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Lower estimates of growth order for the second Painlevé transcendents of higher order

Dedicated to Professor Kazuo Okamoto's 60th birthday

By

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Abstract

In this article, we investigate the Nevanlinna characteristic function of any transcendental meromorphic solutions to some equations of the second Painlevé hierarchy derived by the reduction from the higher order analogue of the Korteweg de Vries equation. Under a certain condition, we give lower estimates of it.

§ 1. Introduction.

Consider the ordinary differential equations

$$\begin{aligned}
&({}_2P_{\mathrm{II}}(\alpha)) && \lambda'' = 2\lambda^3 + t\lambda + \alpha,\\
&({}_4P_{\mathrm{II}}(\alpha)) && \lambda^{(4)} = 10\lambda''\lambda^2 + 10(\lambda')^2\lambda - 6\lambda^5 - t\lambda - \alpha,\\
&({}_6P_{\mathrm{II}}(\alpha)) && \lambda^{(6)} = 14\lambda^{(4)}\lambda^2 + 56\lambda^{(3)}\lambda'\lambda + 42(\lambda'')^2\lambda\\
& && \qquad + 70\lambda''(\lambda')^2 - 70\lambda''\lambda^4 - 140(\lambda')^2\lambda^3 + 20\lambda^7 + t\lambda + \alpha,\\
&({}_8P_{\mathrm{II}}(\alpha)) && \lambda^{(8)} = 18\lambda^{(6)}\lambda^2 + 108\lambda^{(5)}\lambda'\lambda - 6(21\lambda^4 - 35(\lambda')^2 - 38\lambda''\lambda)\lambda^{(4)}\\
& && \qquad + 138(\lambda^{(3)})^2\lambda - 252\lambda^{(3)}\lambda'(4\lambda^3 - 3\lambda'') + 182(\lambda'')^3\\
& && \qquad - 756(\lambda'')^2\lambda^3 + 84\lambda''\lambda^2(5\lambda^4 - 37(\lambda')^2)\\
& && \qquad - 798(\lambda')^4\lambda + 1260(\lambda')^2\lambda^5 - 70\lambda^9 - t\lambda - \alpha.
\end{aligned}$$

Each of them is an equation of the second Painlevé hierarchy ([1]). The first one is well-known as the second Painlevé equation P_{II}. And, in this article, each of the others is called the second Painlevé equation of higher order.

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Each equation of P_{II} hierarchy is obtained from the higher order analogue of the Korteweg de Vries equation ([7],[3, Section 22]):

$$({}_l\mathrm{KdV}) \qquad (2l-1)\frac{\partial u}{\partial t} = X_l u \quad \text{for } l = 1, 2, \ldots.$$

where X_l is given by

$$X_1 u = Du = u_x, \tag{1.1a}$$

$$X_l u = L_u X_{l-1} u \quad \text{for } l \geqslant 2, \tag{1.1b}$$

$L_u = 2u+2DuD^{-1}-D^2$, $D = \partial/\partial x$. By the change of variables $z = xt^{-1/(2l-1)}$, $\lambda(z) = t^{2/(2l-1)}u$, $({}_l\mathrm{KdV})$ is reduced to the ordinary differential equation $X_l\lambda+2\lambda+z\partial\lambda/\partial z = 0$. Replacing z by t, for $l = 2$, we obtain the second Painlevé equation $({}_2P_{\mathrm{II}}(\alpha))$, and for $l = 3, 4, 5$, we obtain $({}_4P_{\mathrm{II}}(\alpha))$, $({}_6P_{\mathrm{II}}(\alpha))$, $({}_8P_{\mathrm{II}}(\alpha))$, respectively.

A lower estimate for the growth of the second Painlevé transcendents is established by Shimomura. Let us introduce some notations. The characteristic function $T(r, f) = m(r, f) + N(r, f)$ is defined by

$$m(r, f) \underset{\text{def.}}{=} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\mathrm{e}^{\sqrt{-1}\theta})|\mathrm{d}\theta, \quad \log^+ x \underset{\text{def.}}{=} \max\{\log x, 0\},$$

$$N(r, f) \underset{\text{def.}}{=} \int_0^r \{n(t, f) - n(0, f)\}\frac{\mathrm{d}t}{t} + n(0, f)\log r,$$

where $n(r, f)$ denotes the number of poles of f in $\{t||t| \leqslant r\}$; each counted according to its multiplicity. And we define

$$\sigma(f) \underset{\text{def.}}{=} \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

These are ordinary notations of the Nevanlinna theory. For details, see [6].

