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Analytic-Liouville-nonintegrable Hamiltonian systems

By

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Abstract

This paper deals with analytic-Liouville-nonintegrable and formally Liouville-integrable Hamiltonian systems. In the study of the integrability in a sectorial subset of a neighborhood of a singular point we will show that so-called hyperasymptotic expansions naturally enter in the analysis.

§ 1. Introduction

A Hamiltonian system in n degrees of freedom is said to be C^∞-Liouville-integrable if there are n smooth first integrals in involution which are functionally independent on an open dense set. If the first integrals are analytic, then we say that it is analytic-Liouville-integrable. In the paper [1], Gorni and Zampieri showed the existence of a Hamiltonian system with two degrees of freedom which is not analytic-Liouville-integrable, while it is C^∞-Liouville-integrable. Their example shows a new phenomenon which was not studied in the preceeding works. (cf. [2] and [3]). The proof of analytic-nonintegrability relies on the power series expansion of a first integral. On the other hand, the C^∞- integrability was proved by constructing concretely a smooth first integral. In order to prove the theorem they assumed that the Hamiltonian vector field has a special subsystem depending on fewer variables. (cf. Remark after Corollary 2.2.) We are interested in the analytical structures which yield nonintegrability and we are also interested in whether similar phenoma occur for more general Hamiltonians. In this note we extend the analytic-nonintegrable Hamiltonian in [1] to a certain class of

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Hamiltonians (2.2) where r depends on p_1 and p_2. For these Hamiltonians, we cannot construct a nonanalytic first integral concretely, and instead we make use of a hyper-asymptotic expansion in order to construct such an integral. Hyperasymptotic series in the above situation is closely related with the characteristic structure of a certain vector field obtained by restricting a given Hamiltonian vector field to an invariant manifold. The introduction of hyperasymptotic series is also convenient in constructing nonanalytic first integrals at least formally although there still remains a problem of convergence for general Hamiltonians. We will discuss the problem in a future.

§ 2. Analytic nonintegrability

Let $\sigma \geq 1$ be an integer and let $c_1 > 0$ and $c_2 > 0$ be positive numbers. Let $r(q_1, q_2, p_1, p_2)$ be an analytic function of $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$ in some neighborhood of the origin $0 \in \mathbb{R}^4$ such that

$$r \equiv r(q_1, q_2, p_1, p_2) = c_1 q_1^{2\sigma} + c_2 q_2^2 + \tilde{r}(q_1, q_2, p_1, p_2) q_2^3, \tag{2.1}$$

where $\tilde{r}(q_1, q_2, p_1, p_2)$ is analytic at the origin. We are interested in the following analytic Hamiltonian in $\mathbb{R}^4$ with two degrees of freedom

$$\mathcal{H} = -q_2 p_2 \partial_{q_1} r + \left(r^2 + q_2 \partial_{q_2} r\right) p_1, \tag{2.2}$$

where $\partial_{q_1} = \frac{\partial}{\partial q_1}$ and $\partial_{q_2} = \frac{\partial}{\partial q_2}$. The associated Hamiltonian system is given by

$$\begin{cases} \dot{q}_1 = \partial\mathcal{H}/(\partial p_1) &= r^2 + q_2 \partial_{q_2} r + (2r\partial_{p_1} r + q_2 \partial_{q_2}\partial_{p_1} r) p_1 - q_2 p_2 \partial_{p_1}\partial_{q_1} r, \\ \dot{q}_2 = \partial\mathcal{H}/(\partial p_2) &= -q_2 \partial_{q_1} r - q_2 p_2 \partial_{q_1}\partial_{p_2} r + p_1(2r\partial_{p_2} r + q_2 \partial_{q_2}\partial_{p_2} r), \\ \dot{p}_1 = -\partial\mathcal{H}/(\partial q_1) &= q_2 p_2 \partial_{q_1}^2 r - (2r\partial_{q_1} r + q_2 \partial_{q_1}\partial_{q_2} r) p_1, \\ \dot{p}_2 = -\partial\mathcal{H}/(\partial q_2) &= p_2 \partial_{q_1} r + q_2 p_2 \partial_{q_1}\partial_{q_2} r - \left(2r\partial_{q_2} r + \partial_{q_2} r + q_2 \partial_{q_2}^2 r\right) p_1. \end{cases} \tag{2.3}$$

