1-y)}}{r-|\xi|}\frac{i\xi_1}{|\xi|}\Bigg\{\left(2i\xi_1 r + C_a\frac{i\xi_1|\xi|^2}{\lambda}\right)\left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|}c_{1k}\widetilde{a_k^{(\ell,m)}}\right) \\
&+\left(2i\xi_2 r + C_a\frac{i\xi_2|\xi|^2}{\lambda}\right)\left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|}c_{2k}\widetilde{a_k^{(\ell,m)}}\right) - \left(r^2+|\xi|^2\right)\left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|}c_{3k}\widetilde{a_k^{(\ell,m)}}\right)\Bigg\} \\
&+e^{-r(1-y)}\frac{|\xi|}{\mathcal{D}}\Bigg\{\left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|}c_{1k}\widetilde{a_k^{(\ell,m)}}\right)\left(-\frac{\xi_1\xi_2(r-|\xi|)(3r+|\xi|)}{r|\xi|} - C_a\frac{\xi_1\xi_2|\xi|(r-|\xi|)}{\lambda r}\right) \\
&+\left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|}c_{2k}\widetilde{a_k^{(\ell,m)}}\right)\Bigg(\frac{(r-|\xi|)\left(r^3+|\xi|r^2+3\xi_1^2 r-\xi_1^2|\xi|\right)}{r|\xi|} \\
&\qquad +C_a\frac{|\xi|(r-|\xi|)\left(|\xi|r+\xi_1^2\right)}{r\lambda}\Bigg) \\
&-\left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|}c_{3k}\widetilde{a_k^{(\ell,m)}}\right)\frac{i\xi_2\left(r-|\xi|\right)^2}{|\xi|}\Bigg\} \\
&+\frac{i\xi_2}{|\xi|}\frac{e^{-ry}-e^{-|\xi|y}}{r-|\xi|}\left(-(r+|\xi|)e^{-|\xi|}C_1 + 2i\xi_1 e^{-r}C_1^1 + 2i\xi_2 e^{-r}C_1^2\right) \\
&-\left(\frac{i\xi_2}{|\xi|}e^{-|\xi|}C_1 + e^{-r}C_1^2\right)e^{-ry}\;,
\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
v_3^{(\ell,m)} &= -\frac{|\xi|(r-|\xi|)}{\mathcal{D}} \frac{e^{-r(1-y)}-e^{|\xi|(1-y)}}{r-|\xi|} \left\{ \left(2i\xi_1 r + C_a \frac{i\xi_1|\xi|^2}{\lambda}\right) \left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|} c_{1k} \widetilde{a_k^{(\ell,m)}}\right) \right. \\
&\left. + \left(2i\xi_2 r + C_a \frac{i\xi_2|\xi|^2}{\lambda}\right) \left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|} c_{2k} \widetilde{a_k^{(\ell,m)}}\right) - \left(r^2+|\xi|^2\right) \left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|} c_{3k} \widetilde{a_k^{(\ell,m)}}\right) \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+e^{-r(1-y)} \frac{|\xi|}{\mathcal{D}} \left\{ -\frac{i\xi_1}{|\xi|}(r-|\xi|)^2 \left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|} c_{1k} \widetilde{a_k^{(\ell,m)}}\right) - \frac{i\xi_2}{|\xi|}(r-|\xi|)^2 \left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|} c_{2k} \widetilde{a_k^{(\ell,m)}}\right) \right. \\
&\qquad \left. - \left(r^2-|\xi|^2\right) \left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|} c_{3k} \widetilde{a_k^{(\ell,m)}}\right) \right\} \\
&\qquad - \frac{e^{-ry}-e^{-|\xi|y}}{r-|\xi|} \left(-(r+|\xi|)e^{-|\xi|} C_1 + 2i\xi_1 e^{-r} C_1^1 + 2i\xi_2 e^{-r} C_1^2\right) \\
&\qquad + e^{-ry} \left\{ \left(e^{-r}-e^{-|\xi|}\right) C_1 - e^{-r} \left(C_1 - \frac{i\xi_1}{r} C_1^1 - \frac{i\xi_2}{r} C_1^2\right) \right\} ,
\end{aligned} \tag{3.29}
$$

$$
\begin{aligned}
p^{(\ell,m)} &= -\frac{\lambda e^{-|\xi|(1-y)}}{P_r \mathcal{D}} \left\{ \left(2i\xi_1 r + C_a \frac{i\xi_1|\xi|^2}{\lambda}\right) \left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|} c_{1k} \widetilde{a_k^{(\ell,m)}}\right) \right. \\
&\left. + \left(2i\xi_2 r + C_a \frac{i\xi_2|\xi|^2}{\lambda}\right) \left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|} c_{2k} \widetilde{a_k^{(\ell,m)}}\right) - \left(r^2+|\xi|^2\right) \left(\widetilde{a_3^{(\ell,m)}} + e^{-|\xi|} c_{3k} \widetilde{a_k^{(\ell,m)}}\right) \right\} \\
&\qquad + \frac{\lambda e^{-|\xi|y}}{P_r|\xi|(r-|\xi|)} \left(-(r+|\xi|)e^{-|\xi|} C_1 + 2i\xi_1 e^{-r} C_1^1 + 2i\xi_2 e^{-r} C_1^2\right) ,
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
h^{(\ell,m)} &= -\frac{i\xi_1(r-|\xi|)^2}{\lambda\mathcal{D}} \left(\widetilde{a_1^{(\ell,m)}} + e^{-|\xi|} c_{1k} \widetilde{a_k^{(\ell,m)}}\right) - \frac{i\xi_2(r-|\xi|)^2}{\lambda\mathcal{D}} \left(\widetilde{a_2^{(\ell,m)}} + e^{-|\xi|} c_{2k} \widetilde{a_k^{(\ell,m)}}\right) \\
&\qquad - \frac{|\xi|}{P_r\mathcal{D}} \left(a_3^{(\ell,m)} + e^{-|\xi|} c_{3k} \widetilde{a_k^{(\ell,m)}}\right) + \frac{1}{\lambda}\left(1 - \frac{C_a}{P_r}\frac{|\xi|^3}{\mathcal{D}}\right) a_0^{(\ell,m)} \\
&\qquad + \frac{e^{-r}}{\lambda} \left(-e^{-|\xi|} C_1 + \frac{i\xi_1}{r} e^{-r} C_1^1 + \frac{i\xi_2}{r} e^{-r} C_1^2\right) \\
&\qquad - \frac{e^{-r}-e^{-|\xi|}}{\lambda(r-|\xi|)} \left(-(r+|\xi|)e^{-|\xi|} C_1 + 2i\xi_1 e^{-r} C_1^1 + 2i\xi_2 e^{-r} C_1^2\right) .
\end{aligned} \tag{3.31}
$$