Proposition 1.1. ([9]) *For every transcendental solution $\lambda(t)$ of P_{II}, we have $\frac{3}{2} \leqslant \sigma(\lambda) \leqslant 3$.*

In this article, we will give lower estimates for the growth of transcendental meromorphic solutions of the equations of the second Painlevé hierarchy $({}_4P_{\mathrm{II}}(\alpha))$, $({}_6P_{\mathrm{II}}(\alpha))$ and $({}_8P_{\mathrm{II}}(\alpha))$, with $\alpha \in \mathbb{Z}$. We can use similar methods for P_{II}, and the main theorem in this article is

Theorem A. *Let λ be a transcendental meromorphic solution of $({}_{2m}P_{\mathrm{II}}(\alpha))$ $(m = 2, 3, 4)$ with the parameter $\alpha\in\mathbb{Z}$. Then we have $\sigma(\lambda) \geqslant a_m$, where $a_2 = 5/4$, $a_3 = 7/6$, $a_4 = 9/8$, respectively.*

We want to make the following conjecture, however:

Conjecture. *Let λ be a transcendental meromorphic solution of $({}_{2m}P_{\mathrm{II}}(\alpha))$, $\alpha \in \mathbb{Z}$ for an arbitrary positive integer m. Then we have $\sigma(\lambda) \geqslant 1 + 1/2m$.*

The conjecture is valid for $m = 1$ by Proposition 1.1, and for $m = 2, 3, 4$ by Theorem A.

§ 2. Proof of Theorem A.

Notation. $g(r) \ll h(r) \iff_{\text{def.}} g(r) = O(h(r))$ *as* $r \to \infty$.

Proposition 2.1. *If Theorem A is valid for $\alpha = 0$, then the theorem is valid for $\alpha \in \mathbb{Z}$.*

In order to prove this proposition, we remark two lemmas on the characteristic functions in the general theory:

Lemma 2.2. ([6, p.37]) *If f is a transcendental meromorphic function, then we have*

$$T(r, f^{(p)}) \leqslant (p+1)T(r, f) + o(T(r, f))$$

as $r \to \infty$, outside of an exceptional set of finite linear measure.

Lemma 2.3. ([6, Proposition 2.1.11]) *If f is a meromorphic function, then*
$T(r, f^n) = nT(r, f)$, $n \in \mathbb{N}$;
$T(r, \sum_{i=1}^{n} f^i) \leqslant \sum_{i=1}^{n} T(r, f^i) + \log n$ *for* $r \geqslant 1$;
$T(r, (af+b)/(cf+d)) = T(r, f) + O(1)$ *with* $ad - bc \neq 0$ *and* $f \not\equiv -d/c$.

We also remark a proposition which concerns the Bäcklund transformations:

Proposition 2.4. ([2, Section 6]) *Let $\lambda(t)$ be a solution of $({}_{2m}P_{\mathrm{II}}(\alpha))$, such that $t + 2K_m(u) \neq 0$. Then the following transformations, called Bäcklund transformations, derive a solution of $({}_{2m}P_{\mathrm{II}}(\alpha \pm 1))$ from $({}_{2m}P_{\mathrm{II}}(\alpha))$.*

$$T : \lambda \to \tilde{\lambda} = -\lambda - \frac{2\alpha - 1}{t + K_m(r)},$$
$$T^{-1} : \tilde{\lambda} \to \lambda = -\tilde{\lambda} - \frac{2\alpha + 1}{t + K_m(\tilde{r})},$$

where $r = \lambda^2 + \lambda'$, $\tilde{r} = \tilde{\lambda}^2 + \tilde{\lambda}'$, $K_m(u)$ is a differential polynomial defined by $X_m u = DK_m(u)$ and (1.1).

After a simple calculation, we have

$$\begin{aligned}
K_1(u) &= u, \\
K_2(u) &= 3u^2 - u'', \\
K_3(u) &= 10u^3 - 5(u')^2 - 10uu'' + u^{(4)}, \\
K_4(u) &= 35u^4 - 70u(u')^2 - 70u^2u'' + 21(u'')^2 + 28u'u^{(3)} + 14uu^{(4)} - u^{(6)}.
\end{aligned}$$

Then, for example, Bäcklund transformations of $({}_4P_{\mathrm{II}}(\alpha))$ are given concretely;

$$T:\ \tilde{\lambda}=-\lambda+\frac{\alpha-1/2}{(\lambda^2-\lambda')''-3(\lambda^2-\lambda')-t/2},$$
$$T^{-1}:\ \lambda=-\tilde{\lambda}+\frac{\alpha+1/2}{(\tilde{\lambda}^2+\tilde{\lambda}')''-3(\tilde{\lambda}^2+\tilde{\lambda}')-t/2}.$$

By use of Lemmas 2.2, 2.3 and Proporition 2.4, we can verify Proposition 2.1. Suppose that there exists a solution $\lambda(t)$ of $({}_{2m}P_{\mathrm{II}}(\alpha))(m=1,2,3,4)$, satisfying $\sigma(\lambda)\leqslant a_m-\varepsilon$, which implies $T(r,\lambda)\ll r^{a_m-\varepsilon}$. By repeating Bäcklund transformations finitely many times, we can get a solution of $({}_{2m}P_{\mathrm{II}}(0))$ satisfying $\sigma(\tilde{\lambda})\leqslant a_m-\varepsilon$, which indicates that Theorem A for $\alpha=0$ is not valid. Therefore, it is sufficient to prove Theorem A only for the case where $\alpha=0$.

Now we prove Theorem A for the case where $\alpha=0$. In what follows, we fix $\alpha=0$.