Then we have

Theorem 2.1. *The Hamiltonian system (2.3) is not analytic-Liouville-integrable in any neighborhood of the origin. More precisely, for any analytic first integral $u = u(q_1, q_2, p_1, p_2)$ of (2.3) in $\mathbb{R}^4$, there exists a function ϕ of one-variable, being analytic at $0 \in \mathbb{R}$ such that $u = \phi \circ \mathcal{H}$.*

We have the following

Corollary 2.2. *Suppose that $\sigma = 1$, $c_1 = c_2 = 2$ and $r = 2(q_1^2 + q_2^2)$ in (2.1). Then the Hamiltonian system (2.3) is not analytic-Liouville-integrable in any neighborhood of the origin.*

Remark. Theorem 2.1 is a generalization of [4, Theorem 1], where the function r in (2.1) was supposed to be independent of p_1 and p_2. Corollary 2.2 was proved in [1]. In this case, it is not difficult to see that (2.3) in Corollary 2.2 is C^∞-Liouville-integrable, because it has a smooth first integral

$$u = \begin{cases} q_2 \exp\left(-\frac{1}{r}\right) & \text{if } (q_1, q_2) \neq (0,0), \\ 0 & \text{if } (q_1, q_2) = (0,0). \end{cases} \tag{2.4}$$

On the other hand, it is not known whether (2.3) in a general case has a nonanalytic first integral because one cannot construct the first integral of (2.3) concretely since r also depends on p_1 and p_2. In §4 we will study the integrability from the viewpoint of hyperasymptotic expansions.

§ 3. Proof of theorem

The proof of Theorem 2.1 is done by the argument in [4]. For the sake of completeness we give the proof.

Proof of Theorem 2.1. By the suitable change of the variables q_1 and q_2 one may assume that $c_1 = 1$ and $c_2 = 1$. Let $u = u(q_1, q_2, p_1, p_2)$ be any analytic first integral of (2.3). We note that u is the first integral of the Hamiltonian system (2.3) if and only if u is a solution of the following first order equation

$$\begin{aligned} \{\mathcal{H}, u\} \equiv &\left(q_2 p_2 \partial_{q_1}^2 r - (2r\partial_{q_1} r + q_2 \partial_{q_1}\partial_{q_2} r)\, p_1\right) \frac{\partial u}{\partial p_1} \\ &+ \left(p_2 \partial_{q_1} r + q_2 p_2 \partial_{q_1}\partial_{q_2} r - \left(2r\partial_{q_2} r + \partial_{q_2} r + q_2 \partial_{q_2}^2 r\right) p_1\right) \frac{\partial u}{\partial p_2} \\ &+ \left(r^2 + q_2 \partial_{q_2} r + (2r\partial_{p_1} r + q_2 \partial_{q_2}\partial_{p_1} r) p_1 - q_2 p_2 \partial_{p_1}\partial_{q_1} r\right) \frac{\partial u}{\partial q_1} \\ &+ \left(-q_2 \partial_{q_1} r - q_2 p_2 \partial_{q_1}\partial_{p_2} r + p_1 (2r\partial_{p_2} r + q_2 \partial_{q_2}\partial_{p_2} r)\right) \frac{\partial u}{\partial q_2} = 0. \end{aligned} \tag{3.1}$$

We define

$$v \equiv v(q_1, p_1, p_2) := u(q_1, 0, p_1, p_2). \tag{3.2}$$

By setting $q_2 = 0$ in (3.1) and noting that $\partial_{q_2} r(q_1, 0) \equiv 0$ and $r(q_1, 0) = q_1^{2\sigma}$ by (2.1) with $c_1 = 1$, we obtain

$$2\sigma p_2 \frac{\partial v}{\partial p_2} - 4\sigma q_1^{2\sigma} p_1 \frac{\partial v}{\partial p_1} + q_1^{2\sigma+1} \frac{\partial v}{\partial q_1} = 0. \tag{3.3}$$

We expand v into the power series of p_2, $v = \sum_{j=0}^\infty v_j(q_1, p_1) p_2^j$. Then we see that $v_j(q_1, p_1)$ $(j = 0, 1, \ldots)$ satisfy

$$2\sigma j v_j - 4\sigma q_1^{2\sigma} p_1 \frac{\partial v_j}{\partial p_1} + q_1^{2\sigma+1} \frac{\partial v_j}{\partial q_1} = 0, \quad j = 0, 1, 2, \ldots \tag{3.4}$$