Here and hereafter we use the convention that a repeated index is summed. For the fundamental solutions $e_0(y) = e^{-ry}$, $e_1(y) = \dfrac{e^{-ry} - e^{-|\xi|y}}{r - |\xi|}$ we have the estimates

Lemma 3.3. *For any* $(\xi_1, \xi_2) \neq (0,0)$, $\mathrm{Re}\,\lambda \geq 1$, *we have*

$$
\int_0^\infty |e_1(y)|^2 \, dy \leq \frac{1}{|r|^2|\xi|} , \quad \int_0^\infty \left|\frac{d^j e_1(y)}{dy^j}\right|^2 dy \leq C \frac{|r|^{2j-1} + |\xi|^{2j-1}}{|r|^2} , \ j = 1,2,3,\dots ,
$$

$$
\int_0^\infty \left|\frac{d^j e_0(y)}{dy^j}\right|^2 dy \leq \frac{1}{\sqrt{2}}|r|^{2j-1} \quad , \ j = 0,1,2,\dots
$$

For these estimates, see Lemma 3.1 in [12]. Using Lemma 3.2 and Lemma 3.3 we can estimate (3.27), (3.28), (3.29), (3.30) and (3.31) in just the same way as in Proposition 4.1 in [9] or Proposition 4.1 in [10]. Summing these in (ξ_1, ξ_2) we can prove

Proposition 3.4. *Assume* $\mathrm{Re}\,\lambda \geq 1$. *Let* $\ell \geq 2$. *Let* a_0 *be given in* $H_0^{\ell-\frac{1}{2}}(\mathbb{T}^2)$ *and let* a_1 , a_2 *and* a_3 *be given in* $H_0^{\ell-\frac{3}{2}}(\mathbb{T}^2)$. *Then there is a solution* h, v, p *to* (3.1), (3.2), (3.3), (3.4), (3.5) *and* (3.6), *which* *satisfies*

$$
\begin{aligned}
&|v|_{H^\ell(\Omega)} \;+\; |\lambda|^{\frac{\ell}{2}}|v|_{H^0(\Omega)} + |\nabla p|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}}\,|\nabla p|_{H^0(\Omega)} + \left|p|_{S_F}\right|_{H^{\ell-\frac{3}{2}}(\mathbb{T}^2)} \\
&\quad + |\lambda|^{\frac{\ell-3/2}{2}}\left|p|_{S_F}\right|_{H^0(\mathbb{T}^2)} + \;|h|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} \;+\; |\lambda|^{\frac{\ell+1/2}{2}}|h|_{H^0(\mathbb{T}^2)} \\
&\le\; C\left(\left(\sum_{j=1}^{3}|a_j|_{H^{\ell-\frac{3}{2}}(\mathbb{T}^2)} \;+\; |\lambda|^{\frac{\ell-3/2}{2}}|a_j|_{H^0(\mathbb{T}^2)}\right) + \;|a_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} \;+\; |\lambda|^{\frac{\ell-1/2}{2}}|a_0|_{H^0(\mathbb{T}^2)}\right)\;. \qquad (3.32)
\end{aligned}
$$

Further, h belongs to $H_0^{\ell+\frac{1}{2}}\left(\mathbb{T}^2\right)$ and satisfies

$$
|h|_{H_0^{\frac{5}{2}}(\mathbb{T}^2)} + |\lambda||h|_{H_0^{\frac{3}{2}}(\mathbb{T}^2)} \le C\left(\sum_{j=1}^{3}|a_j|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \;+\; |a_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right)\;. \qquad (3.33)
$$

4 Construction of the resovent operator

Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$. Let us consider the equation of the form

$$
\lambda\begin{pmatrix} h \\ \boldsymbol{v} \\ \theta \end{pmatrix} \;-\; G\begin{pmatrix} h \\ \boldsymbol{v} \\ \theta \end{pmatrix} \;=\; \begin{pmatrix} g_0 \\ \boldsymbol{f} \\ f_0 \end{pmatrix} \qquad (4.1)
$$

Here $g_0\,,\,\boldsymbol{f}\,,\,f_0$ are given in $H_0^{3/2}\left(\mathbb{T}^2\right)$, $\mathbb{P}^0\left(H^0\left(\Omega\right)\right)^3$ and $H^0\left(\Omega\right)$ respectively. The domain $\mathcal{D}(G)$ is defined by

$$
\begin{aligned}
\mathcal{D}(G) = \Big\{&(h, \boldsymbol{u}, \theta) \in H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right) \times \mathbb{P}^0\left(H^0(\Omega)\right)^3 \times H^0(\Omega); \\
&h \in H_0^{\frac{5}{2}}\left(\mathbb{T}^2\right)\;, \boldsymbol{u} \in \left(H^2(\Omega)\right)^3\;,\; \theta \in H^2(\Omega)\;, \boldsymbol{u} = \boldsymbol{0}\;,\; \theta = 0 \text{ on } S_B\;, \\
&\frac{\partial \theta}{\partial x_3} + B_i\left(\theta - h\right) = 0\;,\;\; \frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} + M_a\frac{\partial}{\partial x_j}\left(\theta - h\right) = 0 \text{ on } S_F\;,\; j = 1,2\Big\}\;.
\end{aligned}
$$

See, for example, [8] for the proposition below.