In order to prove $\sigma(\lambda)\geqslant a_m$, suppose the contrary:

$$\text{(H)}\qquad \sigma(\lambda)=\limsup_{r\to\infty}\frac{\log T(r,\lambda)}{\log r}<a_m$$

for some transcendental meromorphic solution of $({}_{2m}P_{\mathrm{II}}(0))$ $(m=1,2,3,4)$, namely, there exists $\varepsilon>0$ such that $T(r,\lambda)\ll r^{a_m-\varepsilon}$, which implies

$$n(r,\lambda)\ll\int_r^{2r}n(t,\lambda)t^{-1}\mathrm{d}t\ll N(2r,\lambda)\ll r^{a_m-\varepsilon}.$$

Painlevé analysis tells us that

Proposition 2.5. *An arbitrary transcendental meromorphic solution λ of $2m$-th order Painlevé equation $({}_{2m}P_{\mathrm{II}}(\alpha))$ $(m=1,2,3,4)$ has only simple poles with residues $\pm1,\cdots,\pm m$.*

Painlevé analysis is a procedure to get a solution which has Laurent series expansion at $t=t_0$.

Let $\{b_j\}_{j=1}^{\infty}$ be a sequence of poles of λ arranged as $|b_1|<|b_2|<\cdots$. Denote by e_j the residue at $t=b_j$. Define

$$\Psi(t):=\sum_{j=1}^{\infty}e_j\left\{\frac{1}{t-b_j}+\frac{1}{b_j}\right\}.$$

If $b_1=0$, the first term of the sum is to be replaced by e_1/t. By the Mittag-Leffler theorem, there exists an entire function $\psi(t)$ such that $\lambda(t)=\Psi(t)+\psi(t)$.

Define

$$\chi(r,t):=\sum_{r\leq|b_j|}\left|\frac{1}{t-b_j}+\frac{1}{b_j}\right|,\quad \gamma(r):=\sum_{0<|b_j|\leq r}\frac{1}{|b_j|}$$

and $\Delta_0(r):=\{t||t|<r\}$. Then we obtain

Lemma 2.6. *Under supposition* (H), *for every $t \in \Delta_0(r)$ with $r \geqslant 1$, we have* $\chi(2r,t) \ll r^{a_m-1-\varepsilon}$, $\gamma(2r) \ll r^{a_m-1-\varepsilon}$.

Proof. Since

$$\left|\frac{1}{t-b_j}+\frac{1}{b_j}\right| = \frac{|t|}{|b_j|^2|1-t/b_j|} \leqslant \frac{2r}{|b_j|^2}$$

for $|t/b_j| \leqslant 1/2$, we have

$$\chi(2r,t) \leqslant 2r\sum_{2r\leqslant|b_j|}\frac{1}{|b_j|^2} = 2r\int_{2r}^{\infty}\frac{\mathrm{d}n(t,\lambda)}{t^2} \leqslant 4r\int_{2r}^{\infty}n(t,\lambda)\frac{\mathrm{d}t}{t^3}$$
$$\ll r\int_{2r}^{\infty}n(t,\lambda)\frac{\mathrm{d}t}{t^3} \ll r\int_{2r}^{\infty}t^{a_m-3-\varepsilon}\mathrm{d}t \ll r^{a_m-1-\varepsilon}.$$

Moreover,

$$\gamma(2r) = \sum_{0<|b_j|\leq 2r}\frac{1}{|b_j|} = const. + \sum_{1<|b_j|<2r}\frac{1}{|b_j|} = const. + \int_1^{2r}\frac{\mathrm{d}n(t,\lambda)}{t}$$
$$\leqslant const. + \frac{n(2r,\lambda)}{2r} + \int_1^{2r}n(t,\lambda)\frac{\mathrm{d}t}{t^2} \ll r^{a_m-1-\varepsilon}.$$

□

Define

$$E^* := (0,|b_1|+1) \cup \bigcup_{j=2}^{\infty}\left(|b_j|-\frac{1}{|b_j|^2}, |b_j|+\frac{1}{|b_j|^2}\right).$$

Then we obtain

Lemma 2.7. *Under supposition* (H), *E^* is of finite linear measure. Moreover, for every t satisfying $|t| \in (0,\infty)\backslash E^*$,*

$$\sum_{0<|b_j|<\infty}\left|\frac{1}{t-b_j}+\frac{1}{b_j}\right| \ll |t|^{a_m+2}.$$

Proof.

$$2\sum_{j=2}^{\infty}\frac{1}{|b_j|^2} = 2\int_{t>|b_1|}^{\infty}\frac{\mathrm{d}n(t,\lambda)}{t^2} \leqslant 2\int_{t>|b_1|}^{\infty}\frac{n(t,\lambda)}{t^3}\mathrm{d}t$$
$$\leqslant \text{const.}\int_{|b_1|}^{\infty}t^{a_m-3-\varepsilon}\mathrm{d}t = \text{const.}|b_1|^{a_m-2-\varepsilon},$$

which implies

$$\mu(E^*) \leqslant |b_1|+1+2\sum_{j=2}^{\infty}\frac{1}{|b_j|^2} < \infty.$$

For $t \notin E^*$ and $0 < |b_j| < 2|t|$,

$$\left|\frac{1}{t-b_j}+\frac{1}{b_j}\right| < 2|t|^2.$$

And, supposition (H) gives us $n(2|t|,\lambda) \ll |t|^{a_m-\varepsilon}$. Applying Lemma 2.6, we obtain

$$\left\{\sum_{0<|b_j|<2|t|}+\sum_{2|t|<|b_j|}\right\}\left|\frac{1}{t-b_j}+\frac{1}{b_j}\right| \ll (2|t|^2+1)\, n(2|t|,\lambda)+|t|^{a_m-1-\varepsilon} \ll |t|^{a_m+2}.$$