We want to show that $v_j = 0$ if $j \neq 0$, and $v = \phi(p_1 q_1^{4\sigma})$ for some analytic function $\phi(t)$ of one variable. Indeed, we expand $v_j(q_1, p_1) = \sum_{\nu=0}^{\infty} v_{j,\nu}(q_1)p_1^{\nu}$. Then, by substituting the expansion of v_j in (3.4) we obtain

$$2\sigma j v_{j,\nu} - 4\sigma\nu q_1^{2\sigma} v_{j,\nu} + q_1^{2\sigma+1}\frac{\partial v_{j,\nu}}{\partial q_1} = 0, \quad j = 0, 1, 2, \ldots \tag{3.5}$$

If we expand $v_{j,\nu}$ into the power series of q_1, then we can easily see that $v_{j,\nu} \equiv 0$ for all $\nu = 0, 1, \ldots$, if $j \neq 0$. Hence we have $v_j = 0$ if $j \neq 0$. It follows that $v = v_0(q_1, p_1)$. Moreover, by (3.4) v satisfies the equation

$$-4\sigma p_1 \frac{\partial v}{\partial p_1} + q_1 \frac{\partial v}{\partial q_1} = 0.$$

If we substitute the expansion of v into the equation, then, by simple computations, we easily see that $v = \phi(p_1 q_1^{4\sigma})$ for some analytic function $\phi(t)$ of one variable. This proves the assertion.

It follows from (2.2) that $v = v_0 = \phi(p_1 q_1^{4\sigma}) = \phi(\mathcal{H}|_{q_2=0})$. We define

$$g(q_1, q_2, p_1, p_2) := u(q_1, q_2, p_1, p_2) - \phi(\mathcal{H}). \tag{3.6}$$

By (3.2) and by recalling that $\mathcal{H}$ is a first integral we see that g is an analytic solution of (3.1) such that $g(q_1, 0, p_1, p_2) \equiv 0$. In order to prove Theorem 2.1 we shall show $g(q_1, q_2, p_1, p_2) \equiv 0$ in some neighborhood of the origin. First we will show that

$$g(q_1, q_2, p_1, p_2) = \phi_1(p_1 q_1^{4\sigma}) p_2 q_2 + h_2(q_1, p_1, p_2) q_2^2 + \tilde{h}_3(q_1, q_2, p_1, p_2) q_2^3, \tag{3.7}$$

for some analytic function ϕ_1 of one variable and analytic functions h_2 and $\tilde{h}_3$. Because g is analytic we have the expansion

$$g(q_1, q_2, p_1, p_2) = g_1(q_1, p_1, p_2) q_2 + h_2(q_1, p_1, p_2) q_2^2 + \tilde{h}_3(q_1, q_2, p_1, p_2) q_2^3. \tag{3.8}$$

We substitute (3.8) with $u = g$ into (3.1) and compare the coefficients of q_2. By (2.1) we have

$$-4\sigma q_1^{2\sigma} p_1 \frac{\partial g_1}{\partial p_1} + 2\sigma \left(p_2 \frac{\partial g_1}{\partial p_2} - g_1 \right) + q_1^{2\sigma+1} \frac{\partial g_1}{\partial q_1} = 0. \tag{3.9}$$

By substituting the expansion $g_1(q_1, p_1, p_2) = \sum_{m=0}^{\infty} g_{1,m}(q_1, p_1) p_2^m$ into (3.9) and by comparing the coefficients of p_2^m we obtain

$$-4\sigma q_1^{2\sigma} p_1 \frac{\partial g_{1,m}}{\partial p_1} + 2\sigma(m-1) g_{1,m} + q_1^{2\sigma+1} \frac{\partial g_{1,m}}{\partial q_1} = 0. \tag{3.10}$$

If $m \neq 1$, then we obtain a similar equation as for v_j in (3.4). Hence we have $g_{1,m} = 0$ if $m \neq 1$. It follows that $g_1 = g_{1,1} p_2$, and $g_{1,1}$ satisfies the equation $-4\sigma p_1 \frac{\partial g_{1,1}}{\partial p_1} + q_1 \frac{\partial g_{1,1}}{\partial q_1} =$

0. By the same argument as in the above, we see that $g_1 = g_{1,1}p_2 = \phi_1(p_1 q_1^{4\sigma})p_2$ for some analytic function ϕ_1 of one variable. This proves the assertion.

