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On Strichartz estimates for hyperbolic equations with constant coefficients

By

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Abstract

In this note we will review how one can carry out comprehensive analysis of the dispersive and Strichartz estimates for general hyperbolic equations and systems with constant coefficients. We will describe what geometric and other ingredients are responsible for time decay rates of solutions, how multiple roots influence decay rates, and give applications to Grad systems and to Fokker–Planck equations. The note is based on the work by the author [Ruzh06, Ruzh07] as well as the joint work with James Smith [RS05, RS07].

§ 1. Introduction

We consider the general m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem

$$(1.1)\qquad \begin{cases} \overbrace{D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u}^{\text{homogeneous principal part}} + \overbrace{\sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u}^{\text{general lower order terms}} = 0, \quad t>0, \\ D_t^l u(0,x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l=0,\dots,m-1,\ x\in\mathbb{R}^n\,. \end{cases}$$

Here

$$(1.2)\qquad \overbrace{D_t^m + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j}}^{\text{homogeneous principal part}}$$

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is the principal part, homogeneous of order m, and

$$\overbrace{\sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r}^{\text{general lower order terms}} \tag{1.3}$$

are lower order terms of general form. Symbol $P_j(\xi)$ of $P_j(D_x)$ is assumed to be a homogeneous polynomial of order j, and $c_{\alpha,r}$ are complex constants. Here, as usual, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $D_{x_k} = \frac{1}{i}\partial_{x_k}$, $D_t = \frac{1}{i}\partial_t$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. The partial differential operator in (1.1) will be denoted by $L(D_t, D_x)$.

In order for the Cauchy problem (1.1) to be well-posed, we will assume that the principal part (1.2) is hyperbolic. Since we are interested in time decay rates of solutions $u(t,x)$ to (1.1) and their dependence on lower order terms (1.3), we do not want to worry whether the Cauchy problem (1.1) is well-posed for a particular choice of lower order terms. Thus, we will assume that the principal part (1.2) is *strictly hyperbolic*. This means that the characteristic roots of the principal part must be real and distinct. More precisely, let

$$L(\tau,\xi) = \tau^m + \sum_{j=1}^m P_j(\xi)\tau^{m-j} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r}\xi^\alpha \tau^r$$

be the full symbol of the partial differential operator in (1.1). Hyperbolicity means that for each $\xi \in \mathbb{R}^n$, the symbol of the principal part (1.2)

$$L_m(\tau,\xi) = \tau^m + \sum_{j=1}^m P_j(\xi)\tau^{m-j},$$

has m real-valued roots with respect to τ, and strict hyperbolicity means that at each $\xi \in \mathbb{R}^n \backslash \{0\}$, these roots are pairwise distinct. We denote the roots of $L_m(\tau,\xi)$ with respect to τ by $\varphi_1(\xi) \le \cdots \le \varphi_m(\xi)$, and if L is strictly hyperbolic the above inequalities are strict for $\xi \neq 0$.

Our results will show how different properties of the characteristic roots

$$\tau_1(\xi), \ldots, \tau_m(\xi)$$

of the full symbol affect the rate of decay of the solution $u(t,x)$ with respect to t.

First of all, it is natural to impose the stability condition, namely that for all $\xi \in \mathbb{R}^n$ we have

$$\operatorname{Im} \tau_k(\xi) \ge 0 \quad \text{for } k = 1, \ldots, m\,; \tag{1.4}$$

this is equivalent to requiring the characteristic polynomial of the operator to be stable at all points $\xi \in \mathbb{R}^n$, and thus cannot be expected to be lifted. In fact, certain microlocal decay estimates are possible even without this condition but we will assume (1.4) here for simplicity.

It turns out to be sensible to divide the considerations of how characteristic roots behave into two parts: their behaviour for large values of $|\xi|$ and for bounded values of $|\xi|$. These two cases are then subdivided further; in particular the following are the key properties to consider:

- multiplicities of roots (this only occurs in the case of bounded frequencies $|\xi|$);
- whether roots lie on the real axis or are separated from it;
- behaviour as $|\xi| \to \infty$ (only in the case of large $|\xi|$);
- how roots meet the real axis (if they do);
- properties of the Hessian of the root, $\operatorname{Hess} \tau_k(\xi)$;
- a convexity-type condition, as in the case of homogeneous roots.