Proposition 4.1. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$. Let $\ell \ge 2$. For arbitrarily given f_0 in $H^{\ell-2}\left(\Omega\right)$ and b_0 in $H^{\ell-3/2}\left(\mathbb{T}^2\right)$ there is a unique $\theta \in H^\ell\left(\Omega\right)$ satisfying*

$$
\lambda\,\theta \;-\; \Delta\theta \;=\; f_0 \;\text{ in }\Omega\;,\quad \theta = 0 \text{ on } S_B\;,\quad \frac{\partial\theta}{\partial x_3} + B_i\theta = b_0 \text{ on } S_F\;,
$$

$$
|\theta|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}}\,|\theta|_{H^0(\Omega)} \;\le\; C\left(|f_0|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell-2}{2}}\,|f_0|_{H^0(\Omega)} + |b_0|_{H^{\ell-3/2}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell-3/2}{2}}\,|b_0|_{H^0(\mathbb{T}^2)}\right)\;.
$$

In particular we have $\quad |\theta|_{H^2(\Omega)} \;\le\; C\,|b_0|_{H^{1/2}(\mathbb{T}^2)}\;.$

Let h_0 and g_0 be given in $H_0^{3/2}\left(\mathbb{T}^2\right)$. For $\boldsymbol{f}$ given above we take the solution $\boldsymbol{v}^{(0)}$, $q^{(0)}$ given in Proposition 2.3. For f_0 given above we take the solution θ_0 given in Proposition 4.1 with b_0 replaced by $B_i h_0$. We next consider the following system

$$
\lambda h \;-\; v_3 \;=\; v_3^{(0)} + g_0 \qquad \text{on } S_F\;, \qquad (4.2)
$$

$$
\frac{\lambda}{P_r}\boldsymbol{v} \;-\Delta\boldsymbol{v} \;+\; \nabla p \;=\; \boldsymbol{0}\;, \qquad (4.3)
$$

$$
\operatorname{div}\boldsymbol{v} \;=\; 0 \qquad\qquad \text{in } \Omega\;, \qquad (4.4)
$$

$$
\boldsymbol{v} \;=\; \boldsymbol{0} \qquad \text{on } S_B\;, \qquad (4.5)
$$

$$
\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} = -M_a\left(\frac{\partial\theta_0}{\partial x_j} - \frac{\partial h_0}{\partial x_j}\right)\;,\; j = 1,2, \qquad (4.6)
$$

$$
p - 2\frac{\partial v_3}{\partial x_3} + C_a\Delta_F\,h \;=\; G_a h_0\;, \quad \text{on } S_F\;. \qquad (4.7)
$$

Note that $R\boldsymbol{v}^{(0)}$ belongs to $H_0^{3/2}\left(\mathbb{T}^2\right)$ by the solenoidal condition. By Proposition 3.4 we can construct $\boldsymbol{v}^{(1)}$, $p^{(1)}$ and $\tilde{h}$ satisfying (4.2), (4.3), (4.4), (4.5), (4.6) (4.7). The esitimates (3.32), (3.33) with $\ell = 2$ also hold for the right hand sides replaced by those in (4.2), (4.6) and (4.7) respectively. By this we can define the mapping

$$H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right) \ni h_0 \to \tilde{h} \in H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$$

for each g_0. For given h_0, h_0' in $H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right)$, the estimate (3.33) and the result in Proposition 4.1 yield

$$\begin{aligned} |\lambda|\left|\tilde{h}-\tilde{h}'\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} &\leq C\left(\sum_{j=1}^{2}\left|\frac{\partial\theta_0}{\partial x_j}-\frac{\partial\theta_0'}{\partial x_j}\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}+\left|h_0-h_0'\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right) \\ &\leq C\left|h_0-h_0'\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} . \end{aligned}$$

Here θ_0' is the solution corresponding to h_0'. We can choose $\gamma_1 > 0$ so that, if $\operatorname{Re}\lambda \geq \gamma_1$, then

$$\frac{C}{|\lambda|} \leq \frac{C}{\operatorname{Re}\lambda} \leq \frac{C}{\gamma_1} \leq \frac{1}{2} .$$