Let us now suppose that

$$
\begin{aligned}
g(q_1,q_2,p_1,p_2) = {} & \phi_{n-1}(p_1 q_1^{4\sigma}) p_2^{n-1} q_2^{n-1} \\
& + h_n(q_1,p_1,p_2) q_2^n + \tilde{h}_{n+1}(q_1,q_2,p_1,p_2) q_2^{n+1},
\end{aligned} \tag{3.11}
$$

for some $n \geq 2$, some analytic function ϕ_{n-1} of one variable and analytic functions $h_n(q_1,p_1,p_2)$ and $\tilde{h}_{n+1}(q_1,q_2,p_1,p_2)$. Then we substitute (3.11) into (3.1) with $u = g$ and we compare the coefficients of q_2^n. By (2.1) we have

$$
\begin{aligned}
& 2p_2^n \sigma(2\sigma-1) q_1^{6\sigma-2}\phi'_{n-1} - 4\sigma q_1^{4\sigma-1} p_1 \frac{\partial h_n}{\partial p_1} - 4(q_1^{2\sigma}+1)(n-1) p_1 p_2^{n-2}\phi_{n-1} \\
& +2\sigma q_1^{2\sigma-1}\left(p_2 \frac{\partial h_n}{\partial p_2} - n h_n\right) + q_1^{4\sigma}\frac{\partial h_n}{\partial q_1} = 0.
\end{aligned} \tag{3.12}
$$

By substituting the expansion $h_n(q_1,p_1,p_2) = \sum_{m=0}^{\infty} h_{n,m}(q_1,p_1) p_2^m$ into (3.12) and by comparing the coefficients of p_2^{n-2} we obtain

$$
\begin{aligned}
& -4\sigma q_1^{4\sigma-1} p_1 \frac{\partial h_{n,n-2}}{\partial p_1} - 4(q_1^{2\sigma}+1)(n-1) p_1 \phi_{n-1} \\
& \qquad - 4\sigma q_1^{2\sigma-1} h_{n,n-2} + q_1^{4\sigma}\frac{\partial h_{n,n-2}}{\partial q_1} = 0.
\end{aligned} \tag{3.13}
$$

We will show that

$$
h_{n,n-2} = 0, \quad \phi_{n-1} = 0. \tag{3.14}
$$

If we can prove $\phi_{n-1} = 0$, then it follows from (3.13) that $v := h_{n,n-2}$ satisfies a similar equation as (3.4). Hence, by the same argument as for (3.4) we have $h_{n,n-2} = 0$. In order to show $\phi_{n-1} = 0$ we insert the expansions

$$
\phi_{n-1}(p_1 q_1^{4\sigma}) = \sum_{k=0}^{\infty} \phi_{n-1,k}\, p_1^k q_1^{4\sigma k}, \quad h_{n,n-2}(q_1,p_1) = \sum_{k=0}^{\infty} h_{n,n-2,k}(q_1) p_1^k \tag{3.15}
$$

into (3.13) and compare the coefficients of p_1^k. Then we obtain, for $k \geq 0$

$$
\begin{aligned}
& -4\sigma q_1^{4\sigma-1} k h_{n,n-2,k} - 4\sigma q_1^{2\sigma-1} h_{n,n-2,k} + q_1^{4\sigma}\frac{\partial h_{n,n-2,k}}{\partial q_1} \\
& \qquad = 4(q_1^{2\sigma}+1)(n-1)\phi_{n-1,k-1} q_1^{4\sigma(k-1)},
\end{aligned} \tag{3.16}
$$

where we set $\phi_{n-1,-1} = 0$. If we set $q_1 = 0$ and $k = 1$ in (3.16), then we obtain $0 = 4(n-1)\phi_{n-1,0}$, which implies $\phi_{n-1,0} = 0$.