For some frequencies away from multiplicities we can actually establish independently interesting estimates for the corresponding oscillatory integrals that contribute to the solution. Around multiplicities we need to take extra care of the structure of solutions. This will be done by dividing the frequencies into zones each of which will give a certain decay rate. Combined together they will yield the total decay rate for solution to (1.1).

§ 2. Brief history

For the homogeneous linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, \quad (t,x) \in \mathbb{R}^n \times (0,\infty)\,, \\ u(0,x) = \phi(x),\ \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}^n\,, \end{cases} \tag{2.1}$$

where the initial data ϕ and ψ lie in suitable function spaces such as $C_0^\infty(\mathbb{R}^n)$, the decay rate of solutions was established by Strichartz [Str70a, Str70b]. He showed that the a priori estimate

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x \phi, \psi)\|_{W_p^{N_p}} \tag{2.2}$$

holds when $n \ge 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \le 2$ and $N_p \ge n(\frac{1}{p} - \frac{1}{q})$. See also [Pec76] or [vW71] with different methods of proof.

Let us now compare the time decay rate for the wave equation with equations with lower order terms. An important example is the Klein–Gordon equation, where $u = u(t, x)$ satisfies the initial value problem

$$\begin{cases} \partial_t^2 u - \Delta_x u + \mu^2 u = 0, & (t, x) \in \mathbb{R}^n \times (0, \infty)\,, \\ u(0, x) = \phi(x),\ u_t(0, x) = \psi(x), & x \in \mathbb{R}^n\,, \end{cases} \tag{2.3}$$

where $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ and $\mu \neq 0$ is a constant (representing a *mass term*); then

$$\|(u(t,\cdot), u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x \phi, \psi)\|_{W_p^{N_p}}, \tag{2.4}$$

where p, q, N_p are as before (see [vW71] or [Pec76]).

Another second order problem of interest is the Cauchy problem for the dissipative wave equation,

$$\begin{cases} \partial_t^2 u - \Delta_x u + u_t = 0\,, & (t, x) \in \mathbb{R}^n \times (0, \infty), \\ u(0, x) = \phi(x),\ u_t(0, x) = \psi(x), & x \in \mathbb{R}^n\,, \end{cases} \tag{2.5}$$

where $\psi, \phi \in C_0^\infty(\mathbb{R}^n)$, say. In this case,

$$\|\partial_t^r \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-r-\frac{|\alpha|}{2}} \|(\phi, \nabla\psi)\|_{W_p^{N_p}}\,, \tag{2.6}$$

with some $N_p = N_p(n, \alpha, r)$ (see [Mat77]).

The case of equations of high orders with homogeneous symbols (i.e. with no lower order terms (1.3)) has been extensively studied as well. Assume here that the operator in (1.1) has only homogeneous part (1.2) with no lower order terms (1.3), so that the problem becomes

$$\begin{cases} L_m(D_t, D_x) u = 0, & (t, x) \in \mathbb{R}^n \times (0, \infty), \\ D_t^l u(0, x) = f_l(x), & l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,, \end{cases} \tag{2.7}$$

where L_m is a homogeneous m^{th} order constant coefficient strictly hyperbolic differential operator in (1.2); the symbol of L_m may be written in the form

$$L_m(\tau, \xi) = (\tau - \varphi_1(\xi)) \dots (\tau - \varphi_m(\xi)), \text{ with } \varphi_1(\xi) < \cdots < \varphi_m(\xi) \quad (\xi \neq 0).$$

It can be easily shown that the question of time decay rates of solution $u(t, x)$ to (2.7) can be reduced to the question of the L^p–L^q boundedness of operators of the form

$$M_r(D) := \mathcal{F}^{-1} e^{i\varphi(\xi)} |\xi|^{-r} \chi(\xi) \mathcal{F}\,,$$

where $\varphi(\xi) \in C^\omega(\mathbb{R}^n \setminus \{0\})$ is homogeneous of order 1 and $\chi \in C^\infty(\mathbb{R}^n)$ is equal to 1 for large ξ and zero near the origin, and $\mathcal{F}$ is the Fourier transform.