Hence, for λ with $\operatorname{Re}\lambda \geq \gamma_1$ we have the unique fixed point $h_0 = \tilde{h}$, which solves

$$\lambda h_0 - v_3^{(1)} = v_3^{(0)} + g_0 \qquad \text{on } S_F , \tag{4.8}$$

$$\frac{\lambda}{P_r}\boldsymbol{v}^{(1)} - \Delta\boldsymbol{v}^{(1)} + \nabla p^{(1)} = \mathbf{0} , \quad \operatorname{div}\boldsymbol{v}^{(1)} = 0 \qquad \text{in } \Omega , \tag{4.9}$$

$$\boldsymbol{v}^{(1)} = \mathbf{0} \qquad \text{on } S_B , \tag{4.10}$$

$$\frac{\partial v_3^{(1)}}{\partial x_j} + \frac{\partial v_j^{(1)}}{\partial x_3} + M_a\left(\frac{\partial\theta_0}{\partial x_j} - \frac{\partial h_0}{\partial x_j}\right) = 0 , \; j = 1, 2, \tag{4.11}$$

$$p^{(1)} - 2\frac{\partial v_3^{(1)}}{\partial x_3} - \left(G_a - C_a\mathit{\Delta}_F\right) h_0 = 0 , \quad \text{on } S_F . \tag{4.12}$$

together with the corresponding $\boldsymbol{v}^{(1)}$ and $p^{(1)}$. For this h_0, by the estimate (3.33), it holds that

$$|h_0|_{H^{\frac{5}{2}}(\mathbb{T}^2)} + |\lambda||h_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \leq C_{\gamma_1}\left(\left|R\boldsymbol{v}^{(0)}\right|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |g_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + |\nabla_F\theta_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right) . \tag{4.13}$$

The constant C_{γ_1} remains bounded for $\operatorname{Re}\lambda \geq \gamma_1$. By the usual trace theorem, according to Propositions 2.3 and 4.1, we have

$$|h_0|_{H^{\frac{5}{2}}(\mathbb{T}^2)} + |\lambda||h_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \leq C_{\gamma_1}\left(|\boldsymbol{f}|_{H^0(\Omega)} + |f_0|_{H^0(\Omega)} + |g_0|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right) . \tag{4.14}$$

Thus, if we put $\boldsymbol{v} = \boldsymbol{v}^{(0)} + \boldsymbol{v}^{(1)}$ and $p = q^{(0)} + p^{(1)}$, we see that $(h_0, \boldsymbol{v}, \theta_0) \in \mathcal{D}(G)$ and

$$\lambda\begin{pmatrix} h_0 \\ \boldsymbol{v} \\ \theta_0 \end{pmatrix} - G_0\begin{pmatrix} h_0 \\ \boldsymbol{v} \\ \theta_0 \end{pmatrix} = \begin{pmatrix} g_0 \\ \boldsymbol{f} \\ f_0 \end{pmatrix} \tag{4.15}$$

where

$$G_0 = \begin{pmatrix} 0 & R & 0 \\ -P_r R^*\left(G_a - C_a\mathit{\Delta}_F\right) & -A & 0 \\ 0 & 0 & \Delta \end{pmatrix} .$$

Since the difference $G - G_0$ consists of lower order terms, by the standard argument we can show

Proposition 4.2. *There is a $\gamma > 0$ such that, if $\operatorname{Re}\lambda \geq \gamma$ there exists the inverse $(\lambda - G)^{-1}$ in X with $(\lambda - G)^{-1} X = \mathcal{D}(G)$ and its operator norm satisfying*

$$\left|(\lambda - G)^{-1}\right|_X \leq \frac{C}{|\lambda|}$$

where $X = H_0^{\frac{3}{2}}\left(\mathbb{T}^2\right) \times \mathbb{P}^0\left(H^0(\Omega)\right)^3 \times H^0(\Omega)$.

From this we see that the resolvent operator of G is compact in X. As a consequence of this proposition we can show

Corollary 4.3. *We can take* $\theta \in \left(\frac{\pi}{2}, \pi\right)$ *so that, for* $\lambda \in \mathbb{C}$ *with* $|\arg(\lambda - \gamma)| \leq \theta$, $(\lambda - G)^{-1}$ *exists and satisfies*

$$\left|(\lambda - G)^{-1}\right|_X \leq \frac{C}{|\lambda|}$$

with a different constant C.

If the data from X is more regular, the solution gets the higher regularity.

Proposition 4.4. *Let* $\ell \geq 2$. *Assume that* λ *satisfies the same condition in Corollary 4.3. Suppose* $g_0 \in H_0^{\ell - \frac{1}{2}}(\mathbb{T}^2)$, $\boldsymbol{f} \in \mathbb{P}^0 \left(H^{\ell-2}(\Omega)\right)^3$, $f_0 \in H^{\ell-2}(\Omega)$. *Then the solution*

$$\begin{pmatrix} h \\ \boldsymbol{v} \\ \theta \end{pmatrix} = (\lambda - G)^{-1} \begin{pmatrix} g_0 \\ \boldsymbol{f} \\ f_0 \end{pmatrix} \tag{4.16}$$

satisfies

$$\begin{aligned} &|\boldsymbol{v}|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}} |\boldsymbol{v}|_{H^0(\Omega)} + |h|_{H^{\ell+\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}+\frac{1}{4}} |h|_{H^0(\mathbb{T}^2)} + |\theta|_{H^\ell(\Omega)} + |\lambda|^{\frac{\ell}{2}} |\theta|_{H^0(\Omega)} \\ &\leq C\Big(|\boldsymbol{f}|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell}{2}-1} |\boldsymbol{f}|_{H^0(\Omega)} + |f_0|_{H^{\ell-2}(\Omega)} + |\lambda|^{\frac{\ell}{2}-1} |f_0|_{H^0(\Omega)} \\ &\qquad + |g_0|_{H^{\ell-\frac{1}{2}}(\mathbb{T}^2)} + |\lambda|^{\frac{\ell}{2}-\frac{1}{4}} |g_0|_{H^0(\mathbb{T}^2)} \Big) . \end{aligned} \tag{4.17}$$