□

Now we prove

Lemma 2.8. *Under supposition* (H)*, the entire part $\psi = \lambda - \Psi$ is a polynomial.*

In order to verify this lemma, we recall several lemmas in general theory:

Lemma 2.9. ([6, Lemma 2.4.1]) *Let $f(z)$ be a meromorphic function. Suppose that $Q_n(f)$ is a polynomial of a meromorphic function and its derivatives with degree at most n, each coefficient of which is a meromorphic function. Let $U(r)$ be the maximum of characteristics of those coefficient functions. Then*

$$\frac{1}{2\pi}\int_{|f|>1}\log^+\left|\frac{Q_n(f)}{f^n}\right|\mathrm{d}\theta = O(\log r+\log T(r,f)+U(r))$$

as $r\to\infty$, outside of an exceptional set of finite linear measure.

Lemma 2.10. ([6, Lemma 2.1.3]) $\log^+\left(\sum_{i=1}^{n}\alpha_i\right) \leqslant \log n + \sum_{i=1}^{n}\log^+\alpha_i.$

Lemma 2.11. ([6, Lemma 1.1.1]) *Let $g, h : (0,+\infty)\to\mathbb{R}$ be monotone increasing functions such that $g(r) \leqslant h(r)$ outside of an exceptional set E^* of finite linear measure. Then, for any $K > 1$, there exists $r_0 > 0$ such that $g(r) \leqslant h(Kr)$ for all $r > r_0$.*

Proposition 2.12. ([6, Theorem 2.2.3]) *Suppose f be a meromorphic function. Then f is a rational function if and only if $T(r,f) = O(\log r)$.*

Proof of Lemma 2.8. Each equation $({}_{2m}P_{\mathrm{II}}(0))$ $(m = 1,2,3,4)$ can be written in the form $\lambda^{2m+1} = Q_{2m}(\lambda)$, where $Q_{2m}(\lambda)$ is a polynomial of λ and its derivatives with degree at most $2m$, and each coefficient is constant or linear w.r.t. t. Applying Lemma 2.9, we have

$$\frac{1}{2\pi}\int_{|\lambda|>1}\log^+|\lambda|\mathrm{d}\theta = \frac{1}{2\pi}\int_{|\lambda|>1}\log^+\left|\frac{Q_{2m}(\lambda)}{\lambda^{2m}}\right|\mathrm{d}\theta = O(\log r+\log T(r,\lambda)+U(r))$$

as $r \to \infty$, outside of an exceptional set of finite linear measure. Since $U(r) = O(\log r)$ as $r \to \infty$, we have

$$\frac{1}{2\pi}\int_{|\lambda|>1} \log^+ |\lambda| \mathrm{d}\theta = O(\log r) \quad \text{as } r \to \infty.$$

Lemma 2.7 indicates $|\Psi| \ll |t|^{a_m+2}$ for $|t| \notin E^*$, which implies

$$m(r,\Psi) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\Psi| \mathrm{d}\theta \ll O(1).$$

By Lemma 2.10, $\log^+ |\psi| = \log^+ |\lambda - \Psi| \leqslant \log 2 + \log^+ |\lambda| + \log^+ |\Psi|$. Integrating this inequality, we have $m(r,\psi) \leqslant \log 2 + m(r,\lambda) + m(r,\Psi)$. On the other hand, since ψ is entire, $T(r,\psi) = m(r,\psi)$. Now we have

$$T(r,\psi) = m(r,\psi) \leqslant \log 2 + m(r,\lambda) + m(r,\Psi) = O(\log r)$$

as $r \to \infty$, $r \notin E^*$. And, the finiteness of $\mu(E^*)$ and Lemma 2.11 tells us that $T(r,\lambda) = O(\log r)$ for sufficiently large r. Therefore, by Proposition 2.12, ψ is a polynomial. This completed the proof of Lemma 2.8.

Next, we give estimates of the logarithmic derivatives of λ. To do so, we use three lemmas; Lemma 2.13, 2.14 and 2.15.