It has long been known that the values of r for which $M_r(D)$ is $L^p - L^q$ bounded depend on the geometry of the level set

$$\Sigma_\varphi = \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\} .$$

In [Lit73] and [Bre75] it is shown that if the Gaussian curvature of Σ_φ is never zero then $M_r(D)$ is $L^p - L^q$ bounded provided that $r \geq \frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$. This is extended in [Bre77] where it is proven that $M_r(D)$ is $L^p - L^q$ bounded provided that $r \geq \frac{2n-\rho}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, where $\rho = \min_{\xi\neq 0} \operatorname{rank} \operatorname{Hess} \varphi(\xi)$.

Sugimoto extended this further in [Sug94], where he showed that if Σ_φ is convex then $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \left(n - \frac{n-1}{\gamma(\Sigma_\varphi)}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$; here,

$$\gamma(\Sigma) := \sup_{\sigma\in\Sigma} \sup_{P} \gamma(\Sigma; \sigma, P) , \quad \Sigma \subset \mathbb{R}^n \text{ a hypersurface} ,$$

where P is a plane containing the normal to Σ at σ and $\gamma(\Sigma; \sigma, P)$ denotes the order of the contact between the line $T_\sigma \cap P$, T_σ is the tangent plane at σ, and the curve $\Sigma \cap P$.

If this convexity assumption does not hold the L^p–L^q estimate fails. In fact, in [Sug96] and [Sug98] it is shown that in general, $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \left(n - \frac{1}{\gamma_0(\Sigma_\varphi)}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$, where

$$\gamma_0(\Sigma) := \sup_{\sigma\in\Sigma} \inf_{P} \gamma(\Sigma; \sigma, P) \leq \gamma(\Sigma).$$

For $n = 2$, $\gamma_0(\Sigma) = \gamma(\Sigma)$, so, the convexity condition may be lifted in that case.

The main question of this note is what happens with time decay rates of solutions $u(t, x)$ to the Cauchy problem (1.1) for general strictly hyperbolic operators with arbitrary lower order terms. It is clear that one needs to classify different behaviour of lower order terms that influences the decay rates. Since the equation in question may be rather complicated, one is motivated to look at certain key properties of the symbol that could be verified relatively easily. Let us look at some examples that may motivate such a general theory.

To this end, we mention briefly an example of a system that arises as the linearisation of the 13–moment Grad system of non-equilibrium gas dynamics in two dimensions (other Grad systems are similar). The dispersion relation (the determinant) of this system is a polynomial of the 9^{th} order that can be written as

$$P = Q_9 - iQ_8 - Q_7 + iQ_6 + Q_5 - iQ_4,$$

with polynomials $Q_j(\omega,\xi)$ defined by

$$\begin{aligned}
Q_9(\omega,\xi) =&|\xi|^9\omega^3\left[\omega^6-\frac{103}{25}\omega^4+\frac{21}{5}\omega^2\Big(1-\frac{912}{2625}\alpha\beta\Big)-\frac{27}{25}\Big(1-\frac{432}{675}\alpha\beta\Big)\right],\\
Q_8(\omega,\xi) =&|\xi|^8\omega^2\left[\frac{13}{3}\omega^6-\frac{1094}{75}\omega^4+\frac{1381}{125}\omega^2\Big(1-\frac{2032}{6905}\alpha\beta\Big)-\frac{264}{125}\Big(1-\frac{143}{330}\alpha\beta\Big)\right],\\
Q_7(\omega,\xi) =&|\xi|^7\omega\left[\frac{67}{9}\omega^6-\frac{497}{25}\omega^4+\frac{3943}{375}\omega^2\Big(1-\frac{832}{3943}\alpha\beta\Big)-\frac{159}{125}\Big(1-\frac{48}{159}\alpha\beta\Big)\right],\\
Q_6(\omega,\xi) =&|\xi|^6\left[\frac{19}{3}\omega^6-\frac{2908}{225}\omega^4+\frac{13}{3}\omega^2\Big(1-\frac{32}{325}\alpha\beta\Big)-\frac{6}{25}\right],\\
Q_5(\omega,\xi) =&|\xi|^5\omega\left[\frac{8}{3}\omega^4-\frac{178}{45}\omega^2+\frac{2}{3}\right],\\
Q_4(\omega,\xi) =&\frac{4}{9}|\xi|^4\omega^2\big(\omega^2-1\big),
\end{aligned}$$

where

$$\omega(\xi)=\frac{\tau(\xi)}{|\xi|},\ \alpha=\frac{\xi_1^2}{|\xi|^2},\ \beta=\frac{\xi_2^2}{|\xi|^2}.$$