5 Linear nonstationary problem

In this section, under certain assumptions on the physical constants, we solve globally in time the linear nonstationary problem

$$\frac{\partial h}{\partial t} - v_3 = 0 \qquad \text{on } S_F , \tag{5.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta - v_3 = f_0 , \tag{5.2}$$

$$\frac{1}{P_r}\frac{\partial \boldsymbol{v}}{\partial t} - \Delta \boldsymbol{v} + \nabla p - R_a\, \theta \boldsymbol{e} = \boldsymbol{f} , \quad \operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } \Omega , \tag{5.3}$$

$$\theta = 0 \quad , \quad \boldsymbol{v} = \boldsymbol{0} \qquad \text{on } S_B , \tag{5.4}$$

$$\frac{\partial \theta}{\partial x_3} + B_i\, (\theta - h) = 0 , \tag{5.5}$$

$$\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} + M_a \left(\frac{\partial \theta}{\partial x_j} - \frac{\partial h}{\partial x_j}\right) = 0 , \ j = 1, 2, \tag{5.6}$$

$$p - 2\frac{\partial v_3}{\partial x_3} - (G_a - C_a \Delta_F)\, h = 0 , \text{ on } S_F \tag{5.7}$$

Besides $\mathbb{P}^0$ we need the projection $\mathbb{P} : \left(L^2(\Omega)\right)^3 \to \mathcal{G}^\perp$, where $\mathcal{G} = \left\{\nabla\phi \; ; \; \phi \in H^1(\Omega)\right\}$. Since $\mathcal{G}^\perp \subsetneqq \left(\mathcal{G}^0\right)^\perp$, $\mathbb{P}\mathbb{P}^0 = \mathbb{P}$.

Proposition 5.1. *If* R_a *and* $|M_a|$ *are sufficiently small, then there is a* $d_0 > 0$ *such that the set* $\{\lambda \in \mathbb{C}; \ |\lambda| \leqq 2d_0\}$ *belongs to the resovent set* $\rho(G)$ *of* G.

Proof. It is enoough to show $0 \in \rho(G)$. Suppose $(h, \boldsymbol{v}, \theta) \in \mathcal{D}(G)$ satisfies

$$v_3 \;=\; 0 \qquad \text{on } S_F \;, \tag{5.8}$$

$$-\Delta\theta \;-\; v_3 \;=\; 0 \;, \tag{5.9}$$

$$P_r R^* \left(G_a - C_a \Delta_F\right) h \;+\; A\boldsymbol{v} - \; P_r R_a \, \mathbb{P}^0(\theta \boldsymbol{e}) = \; \boldsymbol{0} \text{ in } \Omega \;. \tag{5.10}$$

We first derive the estimate for θ

$$|\theta|_{H^2(\Omega)} \le C_1 \left(|v_3|_{H^0(\Omega)} + |h|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\right) \;. \tag{5.11}$$

Applying $\mathbb{P}$ to (5.10), we see that $\boldsymbol{v}$ solves the boundary value problem

$$-\mathbb{P}\Delta \boldsymbol{v} = R_a \mathbb{P}(\theta \boldsymbol{e}) \text{ in } \Omega \;, \tag{5.12}$$

$$\boldsymbol{v} = 0 \text{ on } S_B \;, \quad v_3 = 0 \text{ on } S_F, \tag{5.13}$$

$$\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} = -M_a \left(\frac{\partial \theta}{\partial x_j} - \frac{\partial h}{\partial x_j}\right) \;, \; j = 1, 2, \text{ on } S_F. \tag{5.14}$$

This fact yields the estimate

$$|\boldsymbol{v}|_{H^2(\Omega)} \le C_2 \left(R_a \, |\theta|_{H^0(\Omega)} + |M_a| \sum_{j=1}^{2} \left(\left|\frac{\partial \theta}{\partial x_j}\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)} + \left|\frac{\partial h}{\partial x_j}\right|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\right)\right) \;. \tag{5.15}$$

See [11] for this estimate. Combining this and (5.11), by the usual trace estimate, we have

$$|\boldsymbol{v}|_{H^2(\Omega)} \le C_3 \left(R_a + |M_a|\right) \left(|v_3|_{H^0(\Omega)} + |h|_{H^{\frac{3}{2}}(\mathbb{T}^2)}\right). \tag{5.16}$$

Hence, if R_a and $|M_a|$ are sufficiently small, it follows that

$$|\boldsymbol{v}|_{H^2(\Omega)} \le C_4 \left(R_a + |M_a|\right) |h|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \;. \tag{5.17}$$

Subtracting the equality $\mathbb{P}A\boldsymbol{v} - P_r R_a \mathbb{P}(\theta \boldsymbol{e}) = \boldsymbol{0}$ from (5.10) we have

$$R^* \left(G_a - C_a \Delta_F\right) h \;- 2R^* \frac{\partial v_3}{\partial x_3} \;+\; (I - \mathbb{P})\mathbb{P}^0 \left(\Delta \boldsymbol{v} \;-\; R_a(\theta \boldsymbol{e})\right) = \; \boldsymbol{0} \;. \tag{5.18}$$

The way of constructing $\mathbb{P}^0$ and $\mathbb{P}$ implies

$$(I - \mathbb{P})\mathbb{P}^0 \left(\Delta \boldsymbol{v} \;-\; R_a(\theta \boldsymbol{e})\right) = -\nabla(p_1 + q_1) + \nabla(p_1 + p_2).$$

Here p_1 is the solution to the problem

$$\Delta p_1 = \operatorname{div} \; (\Delta \boldsymbol{v} - R_a(\theta \boldsymbol{e})) \in H^{-1}(\Omega) \;, \; p_1 \in H_0^1(\Omega) \;,$$

q_1 is the solution to the problem

$$\Delta q_1 = 0 \text{ in } W \;, \; q_1 = 0 \text{ on } S_F \;, \; \frac{\partial q_1}{\partial n} = \boldsymbol{n} \cdot (\Delta \boldsymbol{v} - R_a(\theta \boldsymbol{e}) - \nabla p_1) \; \text{ on } S_B \;,$$

and p_2 is the solution to the problem

$$\Delta p_2 = 0 \text{ in } \Omega, \; \frac{\partial p_2}{\partial n} = \boldsymbol{n} \cdot (\Delta \boldsymbol{v} - R_a(\theta \boldsymbol{e}) - \nabla p_1) \; \text{ on } \partial\Omega \;, \; \int_\Omega p_2 \, dx = 0 \;.$$