Lemma 2.13. ([10]) *Suppose that $\kappa > 1$. Let $g(t)$ be a meromorphic function, and let $\{\tilde{c}_j\}_{j=1}^{\infty}$ be poles and zeros of $g(t)$ such that $|\tilde{c}_1| < |\tilde{c}_2| < \cdots$, where each zero or pole is counted according to its multiplicity. Then there exists a positive number $C = C(p,\kappa)$ such that for all $p \in \mathbb{N}_{>0}$, for all t satisfying $|t| \geq 1$,*

$$\left|\left(\frac{g'(t)}{g(t)}\right)^{(p-1)}\right| \leqslant C\left(\frac{T(\kappa|t|,g)}{|t|^p} + \sum_{|\tilde{c}_j|<\kappa|t|} \frac{1}{|t-\tilde{c}_j|^p} + 1\right).$$

Proof. For the case where $g(0) \neq 0, \infty$, the lemma is derived from the Poisson-Jensen formula ([4, 5, 8]). The other case, i.e. $g(0) = 0$ or ∞, is reduced to the former case by putting $h(t) = t^d g(t)$, $d \in \mathbb{Z}$, $h(0) \neq 0, \infty$. □

Lemma 2.14. *Take an arbitrary $\kappa > 1$. Let $\{c_j\}_{j=1}^{\infty}$ be a sequence which satisfies $|c_1| \leqslant |c_2| \leqslant \cdots$, and let $\nu(r)$ denote the number of points c_j consisted in $\Delta_0(r)$. If $\nu(r) = O(r^{\Lambda})(\Lambda > 0)$, then for all $r > r_0(\kappa)$ with a sufficiently large positive number $r_0(\kappa)$, there exists a point $t_r \in \Delta_0(r)\backslash\Delta_0(r/\sqrt{2})$ with the properties:*

$$S_0(\{c_j\},\kappa r,t_r) \leqslant 32(\kappa+1)\frac{\nu(\kappa r)}{r},$$
$$S_1(\{c_j\},\kappa r,t_r) \leqslant 6\Lambda_0\nu(\kappa r)\frac{\log r}{r^2},$$

and,

$$S_n(\{c_j\}, \kappa r, t_r) \leqslant \frac{\nu(\kappa r)2^{7+[n/2]}M^{n-1}}{r^2\delta^{n-1}} \quad \textit{for } n = 2, \dots, 7,$$

where $[n]$ *is the largest integer which does not exceed* n, $\Lambda_0 := \max\{1, \Lambda/2\}$, δ *is a sufficiently small positive number, and* $M := \max\{1, |c_1|^{1-\Lambda_0}\}$.

Proof. Let $\Delta_\delta(r)$ and D_δ be the sets as follows:

$$\Delta_\delta(r) := \Delta_0(r)\backslash D_\delta, \quad D_\delta := \bigcup_{j=1}^{\infty}\{t||t-c_j| < \delta\omega(|c_j|)\},$$

with a positive number $\delta < 1$, and $\omega(s) := \min\{1, s^{1-\Lambda_0}\}(s \geqslant 1)$. Define

$$F_r^0 := \{t \in \Delta_0(r)|S_0(\{c_j\}, \kappa r, t) \geqslant 32(\kappa+1)\frac{\nu(\kappa r)}{r}\},$$

$$F_r^1 := \{t \in \Delta_\delta(r)|S_1(\{c_j\}, \kappa r, t) \geqslant 6\Lambda_0\nu(\kappa r)\frac{\log r}{r^2}\}$$

and

$$F_r^n := \{t \in \Delta_\delta(r)|S_n(\{c_j\}, \kappa r, t) \geqslant \frac{\nu(\kappa r)2^{7+[n/2]}M^{n-1}}{r^2\delta^{n-1}}\} \quad \text{for } n = 2, 3, \dots, 2m-1.$$

By [10, Lemma 2.1 and its proof], we may choose δ so small that $\mu(D_\delta \cap \Delta_0(r)) < \pi r^2/32$, and we have $\mu(F_r^0) \leqslant \pi r^2/16$ and $\mu(F_r^1) \leqslant 3\pi r^2/8$ for all $r \geqslant r_0(\kappa)$ with a sufficiently large positive number $r_0(\kappa)$. Here $\mu(\cdot)$ denotes the area of the set.

Put $|t - c_j| = \rho$, $t = \xi + \sqrt{-1}\eta$. We have

$$\sum_{|c_j|<\kappa r} \frac{1}{\omega(|c_j|)} \leqslant \sum_{|c_j|<\kappa r} \frac{1}{\min_j \omega(|c_j|)} \leqslant \nu(\kappa r)M,$$

and hence

$$\begin{aligned}
\iint_{\Delta_\delta(r)} S_n(\{c_j\}, \kappa r, t)\mathrm{d}\xi\mathrm{d}\eta &= \sum_{|c_j|<\kappa r} \iint_{\Delta_\delta(r)} \frac{\mathrm{d}\xi\mathrm{d}\eta}{|t-c_j|^{n+1}} \\
\leqslant \sum_{|c_j|<\kappa r} \iint_{\substack{\delta\omega(|c_j|)\leqslant\rho\leqslant(\kappa+1)r \\ 0\leqslant\theta\leqslant 2\pi}} \frac{\mathrm{d}\rho\mathrm{d}\theta}{\rho^n} &= \sum_{|c_j|<\kappa r} 2\pi\left\{\frac{1}{(\delta\omega(|c_j|))^{n-1}} - \frac{1}{((\kappa+1)r)^{n-1}}\right\} \\
\leqslant \sum_{|c_j|<\kappa r} \frac{2\pi}{(\delta\omega(|c_j|))^{n-1}} &\leqslant 2\pi\nu(\kappa r)\frac{M^{n-1}}{\delta^{n-1}}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\nu(\kappa r)2^{7+[n/2]}M^{n-1}}{r^2\delta^{n-1}}\mu(F_r^n) &= \iint_{F_r^n} \frac{\nu(\kappa r)2^{7+[n/2]}M^{n-1}}{r^2\delta^{n-1}}\mathrm{d}\xi\mathrm{d}\eta \\
\leqslant \iint_{F_r^n} S_n(\{c_j\}, \kappa r, t)\mathrm{d}\xi\mathrm{d}\eta &\leqslant \iint_{\Delta_\delta(r)} S_n(\{c_j\}, \kappa r, t)\mathrm{d}\xi\mathrm{d}\eta \leqslant 2\pi\nu(\kappa r)\frac{M^{n-1}}{\delta^{n-1}},
\end{aligned}$$