Suppose that $\phi_{n-1,k-1} \neq 0$ for some $k \geq 2$. We divide both sides of (3.16) by $q_1^{2\sigma-1}$. Then the right-hand side of (3.16) is divisible by q_1^N, $N = 4\sigma(k-1)+1-2\sigma \geq 2\sigma+1$. Because the operator $-4\sigma k q_1^{2\sigma} + q_1^{2\sigma+1}(d/dq_1)$ in the left-hand side of the equation increases the power of q_1, it follows that $h_{n,n-2,k}$ is divisible by q_1^N. We set $h_{n,n-2,k}(q_1) = q_1^N W(q_1)$. Then we have $q_1(d/dq_1)h_{n,n-2,k} = q_1^N(N + q_1(d/dq_1))W$. It follows from (3.16) that W satisfies

$$(N-4\sigma k)q_1^{2\sigma}W - 4\sigma W + q_1^{2\sigma+1}\frac{dW}{dq_1} = 4(n-1)\phi_{n-1,k-1}(q_1^{2\sigma}+1). \tag{3.17}$$

We set $W = \sum_{j=0}^{2\sigma-1} q_1^j W_j(q_1^{2\sigma})$. Because the right-hand side of (3.17) is a function of $q_1^{2\sigma}$, W_j $(1 \leq j < 2\sigma)$ satisfy

$$q_1^{2\sigma}(N-4\sigma k+j)W_j - 4\sigma W_j + q_1^{2\sigma+1}\frac{dW_j}{dq_1} = 0. \tag{3.18}$$

By a similar argument as for (3.4) we have $W_j = 0$ for $1 \leq j < 2\sigma$. Hence we have $W(q_1) = W_0(q_1^{2\sigma}) =: V(t)$ $(t = q_1^{2\sigma})$. Because $q_1(d/dq_1)V = 2\sigma t(d/dt)V$, it follows from (3.17) that

$$(1-6\sigma)tV - 4\sigma V + 2\sigma t^2\frac{dV}{dt} = 4(n-1)\phi_{n-1,k-1}(t+1).$$

We expand $V = \sum_{j=0}^{\infty} V_j t^j$, and set $c = 4(n-1)\phi_{n-1,k-1}$. Then we can easily see that

$$\begin{aligned} &V_0 = -\frac{c}{4\sigma} \neq 0, \quad V_1 = -\frac{c}{4\sigma} + \frac{6\sigma-1}{4\sigma}\frac{c}{4\sigma} = \frac{2\sigma-1}{16\sigma^2}c \neq 0, \\ &V_j = V_{j-1}\frac{2\sigma(j-1)+1-6\sigma}{4\sigma}. \end{aligned}$$

It follows that V_j grows like $j!$ when j tends to infinity. Therefore $V(t)$ does not converge in any neighborhood of the origin. This is a contradiction. Hence we have $\phi_{n-1,k-1} = 0$. Because k is arbitrary we have $\phi_{n-1} = 0$.

Next we set $\phi_{n-1} = 0$ in (3.12) and consider the coefficients of p_2^m $(m \neq n)$. Then we see that $h_{n,m}$ satisfies a similar equation as for (3.4). Hence we have $h_{n,m} = 0$ if $n \neq m$, and $h_{n,n} = \phi_n(p_1 q_1^{4\sigma})$ for some analytic function ϕ_n of one variable. It follows that $h_n(q_1,p_1,p_2) = h_{n,n}(q_1,p_1)p_2^n = \phi_n(p_1q_1^{4\sigma})p_2^n$. Hence we have (3.11) with n replaced by $n+1$. By induction we obtain (3.11) for an arbitrary integer $n \geq 2$.

It follows from (3.11) with n replaced by $n+2$ that, for every $n \geq 0$ we have $\partial_{q_2}^n g(q_1,0,p_1,p_2) \equiv 0$, where (q_1,p_1,p_2) is in some neighborhood of the origin which may depend on n. On the other hand $\partial_{q_2}^n g(q_1,0,p_1,p_2)$ is analytic in some neighborhood of the origin independent of n. By analytic continuation, we have $\partial_{q_2}^n g(q_1,0,p_1,p_2) \equiv 0$

in some neighborhood of the origin independent of n. By the partial Taylor expansion $g = \sum_n \partial_{q_2}^n g(q_1, 0, p_1, p_2) q_2^n / n!$, we have $g = 0$. This ends the proof. $\square$