See [3] Lemma 3.1 and [13] Theorem 1.5. Thus it holds that $\left(G_a - C_a \Delta_F\right) h \; = 2\dfrac{\partial v_3}{\partial x_3} - p_2$ on S_F. The $H^{1/2}(\mathbb{T}^2)$ norm of the right hand side can be bounded by $C_5(R_a|\theta|_{H^0(\Omega)} + |\boldsymbol{v}|_{H^2(\Omega)})$. Combining (5.17) and (5.11), we now derive the estimate

$$|h|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le C_6 \left(R_a + |M_a|\right) |h|_{H^{\frac{3}{2}}(\mathbb{T}^2)} \;.$$

Hence, if R_a and $|M_a|$ are sufficiently small, we can conclude that $(h, \boldsymbol{v}, \theta) = (0, \boldsymbol{0}, 0)$. This shows $0 \in \rho(G)$ since the resolvent operator of G is compact. ∎

Proposition 5.2. *If R_a and $|M_a|$ are sufficiently small, then there is a $d_1 > 0$ such that the set $\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq -d_1\}$ belongs to $\rho(G)$.*

Proof. Let R_a , M_a and d_0 be as in the above proposition. Since we already have Proposition 5.1 and Corollary 4.3, it is enough to show that λ of $|\lambda| \geq d_0$ together with $\mathrm{Re}\,\lambda \geq 0$ belongs to $\rho(G)$. For λ as such suppose $(h, \boldsymbol{v}, \theta) \in \mathcal{D}(G)$ satisfies

$$\lambda h \ - \ v_3 \ = \ 0 \qquad \text{on } S_F \ , \tag{5.19}$$

$$\lambda \theta - \Delta\theta \ - \ v_3 \ = \ 0 \ , \text{ in } \Omega \tag{5.20}$$

$$\frac{\lambda}{P_r}\boldsymbol{v} - \Delta \boldsymbol{v} + \nabla p - R_a \theta \boldsymbol{e} = \boldsymbol{0} \text{ in } \Omega \tag{5.21}$$

with

$$p - 2\frac{\partial v_3}{\partial x_3} - (G_a - C_a \Delta_F)\, h = 0 \text{ on } S_F \ . \tag{5.22}$$

Multiply (5.20) by θ^*, then integrate this equality over Ω. Integrating by parts and taking the boundary conditions into account, we have

$$\lambda |\theta|^2_{H^0(\Omega)} + \int_\Omega |\nabla\theta|^2\, dx \ = \ \int_\Omega v_3 \theta^* dx - B_i \int_{S_F} |\theta|^2 dx' + B_i \int_{S_F} h\theta^* dx' \ .$$

Taking the real part of the both sides, applying Cauchy–Schwarz inequality, Poincaré's inequality and the usual trace theorem, we can derive

$$\mathrm{Re}\,\lambda\, |\theta|^2_{H^0(\Omega)} + \frac{1}{2}\int_\Omega |\nabla\theta|^2\, dx \ \leq \ C_{11} |h|^2_{H^0(\mathbb{T}^2)} + C_{12}\, |v_3|^2_{H^0(\Omega)} \ . \tag{5.23}$$

We next utilize (2.8). From (5.21), (5.22) and the boundary conditions for tangential stress, we can derive

$$\begin{aligned} &\frac{\lambda}{P_r}\, |\boldsymbol{v}|^2_{H^0(\Omega)} + \langle\ \boldsymbol{v}\ ,\ \boldsymbol{v}\ \rangle + \int_{S_F} (G_a - C_a \Delta_F)\, h v_3^* dx' \\ &+ \sum_{j=1}^{2} M_a \int_{S_F} \left(\frac{\partial\theta}{\partial x_j} - \frac{\partial h}{\partial x_j}\right) v_j^* dx' = R_a \int_\Omega \theta v_3^* dx \ . \end{aligned} \tag{5.24}$$

Since (5.19) holds h can be replaced by v_3/λ. Integrating by parts, applying Cauchy–Schwarz inequality and the trace theorem, we can derive the following inequality

$$\begin{aligned} &\frac{\mathrm{Re}\,\lambda}{P_r}\, |\boldsymbol{v}|^2_{H^0(\Omega)} + \langle\ \boldsymbol{v}\ ,\ \boldsymbol{v}\ \rangle + \mathrm{Re}\left(\frac{1}{\lambda}\right) \int_{S_F} \left(G_a\, |v_3|^2 + C_a\, |\nabla_F\, v_3|^2 \right) dx' \\ &\leq |M_a| \left(C_{13}\, |\nabla\theta|^2_{H^0(\Omega)} + \frac{C_{14}}{|\lambda|}\, |\nabla \boldsymbol{v}|^2_{H^0(\Omega)} \right) + C_{15} R_a \left(|\nabla\theta|^2_{H^0(\Omega)} + |\nabla \boldsymbol{v}|^2_{H^0(\Omega)} \right) \ . \end{aligned} \tag{5.25}$$