which implies that $\mu(F_r^n) \leqslant \pi r^2/2^{6+[n/2]}$ are satisfied for all $r \geqslant r_0(\kappa)$ with a sufficiently large positive number $r_0(\kappa)$.

Gathering the sets above, we get

$$H_r := (D_\delta \cap \Delta_0(r)) \cup \bigcup_{j=0}^{2m-1} F_r^j.$$

Since

$$\begin{aligned}\mu(H_r) &\leqslant \mu(D_\delta \cap \Delta_0(r)) + \sum_{j=0}^{2m-1} \mu(F_r^j)\\ &< \frac{\pi r^2}{32} + \frac{\pi r^2}{16} + \frac{3\pi r^2}{8} + 2\sum_{j=2}^{2m-1} \frac{\pi r^2}{2^{6+[j/2]}}\\ &< \frac{\pi r^2}{2},\end{aligned}$$

there exists a point $t_r \in (\Delta_0(r)\backslash\Delta_0(r/\sqrt{2}))\backslash H_r$ with the desired properties, provided that $r \geqslant r_0(\kappa)$. □

Lemma 2.15. *Under supposition* (H), *defining as* $n^*(r) := n(r,\lambda) + n(r, 1/\lambda)$, *we obtain* $n^*(r) \ll r^{a_m-\varepsilon}$.

Proof. By supposition (H), $n(r,\lambda) \ll r^{a_m-\varepsilon}$. On the other hand,

$$n(r,1/\lambda) \ll N(2r,1/\lambda) \leqslant T(2r,1/\lambda) = T(2r,\lambda) + O(1) \ll r^{a_m-\varepsilon}.$$

Therefore,

$$n^*(r) = n(r,\lambda) + n(r,1/\lambda) \ll r^{a_m-\varepsilon}.$$

□

Now we give the estimates of logarithmic derivatives. Let $\{b'_j\}_{j=1}^\infty$ be poles and zeros of $\lambda(t)$ s.t.$|b'_1| < |b'_2| < \cdots$, each repeated according to its multiplicity, and put $L(t) := \lambda'(t)/\lambda(t)$.

Lemma 2.16. *Under supposition* (H), *for sufficiently large* r, *there exists* $t_r \in \Delta_0(r)\backslash\Delta_0(r/\sqrt{2})$ *such that* $|\Psi(t_r)| \ll r^{a_m-1-\varepsilon}$, $|L(t_r)| \ll r^{a_m-1-\varepsilon}$, *and* $|L^{(p)}(t_r)| \ll 1$ *for* $p = 1, 2, \cdots, 7$.

Proof. Estimate of $|\Psi(t_r)|$. In Lemma 2.13 and Lemma 2.14, put $\kappa = 2$. Take $\nu(2r) = n(2r)$ when we apply Lemma 2.14 to $\{b_j\}_{j=1}^\infty$, and $\nu(2r) = n^*(2r)$ when $\{b'_j\}_{j=1}^\infty$. Note that, if we apply Lemma 2.14 to ${}_{2m}P_{\rm II}(0)$ $(m = 1, 2, 3, 4)$, we can take

$\Lambda = a_m - \varepsilon$ because of $n(2r) \ll r^{a_m-\varepsilon}$, $n^*(2r) \ll r^{a_m-\varepsilon}$. $a_1 = 3/2$, $a_2 = 5/4$, $a_3 = 7/6$, $a_4 = 9/8$, $\Lambda_0 = \max\{1, \Lambda/2\} = 1$ in each cases. Then $M = \max\{1, |c_1|^{1-\Lambda_0}\} = 1$. Each residue e_j of $({}_{2m}P_{\mathrm{II}}(\alpha))$ $(m = 1, 2, 3, 4)$ satisfies $|e_j| \leqslant m$. Lemma 2.15 implies $n^*(r) \ll r^{a_m-\varepsilon}$. Apply Lemma 2.14, then for a sufficiently large r, there exists a point $t_r \in \Delta_0(r)\backslash\Delta_0(r/\sqrt{2})$, such that

$$\begin{aligned}
&S_0(\{b'_j, 2|t_r|, t_r\}) \ll S_0(\{b'_j, 2r, t_r\}) \ll n^*(2r)r^{-1} \ll r^{a_m-1-\varepsilon},\\
&S_1(\{b'_j, 2|t_r|, t_r\}) \ll S_1(\{b'_j, 2r, t_r\}) \ll n^*(2r)r^{-2}\log r \ll 1,\\
&S_n(\{b'_j, 2|t_r|, t_r\}) \ll S_n(\{b'_j, 2r, t_r\}) \ll n^*(2r)r^{-2} \ll 1 \quad \text{for } n = 2, \cdots, 7.
\end{aligned}$$