§ 4. Formal integrability

In this section, we will study the integrability of (2.3) from the viewpoint of hyperasymptotic expansions. First we study the formal integrability. Without loss of generality we may assume that $c_1 = c_2 = 1$ by a suitable change of the coordinates. The terms in (3.1) which preserve the order of q_2 are given by

$$\mathcal{L}u := q_1^{2\sigma-1}\left(2\sigma(p_2\frac{\partial u}{\partial p_2} - q_2\frac{\partial u}{\partial q_2}) + q_1^{2\sigma}(q_1\frac{\partial u}{\partial q_1} - 4\sigma p_1\frac{\partial u}{\partial p_1})\right). \tag{4.1}$$

Then the equation (3.1) can be written in the form

$$\mathcal{L}u + \{\mathcal{H}, u\} - \mathcal{L}u \equiv \mathcal{L}u + Ru = 0, \quad Ru := \{\mathcal{H}, u\} - \mathcal{L}u. \tag{4.2}$$

The Lagrange-Charpit system corresponding to $\mathcal{L}$ is given by

$$\frac{dq_1}{q_1^{4\sigma}} = \frac{dq_2}{-2\sigma q_1^{2\sigma-1}q_2} = \frac{dp_1}{-4\sigma q_1^{4\sigma-1}p_1} = \frac{dp_2}{2\sigma q_1^{2\sigma-1}p_2}. \tag{4.3}$$

We integrate (4.3) by taking q_1 as an independent variable. By simple computations we can easily see that the solution of (4.3) is given by

$$q_2 = q_2^0 \exp\left(q_1^{-2\sigma}\right),\ p_2 = p_2^0 \exp\left(-q_1^{-2\sigma}\right),\ p_1 = p_1^0 q_1^{-4\sigma}, \tag{4.4}$$

where q_2^0, p_2^0 and p_1^0 are certain constants.

Because the solution of the homogeneous equation $\mathcal{L}v = 0$ is given by

$$v = \phi(p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right), q_2 \exp\left(-q_1^{-2\sigma}\right)), \tag{4.5}$$

with ϕ being an arbitrary function, we first construct a solution of $\mathcal{L}v = 0$ in the form

$$u_0 = u_0(p_1^0, p_2^0) \equiv u_0(p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right))$$

such that $\partial u_0 / \partial p_2^0 \neq 0$, where u_0 is an arbitrary function. We then construct a solution u of (4.2) in the form

$$u = \sum_{j=0}^{\infty} u_j(q_1, p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)) \left(\exp\left(-q_1^{-2\sigma}\right) q_2\right)^j, \tag{4.6}$$

where $u_0(q_1, p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)) \equiv u_0(p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right))$.

We note that R in (4.2) has analytic coefficients and R raises the power of q_2 at least by one. On the other hand we have

$$\mathcal{L}\left(u_j\left(\exp\left(-q_1^{-2\sigma}\right)q_2\right)^j\right) = (\mathcal{L}u_j)\left(\exp\left(-q_1^{-2\sigma}\right)q_2\right)^j . \tag{4.7}$$

Hence, if we substitute (4.6) into (4.2) and compare the coefficients of q_2^j of both sides, then we have

$$\mathcal{L}u_j = (\text{ linear functions of } u_k \text{ and their derivatives } (k<j), \quad j=1,2,\ldots \tag{4.8}$$

We note that the right-hand side is a known quantity if we determine u_j recursively.

We will solve $\mathcal{L}v = f$, where

$$v = v\left(q_1, p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)\right).$$

By making the change of variables $(q_1, p_1, p_2) \mapsto (q_1, p_1^0, p_2^0)$ given by (4.4), the equation $\mathcal{L}v = f(q_1, p_1, p_2)$ can be written in the form

$$q_1^{4\sigma}(\partial v/\partial q_1) = g(q_1, p_1^0, p_2^0), \tag{4.9}$$

where

$$g \equiv g(q_1, p_1^0, p_2^0) = f(q_1, p_1^0 q_1^{-4\sigma}, p_2^0 \exp\left(-q_1^{-2\sigma}\right)).$$

Hence the solution of (4.9) is given by

$$v = \int_a^{q_1} s^{-4\sigma} g(s, p_1^0, p_2^0)\,ds, \tag{4.10}$$

where $a \neq 0$ is an arbitrary complex constant. If we go back to the original variables q_1, p_1 and p_2, then we obtain a solution of $\mathcal{L}v = f$. Therefore we have a solution u of (4.2) given by (4.6).