In (5.23) we replace h by v_3/λ. Note that $|\lambda| \geq d_0$. Adding (5.25) and (5.23) multiplied by $\kappa > 0$, we obtain

$$\begin{aligned} &\kappa\, \mathrm{Re}\,\lambda\, |\theta|^2_{H^0(\Omega)} + \frac{\kappa}{2}\, |\nabla\theta|^2_{H^0(\Omega)} + \frac{\mathrm{Re}\,\lambda}{P_r}\, |\boldsymbol{v}|^2_{H^0(\Omega)} + \langle\ \boldsymbol{v}\ ,\ \boldsymbol{v}\ \rangle + \mathrm{Re}\left(\frac{1}{\lambda}\right) \int_{S_F} \left(G_a\, |v_3|^2 + C_a\, |\nabla_F\, v_3|^2 \right) dx' \\ &\leq \ \kappa C_{16} (1 + \frac{1}{d_0^2})\, |\nabla \boldsymbol{v}|^2_{H^0(\Omega)} + C_{17} \left(\frac{|M_a|}{d_0} + R_a \right) |\nabla \boldsymbol{v}|^2_{H^0(\Omega)} + C_{18}\, (R_a + |M_a|)\, |\nabla\theta|^2_{H^0(\Omega)} \ . \end{aligned} \tag{5.26}$$

Here we have used the trace theorem and Poincaré inequality. We assume $C_{18}\, (R_a + |M_a|) \leq \dfrac{\kappa}{4}$. Then we see $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$ and $h = 0$ if κ, R_a and $|M_a|$ are sufficiently small. This shows λ as above belongs to $\rho(G)$. □

Based on this proposition we can show the theorem below. Since we can prove this in a similar way as in Section 3 in [4], we omit the details.

Theorem 5.3. *Let $\ell > 2$ be not a half integer. Let R_a and M_a be as in the above proposition. Suppose that the inhomogeneous term f_0 and $\boldsymbol{f}$ in (5.2) and (5.3) are given in $K^{\ell-2}_{(0)}(\Omega \times \mathbb{R}^+)$. Then there is a unique solution $(h, \boldsymbol{v}, \theta, p)$ to problem (5.1) – (5.7), with*

$$h \in K^{\ell+\frac{1}{2}}_{0,(0)}(\mathbb{T}^2 \times \mathbb{R}^+) \ , \ \boldsymbol{v} \in K^{\ell}_{(0)}(\Omega \times \mathbb{R}^+) \ , \theta \in K^{\ell}_{(0)}(\Omega \times \mathbb{R}^+) \ , \\ \nabla p \in K^{\ell-2}_{(0)}(\Omega \times \mathbb{R}^+) \ , \ p|_{S_F} \in K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2 \times \mathbb{R}^+) \ . \tag{5.27}$$

This solution satisfies

$$|h|_{K^{\ell+\frac{1}{2}}_{0,(0)}(\mathbb{T}^2\times\mathbb{R}^+)} + |\boldsymbol{v}|_{K^{\ell}_{(0)}(\Omega\times\mathbb{R}^+)} + |\theta|_{K^{\ell}_{(0)}(\Omega\times\mathbb{R}^+)} + |\nabla p|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)} + \left|p|_{S_F}\right|_{K^{\ell-\frac{3}{2}}_{(0)}(\mathbb{T}^2\times\mathbb{R}^+)} \\ \leq C\left(|f_0|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)} + |\boldsymbol{f}|_{K^{\ell-2}_{(0)}(\Omega\times\mathbb{R}^+)}\right) \ . \tag{5.28}$$

6 Full nonlinear problem

We finally announce the result of the full nonlinear problem in the unknown domain $\Omega(t) = \{(x', y) \ ; \ x' \in \mathbb{T}^2 \ , 0 < y < 1 + h(x', t)\}$. The unknown free surface is denoted by $y = 1 + h(x', t) \ , x' \in \mathbb{T}^2 \ , \ t > 0$. The equations governing disturbances from the basic heat conductive state are written as follows

$$\frac{1}{P_r}\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}, \nabla)\boldsymbol{u}\right) - \Delta \boldsymbol{u} + \nabla q - R_a T\boldsymbol{e} = 0 \ , \quad \operatorname{div} \boldsymbol{u} = 0 \quad , \tag{6.1}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}, \nabla)T - \Delta T - u_3 = 0 \ \text{ in } \Omega(t) \tag{6.2}$$

The boundary conditions are

$$T = 0 \quad , \quad \boldsymbol{u} = \boldsymbol{0} \qquad \text{on } x_3 = 0 \ , \tag{6.3}$$

$$\frac{\partial T}{\partial n} + B_i(T - h) = -1 + \frac{1}{\sqrt{1 + |\nabla_F h|^2}} \ , \tag{6.4}$$

$$\left(q - G_a \left(h + \frac{1}{2}\alpha\beta h^2 \right) \right) n_j - \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) n_k \tag{6.5}$$

$$+ (C_a - M_a (T - h)) \nabla_F \left(\frac{\nabla_F h}{\sqrt{1 + |\nabla_F h|^2}} \right) n_j$$

$$- M_a \left(\frac{\partial T}{\partial x_j} - \delta_{j3} \right) + M_a \left(\frac{\partial T}{\partial x_k} - \delta_{k3} \right) n_k n_j = 0 \text{ on } S_F \ , \ j = 1, 2, 3 \ . \tag{6.6}$$

On the free surface we impose the kinematic boundary condition

$$\frac{\partial h}{\partial t} = u_3 - u_1 \frac{\partial h}{\partial x_1} - u_2 \frac{\partial h}{\partial x_2} \quad \text{on } y = 1 + h(x', t) \ , \ x' \in \mathbb{T}^2 \ , \ t > 0. \tag{6.7}$$