A natural question of finding dispersive (and subsequent Strichartz) estimates for the Cauchy problem for operator $P(D_t,D_x)$ with symbol $P(\tau,\xi)$ becomes calculationally complicated. Clearly, in this situation it is hard to find the roots explicitly and, therefore, we need to devise some procedure of determining what are the general properties of the characteristics roots, and how to derive the time decay rate from these properties. For example, in [Rad03] and [VR04] it is discussed when such polynomials are stable. In this case, our results will guarantee the decay rate even though the exact formulae for characteristic roots may not be known and even though characteristics become multiple and irregular on some sets.

Once we determine the time decay rates in dispersive estimates, it is quite well known how to derive the corresponding Strichartz estimates (see e.g. [KT98]). The results can be also applied to study the time decay rates to solutions to equations that can be reduced to hyperbolic equations or systems of high orders. For example, following the Grad method for the analysis of the Fokker–Planck equation (e.g. [VR04], [ZR04]), one can obtain the decay rates for the solutions of the Fokker–Planck equation for the distribution function for the Brownian motion (see [Ruzh06]).

The results can be further applied to equations with time dependent coefficients, see the paper of the author with T. Matsuyama [MR07].

§ 3. Decay for the Cauchy problem

Putting together microlocal versions of decay rates in all zones (that we omit here but refer to [RS07] for precise statements) we can obtain the following conclusion about solutions to the Cauchy problem (1.1).

Theorem 3.1. *Suppose $u = u(t,x)$ is the solution of the m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem* (1.1). *Denote the characteristic roots of the operator by $\tau_1(\xi), \dots, \tau_m(\xi)$, and assume that $\operatorname{Im} \tau_k(\xi) \geq 0$ for all $k = 1, \dots, n$, and all $\xi \in \mathbb{R}^n$.*

We introduce two functions, $K^{(l)}(t)$ and $K^{(b)}(t)$, which take values as follows:

- *Consider the behaviour of each characteristic root, $\tau_k(\xi)$, in the region $|\xi| \geq M$, where M is a large enough real number. The following table gives values for the function $K_k^{(l)}(t)$ corresponding to possible properties of $\tau_k(\xi)$; if $\tau_k(\xi)$ satisfies more than one, then take $K_k^{(l)}(t)$ to be function that decays the slowest as $t \to \infty$.*

Location of $\tau_k(\xi)$	Additional Property	$K_k^{(l)}(t)$
away from real axis		$e^{-\delta t}$, some $\delta > 0$
on real axis	$\det \operatorname{Hess} \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	$\operatorname{rank} \operatorname{Hess} \tau_k(\xi) = n-1$	$(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$
	convexity condition γ	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$

Then take $K^{(l)}(t) = \max_{k=1\dots,n} K_k^{(l)}(t)$.

- *Consider the behaviour of the characteristic roots in the bounded region $|\xi| \leq M$; again, take $K^{(b)}(t)$ to be the maximum (slowest decaying) function for which there are roots satisfying the conditions in the following table:*

Location of Root(s)	Properties	$K^{(\mathrm{b})}(t)$
away from axis	no multiplicities	$e^{-\delta t}$, some $\delta > 0$
	L roots coinciding	$(1+t)^L e^{-\delta t}$
on axis, no multiplicities *	$\det \operatorname{Hess} \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	convexity condition γ	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$
on axis, multiplicities*,**	L roots coincide on set of codimension ℓ	$(1+t)^{L-1-\ell}$
meeting axis with finite order s	L roots coincide on set of codimension ℓ	$(1+t)^{L-1-\frac{\ell}{s}(\frac{1}{p}-\frac{1}{q})}$

* *These two cases of roots lying on the real axis require some additional regularity assumptions; we refer to the corresponding microlocal statements in [RS07] for details.*

** *This is the $L^1 - L^\infty$ rate in a shrinking region; there are different versions of L^2 estimates* *possible in this case; the function for the L^p–L^q decay can be then found by interpolation, see [RS07] for details.*

Then, with $K(t) = \max\left(K^{(b)}(t), K^{(l)}(t)\right)$, the following estimate holds:

$$\|D_x^\alpha D_t^r u(t,\cdot)\|_{L^q} \leq C_{\alpha,r} K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}},$$

where $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $N_p = N_p(\alpha, r)$ is a constant depending on p, α and r.