Finally, we will show that u is a formal integral of (2.3) functionally independent of $\mathcal{H}$. Hence our Hamiltonian system is formally Liouville-integrable. Indeed, if this is not the case, then we have $u = \phi(\mathcal{H})$ for some smooth function ϕ of one variable. If we set $q_2 = 0$, then we obtain

$$u_0\left(p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)\right) = \phi(\mathcal{H})|_{q_2=0} = \phi(p_1 q_1^{4\sigma}).$$

This is a contradiction to the assumption that $\partial u_0/\partial p_2^0 \neq 0$. Summing up the above we obtain

Proposition 4.1. *Let $u_0(p_1^0, p_2^0)$ be an arbitrary analytic function of p_1^0 and p_2^0 such that $\partial u_0/\partial p_2^0 \neq 0$. Then (2.3) is formally Liouville-integrable in the sense that (4.6) is a formal integral of (2.3) which is functionally independent of $\mathcal{H}$.*

In the rest of this section we give an example for which the hyperasymptotic expansion (4.6) converges in some subset of a neighborhood of the origin. We consider the following Hamiltonian corresponding to $r = q_1^{2\sigma} + q_2^2$

$$\mathcal{H} = -2\sigma q_1^{2\sigma-1} q_2 p_2 + p_1 \left((q_1^{2\sigma} + q_2^2)^2 + 2q_2^2 \right). \tag{4.11}$$

Clearly, the integral u is the solution of (4.2), where $\mathcal{L}$ and R are given, respectively, by (4.1) and

$$\begin{aligned} Ru = &\left(2\sigma(2\sigma-1) q_2 p_2 q_1^{2\sigma-2} - 4\sigma q_1^{2\sigma-1} q_2^2 p_1 \right) \frac{\partial u}{\partial p_1} \\ &- 4p_1 q_2 (q_1^{2\sigma} + q_2^2 + 1) \frac{\partial u}{\partial p_2} + q_2^2 (q_2^2 + 2q_1^{2\sigma} + 2) \frac{\partial u}{\partial q_1}. \end{aligned} \tag{4.12}$$

For simplicity we write (4.6) in the form

$$u = \sum_{j=0}^{\infty} u_j(q_1, p_1^0, p_2^0)(q_2^0)^j, \tag{4.13}$$

where $u_0 \equiv u_0(p_1^0, p_2^0)$ can be chosen arbitrarily. In the following we take u_0 as a linear function of p_1^0 and p_2^0

$$u_0(p_1^0, p_2^0) = c_1 p_1^0 + c_2 p_2^0, \ c_1, c_2, \text{ constants}. \tag{4.14}$$

It is easy to see, from the construction of the formal solution that all u_j's are linear functions of p_1^0 and p_2^0. Namely we have

$$u = \sum_{j=0}^{\infty} \left(c_{j,1}(q_1) p_1^0 + c_{j,2}(q_1) p_2^0 \right) (q_2^0)^j, \tag{4.15}$$

for some functions $c_{j,1}(q_1)$ and $c_{j,2}(q_1)$. Let ε_0 be a small positive constant. Then we define

$$S_0 := \{q_1 \in \mathbb{C}; |q_1| < \varepsilon_0\} \cap \left\{ q_1 \in \mathbb{C}; \Re\, q_1^{2\sigma} < 0 \right\}, \tag{4.16}$$

where $\Re\, q_1^{2\sigma}$ denotes the real part of $q_1^{2\sigma}$.

Then we have

Proposition 4.2. *Let the Hamiltonian $\mathcal{H}$ be given by (4.11). Then there exist a $\delta > 0$, an $\varepsilon_0 > 0$, neighborhoods V_1, V_2 of the origin in $\mathbb{C}$ and the formal integral u, (4.15) such that u converges when*

$$\{(q_1, q_2, p_1, p_2); q_1 \in S_0, p_1 \in V_1, p_2 \in V_2, |\exp\left(-q_1^{-2\sigma}\right) q_2| < \delta\}.$$

§ 5. Discussions and future problems

In the preceeding section we gave an example of a Hamiltonian system for which (4.6) converges when q_2^0 is in some neighborhood of the origin and q_1 is in a sectorial domain S_0. It is an open question whether (4.6) converges on the set of q_1-plane which contains a positive real axis. As to general Hamiltonians, it seems that the formal integral (4.6) diverges because R in (4.2) has a loss of derivative. It is also an interesting question whether (4.6) is summable with respect to q_2^0. We will study these problems in a future.

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