Here $(n_1, n_2, n_3) = \left(1 + |\nabla_F h|^2\right)^{-\frac{1}{2}} \left(-\frac{\partial h}{\partial x_1}, -\frac{\partial h}{\partial x_2}, 1\right)$ is the outward unit normal to the free surface. We follow [4] to show existence and decay of global in time solutions. As in [4] we transform the problem to the one on the equilibrium domain $\Omega = \left\{(x', x_3); x' \in \mathbb{T}^2 \ , \ 0 < x_3 < 1\right\}$ using the unknown free surface $h(x', t)$. For each $t \geq 0$ we define $\Theta \ : \Omega \to \Omega(t)$ by

$$\Theta(x_1, x_2, x_3 : t) = (\ x_1 \ , \ x_2 \ , \ (\hat{h} + 1)x_3 \ ) \ , \quad 0 < x_3 < 1 \ .$$

Here $\hat{h}$ is an extension of h to $\mathbb{T}^2 \times (0, 1)$ in a suitable way. For each t we define $\boldsymbol{u}$ on $\Omega(t)$ by $u_j = \Theta_{j,k} v_k / J$, where $(\Theta_{j,k})$ is the Jacobian matrix of Θ and J is the Jacobian $\det(\Theta_{j,k})$. Set $p = q \circ \Theta$ and

$\theta = T \circ \Theta$. Substituting these into (6.1) – (6.7) we obtain the problem for unknowns h, $\boldsymbol{v}$ and θ in the fixed domain Ω. Collecting the linear terms we can write down the problem as follows

$$\frac{\partial h}{\partial t} \; - \; v_3 \; = \; 0 \quad \text{on } S_F \; , \tag{6.8}$$

$$\frac{1}{P_r}\frac{\partial \boldsymbol{v}}{\partial t} \; - \Delta \boldsymbol{v} \; + \; \nabla p \; - \; R_a \, \theta \boldsymbol{e} = \; \boldsymbol{F} \; , \quad \operatorname{div} \boldsymbol{v} \; = \; 0 \; \text{ in } \Omega \; , \tag{6.9}$$

$$\frac{\partial \theta}{\partial t} \; - \Delta \theta \; - \; v_3 \; = \; F_0 \; \text{ in } \Omega \; , \tag{6.10}$$

$$\theta \; = \; 0 \quad , \quad \boldsymbol{v} \; = \; \boldsymbol{0} \qquad \text{on } S_B \; , \tag{6.11}$$

$$\frac{\partial v_3}{\partial x_j} + \frac{\partial v_j}{\partial x_3} + M_a \left(\frac{\partial \theta}{\partial x_j} - \frac{\partial h}{\partial x_j} \right) = F_j \; , \; j = 1, 2, \; \text{ on } S_F \; , \tag{6.12}$$

$$p - 2\frac{\partial v_3}{\partial x_3} - (G_a - C_a \Delta_F)\, h \; = \; F_3 \; \text{ on } S_F \; , \tag{6.13}$$

$$\frac{\partial \theta}{\partial x_3} + B_i \, (\theta - h) \; = \; F_4 \; \text{ on } S_F \; . \tag{6.14}$$

$\boldsymbol{F}$, F_0,F_1,F_2,F_3 and F_4 consist of nonlinear terms.

Theorem 6.1. *Let ℓ be chosen with $3 < \ell < \frac{7}{2}$. Assume that R_a and $|M_a|$ are suuficiently small as in Proposition 5.3. There is a $\delta > 0$ such that for h_0 , $\boldsymbol{v}_0$, θ_0 satisfying*

$$|h_0|_{H^\ell(\mathbb{T}^2)} + |\boldsymbol{v}_0|_{H^{\ell-\frac{1}{2}}(\Omega)} + |\theta_0|_{H^{\ell-\frac{1}{2}}(\Omega)} \le \delta$$

and the compatibility conditions

$$\begin{gathered} \operatorname{div} \boldsymbol{v}_0 \; = \; 0 \text{ in } \Omega \; , \; \boldsymbol{v}_0 \; = \; \boldsymbol{0} \; , \; \theta_0 \; = \; 0 \text{ on } S_B \\ \frac{\partial v_{0,3}}{\partial x_j} + \frac{\partial v_{0,j}}{\partial x_3} + M_a \left(\frac{\partial \theta_0}{\partial x_j} - \frac{\partial h_0}{\partial x_j} \right) \; = \; F_j|_{t=0} \; , \; j = 1, 2, \; \text{ on } S_F \; , \\ \frac{\partial \theta_0}{\partial x_3} + B_i \, (\theta_0 - h_0) \; = \; F_4|_{t=0} \; \text{ on } S_F \end{gathered}$$

the following assertion holds: The problem (6.8) – (6.14) *has a solution h , $\boldsymbol{v}$, θ , p with*

$$\begin{aligned} &|h|_{K^{\ell+\frac{1}{2}}(\mathbb{T}^2 \times (0,\infty))} + |\boldsymbol{v}|_{K^\ell(\Omega \times (0,\infty))} + |\theta|_{K^\ell(\Omega \times (0,\infty))} \\ &+ |\nabla p|_{K^{\ell-2}(\Omega \times (0,\infty))} + \left| p|_{S_F} \right|_{K^{\ell-\frac{3}{2}}(\mathbb{T}^2 \times (0,\infty))} \le C\delta \end{aligned}$$

and

$$h(0) = h_0 \; , \; \boldsymbol{v}(0) = \boldsymbol{v}_0 \; , \; \theta(0) = \theta_0 \; .$$

Note that $F_j|_{t=0}$, $j = 1, 2, 4$ consist of the initial data h_0, $\boldsymbol{v}_0$, θ_0. This is proved by the fixed point theorem based on the global in time solvability of the initial value problem (Theorem 5.3). This theorem is proved in a similar way as in [4], so we omit details.

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