Let us now briefly explain how to understand this theorem. Since the decay rates do depend on the behaviour of characteristic roots in different regions and corresponding microlocal theorems can be used to determine the corresponding rates, in Theorem 3.1 we single out properties which determine the final decay rate. Since the same characteristic root, say τ_k, may exhibit different properties in different regions, we look at the corresponding rates $K^{(\mathrm{b})}(t), K^{(\mathrm{l})}(t)$ under each possible condition and then take the slowest one for the final answer. The value of the Sobolev index $N_p = N_p(\alpha, r)$ depends on the regions as well, and it can be found from the corresponding microlocal statements (see [RS07]).

In conditions of Part I of the theorem, it can be shown by the perturbation arguments that only three cases are possible for large ξ, namely, the characteristic root may be uniformly separated from the real axis, it may lie on the axis, or it may converge to the real axis at infinity. If, for example, the root lies on the axis and, in addition, it satisfies the convexity condition with so-called Sugimoto index γ, we get the corresponding decay rate $K^{(\mathrm{l})}(t) = (1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}$. Indices γ and γ_0 in the tables are defined as the maximum of the corresponding indices $\gamma(\Sigma_\lambda)$ and $\gamma_0(\Sigma_\lambda)$, respectively, where $\Sigma_\lambda = \{\xi : \tau_k(\xi) = \lambda\}$, over all k and over all λ, for which ξ lies in the corresponding region. Indices $\gamma(\Sigma_\lambda)$ and $\gamma_0(\Sigma_\lambda)$ are introduced in the previous section.

The statement in Part II is more involved since we may have multiple roots intersecting on rather irregular sets. The number L of coinciding roots corresponds to the number of roots which actually contribute to the loss of regularity. For example, operator $(\partial_t^2 - \Delta)(\partial_t^2 - 2\Delta)$ would have $L = 2$ for both pairs of roots $\pm|\xi|$ and $\pm\sqrt{2}|\xi|$, intersecting at the origin. Meeting the axis with finite order s means that we have the estimate

$$\mathrm{dist}(\xi, Z_k)^s \leq c|\ \mathrm{Im}\ \tau_k(\xi)| \tag{3.1}$$

for all the intersecting roots, where $Z_k = \{\xi :\ \mathrm{Im}\ \tau_k(\xi) = 0\}$. In Part II of Theorem 3.1, the condition that L roots meet the axis with finite order s on a set of codimension ℓ means that all these estimates hold and that there is a (C^1) set $\mathcal{M}$ of codimension ℓ such that $Z_k \subset \mathcal{M}$ for all corresponding k.

In Part II of the theorem, condition ** is formulated in the region of the size decreasing with time: if we have L multiple roots which coincide on the real axis on a set $\mathcal{M}$ of codimension ℓ, we have an estimate

$$|u(t,x)| \le C(1+t)^{L-1-\ell} \sum_{l=0}^{m-1} \|f_l\|_{L^1}, \tag{3.2}$$

if we cut off the Fourier transforms of the Cauchy data to the ϵ-neighbourhood $\mathcal{M}^\epsilon$ of $\mathcal{M}$ with $\epsilon = 1/t$. Here we may relax the definition of the intersection above and say that if L roots coincide in a set $\mathcal{M}$, then they coincide on a set of codimension ℓ if the measure of the ϵ-neighborhood $\mathcal{M}^\epsilon$ of $\mathcal{M}$ satisfies $|\mathcal{M}^\epsilon| \le C\epsilon^\ell$ for small $\epsilon > 0$; here $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, \mathcal{M}) \le \epsilon\}$.