Moreover, these results imply $S_0(\{b_j, 2r, t_r\}) \leqslant S_0(\{b'_j, 2r, t_r\}) \ll r^{a_m-1-\varepsilon}$. Because of these estimates and results of Lemma 2.6, namely, $\chi(2r) \ll r^{a-1-\varepsilon}$, $\gamma(2r) \ll r^{a-1-\varepsilon}$, the function $\Psi(t) := \sum_{j=1}^{\infty} e_j\{(t-b_j)^{-1} + {b_j}^{-1}\}$ is estimated as follows:

$$\begin{aligned}
|\Psi(t_r)| &= \left|\sum_{j=1}^{\infty} e_j \left\{\frac{1}{t_r - b_j} + \frac{1}{b_j}\right\}\right|\\
&\leqslant m \sum_{0<|b_j|\leqslant 2r} \frac{1}{|b_j|} + m \sum_{0<|b_j|\leqslant 2r} \frac{1}{|t_r - b_j|} + m \sum_{2r\leqslant|b_j|} \left|\frac{1}{t_r - b_j} + \frac{1}{b_j}\right|\\
&= m\gamma(2r) + mS_0(\{b_j\}, 2r, t_r) + m\chi(2r, t_r)\\
&\ll r^{a_m-1-\varepsilon} + r^{a_m-1-\varepsilon} + r^{a_m-1-\varepsilon}\\
&\ll r^{a_m-1-\varepsilon}.
\end{aligned}$$

Estimate of $|L^{(p)}(t_r)|$. By supposition (H), $T(2|t_r|, \lambda) \leqslant T(2r, \lambda) \ll r^{a_m-\varepsilon}$. And also we have the estimates obtained in the above, Lemma 2.13 indicates

$$\begin{aligned}
|L(t_r)| &\leqslant C \left(\frac{T(2|t_r|, \lambda)}{|t_r|} + \sum_{|b'_j|<2|t_r|} \frac{1}{|t - b'_j|} + 1 \right)\\
&\ll \frac{T(2|t_r|, \lambda)}{|t_r|} + S_0(\{b'_j, 2|t_r|, t_r\}) + 1\\
&\ll r^{a_m-1-\varepsilon} + r^{a_m-1-\varepsilon} + 1\\
&\ll r^{a_m-1-\varepsilon}
\end{aligned}$$

and

$$\begin{aligned}|L^{(n)}(t_r)| \leqslant & C\left(\frac{T(2|t_r|,\lambda)}{|t_r|^2} + \sum_{|b'_j|<2|t_r|}\frac{1}{|t-b'_j|^{n+1}} + 1\right)\\ & \ll \frac{T(2|t_r|,\lambda)}{|t_r|^{n+1}} + S_n(\{b'_j, 2|t_r|, t_r\}) + 1\\ & \ll 1+1+1\\ & \ll 1 \quad \text{for } n = 1, \cdots, 7.\end{aligned}$$

□

Under supposition (H), we show that the above results depending (H) are reduced to a contradiction.

Note that

Lemma 2.17. ([6, p.10]) *Put*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n,\ a_0 \neq 0,$$

then all of the roots of $P(z)$ *lie in the disk* $\{r \leqslant 1 + \max_{1\leqslant i\leqslant n} |a_i/a_0|\}$.

Lemma 2.18. *Under supposition* (H), $\psi(t) = \lambda(t) - \Psi(t)$ *is a constant.*

Proof. Each equation $({}_{2m}P_{\rm II}(0))$ $(m = 1, 2, 3, 4)$ can be written as

$$a_{2m,2m}(\lambda^2)^m + \sum_{j=1}^{m} a_{2m,2(m-j)}(\lambda^2)^{m-j} + t = 0.$$

Here, $a_{2m,2m}$ is a constant and each $a_{2m,2(m-j)}(j = 1, ..., m)$ is a polynomial of L and its derivatives with degree at most $2j$ and constant coefficients. By Lemma 2.17, $|\lambda^2| \ll 1 + t + \max_{1\leqslant i\leqslant m} |a_{2m,2(m-i)}|$. And, considering it together with the results of the estimates of $|\Psi(t_r)|$ and $|L^{(p)}(t_r)|$, we can estimate the coefficients $a_{2m,2(m-j)}$.

For $({}_2P_{\rm II}(0))$, $2\lambda^2(t_r) + (t_r + a_{20}(t_r)) = 0$. Since $a_m - 1 = 1/2$ and

$$|a_{20}(t_r)| \ll |L(t_r)|^2 \ll r^{2(a_m-1)-\varepsilon} \ll r,$$

we have $|\lambda(t_r)|^2 \ll r$, which implies $|\psi(t_r)| \ll |\Psi(t_r)| + |\lambda(t_r)| \ll r^{1/2}$.