We can then combine it with the remaining cases outside of this neighborhood, where it is possible to establish decay by different arguments. In particular, this is the case of homogeneous equations with roots intersecting at the origin.

§ 4. Strichartz estimates and nonlinear problems

Let us denote by $\kappa_{p,q}(L(D_t, D_x))$ the time decay rate for the Cauchy problem (1.1), so that function $K(t)$ from Theorem 3.1 satisfies $K(t) \simeq t^{-\kappa_{p,q}(L)}$ for large t. Thus, for polynomial decay rates, we have

$$\kappa_{p,q}(L) = -\lim_{t\to\infty} \frac{\ln K(t)}{\ln t}. \tag{4.1}$$

We will also abbreviate the important case $\kappa(L) = \kappa_{1,\infty}(L)$ since by interpolation we have $\kappa_{p,p'} = \kappa_{2,2}\frac{2}{p'} + \kappa_{1,\infty}(\frac{1}{p} - \frac{1}{p'})$, $1 \le p \le 2$. These indices $\kappa(L)$ and $\kappa_{p,p'}(L)$ of operator $L(D_t, D_x)$ will be responsible for the decay rate in the Strichartz estimates for solutions to (1.1), and for the subsequent well-posedness properties of the corresponding semilinear equation which are discussed below.

In order to present an application to nonlinear problems let us first consider the inhomogeneous equation

$$\begin{cases} L(D_t, D_x)u = f, \quad t > 0, \\ D_t^l u(0,x) = 0, \quad l = 0, \dots, m-1,\ x \in \mathbb{R}^n, \end{cases} \tag{4.2}$$

with $L(D_t, D_x)$ as in (1.1) satisfying (1.4). By the Duhamel's formula the solution can be expressed as

$$u(t) = \int_0^t E_{m-1}(t-s) f(s)\, ds, \tag{4.3}$$

where E_{m-1} is the propagator for the homogeneous equation. Let $\kappa = \kappa_{p,p'}(L)$ be the time decay rate of operator L, determined by Theorem 3.1 and given in (4.1). Then Theorem 3.1 implies that we have estimate

$$||E_{m-1}(t)g||_{W^s_{p'}} \leq C(1+t)^{-\kappa}||g||_{W^s_p}.$$

Together with (4.3) this implies

$$||u(t)||_{W^s_{p'}(\mathbb{R}^n_x)} \leq C\int_0^t (t-s)^{-\kappa}||f(s)||_{W^s_p} ds \leq C|t|^{-\kappa} * ||f(t)||_{W^s_p}.$$

By the Hardy–Littlewood–Sobolev theorem this is $L^q(\mathbb{R}) - L^{q'}(\mathbb{R})$ bounded if $1 < q < 2$ and $1 - \kappa = \frac{1}{q} - \frac{1}{q'}$. Therefore, this implies the following Strichartz estimate:

Theorem 4.1. *Let $\kappa_{p,p'}$ be the time decay rate of the operator $L(D_t, D_x)$ in the Cauchy problem* (4.2). *Let $1 < p, q < 2$ be such that $1/p + 1/p' = 1/q + 1/q' = 1$ and $1/q - 1/q' = 1 - \kappa_{p,p'}$. Let $s \in \mathbb{R}$. Then there is a constant C such that the solution u to the Cauchy problem* (4.2) *satisfies*

$$||u||_{L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))} \leq C||f||_{L^q(\mathbb{R}_t, W^s_p(\mathbb{R}^n_x))},$$

for all data right hand side $f = f(t,x)$.

By the standard iteration method we obtain the well-posedness result for the following semilinear equation

$$\begin{cases} L(D_t, D_x)u = F(t,x,u), & t > 0, \\ D^l_t u(0,x) = f_l(x), & l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,. \end{cases} \tag{4.4}$$

Theorem 4.2. *Let $\kappa_{p,p'}$ be the time decay index of the operator $L(D_t, D_x)$ in the Cauchy problem* (4.4). *Let p, q be such that $1/p + 1/p' = 1/q + 1/q' = 1$ and $1/q - 1/q' = 1 - \kappa_{p,p'}$. Let $s \in \mathbb{R}$.*