For $({}_4P_{\rm II}(0))$, $6\lambda^4(t_r) + a_{42}(t_r)\lambda^2(t_r) + (t_r + a_{40}(t_r)) = 0$. Since $a_m - 1 = 1/4$ and

$$\begin{aligned}|a_{42}(t_r)| &\ll |L(t)|^2 \ll r^{2(a_m-1)-\varepsilon} \ll r,\\ |a_{40}(t_r)| &\ll |L(t)|^4 \ll r^{4(a_m-1)-\varepsilon} \ll r,\end{aligned}$$

we have $|\lambda(t_r)|^2 \ll r$, which implies $|\psi(t_r)| \ll |\Psi(t_r)| + |\lambda(t_r)| \ll r^{1/2}$.

For $({}_6P_{\mathrm{II}}(0))$, $20\lambda^6(t_r) + a_{64}(t_r)\lambda^4(t_r) + a_{62}(t_r)\lambda^2(t_r) + (t_r + a_{60}(t_r)) = 0$. Since $a_m - 1 = 1/6$ and

$$\begin{aligned}|a_{64}(t_r)| &\ll |L(t)|^2 \ll r^{2(a_m-1)-\varepsilon} \ll r,\\ |a_{62}(t_r)| &\ll |L(t)|^4 \ll r^{4(a_m-1)-\varepsilon} \ll r,\\ |a_{60}(t_r)| &\ll |L(t)|^6 \ll r^{6(a_m-1)-\varepsilon} \ll r,\end{aligned}$$

we have $|\lambda(t_r)|^2 \ll r$, which implies $|\psi(t_r)| \ll |\Psi(t_r)| + |\lambda(t_r)| \ll r^{1/2}$.

For $({}_8P_{\mathrm{II}}(0))$, $70\lambda^8(t_r)+a_{86}(t_r)\lambda^6(t_r)+a_{84}(t_r)\lambda^4(t_r)+a_{82}(t_r)\lambda^2(t_r)+(t_r+a_{80}(t_r)) = 0$. Since $a_m - 1 = 1/8$ and

$$\begin{aligned}|a_{86}(t_r)| &\ll |L(t)|^2 \ll r^{2(a_m-1)-\varepsilon} \ll r,\\ |a_{84}(t_r)| &\ll |L(t)|^4 \ll r^{4(a_m-1)-\varepsilon} \ll r,\\ |a_{82}(t_r)| &\ll |L(t)|^6 \ll r^{6(a_m-1)-\varepsilon} \ll r,\\ |a_{80}(t_r)| &\ll |L(t)|^8 \ll r^{8(a_m-1)-\varepsilon} \ll r,\end{aligned}$$

we have $|\lambda(t_r)|^2 \ll r$, which implies $|\psi(t_r)| \ll |\Psi(t_r)| + |\lambda(t_r)| \ll r^{1/2}$.

Therefore, for each $({}_{2m}P_{\mathrm{II}}(0))$ $(m = 1, 2, 3, 4)$, $|\psi(t_r)| \ll r^{1/2}$. And, since $\psi(t_r)$ is a polynomial, it must not be anything but a constant. □

Completion of the proof of Theorem A. Under supposition (H), the estimates mentioned above imply a contrary estimate $r/\sqrt{2} \ll r^{1-\varepsilon}$.

Because of $|\lambda(t_r)| \ll |\Psi(t_r)| \ll r^{(a_m-1)-\varepsilon}$, using $t = \sum_{j=0}^{m} a_{2m,2j}\lambda^{2j}$, we can estimate as follows:

For $({}_2P_{\mathrm{II}}(0))$, $r/\sqrt{2} \le |t_r| \ll |L(t_r)|^2 + |\lambda(t_r)|^2 \ll r^{2(a_1-1)-\varepsilon}$, which is a contradiction. Therefore, $a_1 \geqslant 3/2$.

For $({}_4P_{\mathrm{II}}(0))$, $r/\sqrt{2} \le |t_r| \ll |L(t_r)|^4 + |L(t_r)|^2|\lambda(t_r)|^2 + |\lambda(t_r)|^4 \ll r^{4(a_2-1)-\varepsilon}$, which is a contradiction. Therefore, $a_2 \geqslant 5/4$.

For $({}_6P_{\mathrm{II}}(0))$, $r/\sqrt{2} \le |t_r| \ll |L(t_r)|^6+|L(t_r)|^4|\lambda(t_r)|^2+|L(t_r)|^2|\lambda(t_r)|^4+|\lambda(t_r)|^6 \ll r^{6(a_3-1)-\varepsilon}$, which is a contradiction. Therefore, $a_3 \geqslant 7/6$.

For $({}_8P_{\mathrm{II}}(0))$,

$$\begin{aligned}r/\sqrt{2} \le |t_r| &\ll |L(t_r)|^8 + |L(t_r)|^6|\lambda(t_r)|^2 + |L(t_r)|^4|\lambda(t_r)|^4 + |L(t_r)|^2|\lambda(t_r)|^6 + |\lambda(t_r)|^8\\ &\ll r^{8(a_4-1)-\varepsilon},\end{aligned}$$

which is a contradiction. Therefore, $a_4 \geqslant 9/8$.

This is the completion of Theorem A for the case where $\alpha = 0$.

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