Assume that for any $v \in L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))$, the nonlinear term satisfies $F(t,x,v) \in L^q(\mathbb{R}_t, W^s_p(\mathbb{R}^n_x))$. Moreover, assume that for every $\varepsilon > 0$ there exists a decomposition $-\infty = t_0 < t_1 < \cdots < t_k = +\infty$ such that the estimates

$$||F(t,x,u) - F(t,x,v)||_{L^q(I_j, W^s_p(\mathbb{R}^n_x))} \leq \varepsilon ||u - v||_{L^{q'}(I_j, W^s_{p'}(\mathbb{R}^n_x))}$$

hold for the intervals $I_j = (t_j, t_{j+1})$, $j = 0, \dots, k-1$.

Finally, assume that the solution of the corresponding homogeneous Cauchy problem is in the space $L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))$.

Then the semilinear Cauchy problem (4.4) *has a unique solution in $L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))$.*

§ 5. Application to the Fokker–Planck equation

We will here give a result for the Fokker–Planck equation for the distribution function of particles in Brownian motion:

$$\left(\partial_t + \sum_{k=1}^{n} c_k \partial_{x_k}\right) f(t,x,c) = \sum_{k=1}^{n} \partial_{c_k}(c_k + \partial_{c_k}) f.$$

By the Hermite–Grad method (see [VR04], [ZR04]) this equation can be reduced to an infinite hyperbolic system. Indeed, we can write the solution as

$$f(t,x,c) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} m_\alpha(t,x)\psi^\alpha(c),$$

where $\psi^\alpha(c) = (2\pi)^{-n/2}(-\partial_c)^\alpha \exp(-\frac{|c|^2}{2})$ are Hermite functions. The Galerkin approximation f^N of the solution f is

$$f^N(t,x,c) = \sum_{0\leq|\alpha|\leq N} \frac{1}{\alpha!} m_\alpha(t,x)\psi^\alpha(c)\,,$$

with $m(t,x) = \{m_\beta(t,x) :\ 0 \leq |\beta| \leq N\}$ being the unknown function of coefficients. It can be shown that they satisfy a hyperbolic system of partial differential equations of first order:

$$D_t m(t,x) + \sum_j A_j D_{x_j} m(t,x) - iBm(t,x) = 0,$$

where B is a diagonal matrix, $B_{\alpha,\beta} = |\alpha|\delta_{\alpha,\beta}$, and the only non-zero elements of the matrix A_j are $a_j^{\alpha-e_j,\alpha} = \alpha_j$, $a_j^{\alpha+e_j,\alpha} = 1$. Hence, the dispersion equation for the system is

$$P(\tau,\xi) \equiv \det(\tau I + \sum_j A_j\xi_j - iB) = 0, \tag{5.1}$$

which we will call the N^{th} Fokker–Planck polynomial. We will say that $P(\tau,\xi)$ is a stable polynomial if its roots $\tau(\xi)$ satisfy $\operatorname{Im}\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$, and if $\operatorname{Im}\tau(\xi) = 0$ implies $\xi = 0$. Then we will say that $P(\tau,\xi)$ is strongly stable if, moreover, $\operatorname{Im}\tau(\xi) = 0$ implies $\xi = 0$ and $\operatorname{Re}\tau(\xi) = 0$, and if its roots $\tau(\xi)$ satisfy $\liminf_{|\xi|\to\infty} \operatorname{Im}\tau(\xi) > 0$. Thus, the condition of strong stability means that the roots $\tau(\xi)$ may become real only at the origin of the complex plane at $\xi = 0$, and that they do not approach the real axis asymptotically for large ξ.

Theorem 5.1. *If the N^{th} Fokker–Planck polynomial P in* (5.1) *is strongly stable, we have the estimate*

$$||f_N(t,x,c)||_{L^\infty(\mathbb{R}^n_x;L^2_w(\mathbb{R}^n_c))} \leq C(1+t)^{-n/2} + C_N e^{-\epsilon(N)t},$$

where the constant C does not depend on N, with $w = \exp(-|c|^2/2)$ and $\epsilon(N) > 0$.

Here, in general, we may have asymptotically that $\epsilon(N) \to 0$ as $N \to \infty$. For details of this construction we refer to [Ruzh06] or to [RS07].

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