The Elliptic Quantum Group $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$

By

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Abstract

We survey recent results on a formulation of the elliptic quantum group $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$ as an $H$-Hopf algebroid and its representation theory. We put emphasis on a connection of $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$ with the quantum affine algebra $U_q(\hat{\mathfrak{s}\mathfrak{l}}_2)$ and a constructive derivation of both finite and infinite-dimensional representations from those of $U_q(\hat{\mathfrak{s}\mathfrak{l}}_2)$. Included is an announcement of a new result on a criterion for the finiteness of irreducible pseudo-highest weight representations stated in terms of an elliptic analogue of the Drinfeld polynomials. A derivation of the type I and II vertex operators of $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$ and its implication in the algebraic analysis of elliptic solvable lattice models are also explained.

§1. Introduction

Theory of elliptic quantum groups has been developed in the two different approaches, the one based on $H$-Hopf algebroids [10] and the other on quasi-Hopf algebras [8].

The $H$-Hopf algebroid was introduced by Etingof and Varchenko [10], motivated by the work of Felder and Varchenko [12, 13]. There are some structures added by Koelink and Rosengren[19, 31]. See also a survey by van Norden [34]. A similar coalgebra structure was introduced by Lu [30] and Xu [35]. As an $H$-Hopf algebroid, Felder’s elliptic quantum group $E_{r,\eta}(\mathfrak{s}\mathfrak{l}_2)$ was formulated in terms of the $L$ operator satisfying the $RLL$ relation associated with the elliptic dynamical $R$ matrices[12, 13, 10, 20].

The quasi-Hopf algebra formulation was carried out by Jimbo, Konno, Odake and Shiraishi [17] motivated by the works of Drinfeld[8], Babelon, Bernard and Billey [2] and Frønsdal [15]. There are two types of elliptic quantum groups (quasi-Hopf algebras),

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the vertex type $\mathcal{A}_{q,p}(\widehat{\mathfrak{s}\mathfrak{l}}_N)$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, where $\mathfrak{g}$ is an affine Lie algebra. $p$ is a complex parameter giving the nome of the related elliptic functions. $\lambda$ denotes a Cartan subalgebra valued parameter which provides the elliptic nome and the dynamical parameters. Both $\mathcal{A}_{q,p}(\widehat{\mathfrak{s}\mathfrak{l}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ are isomorphic to the corresponding quantum affine algebras $U_q(\mathfrak{g})$ as associative algebras, but their coalgebra structures are deformed from $U_q(\mathfrak{g})$ by the twistors $E(r)$ and $F(\lambda)$, respectively[2, 15, 17]. Here $r$ is related to $p$ by $p = q^{2r}$. Felder’s elliptic quantum group also has a formulation as a quasi-Hopf algebra[9].

The classification of the vertex and the face types is based on the fact that the vector representation of the universal dynamical $R$ matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{s}\mathfrak{l}}_2)$ yields Baxter’s elliptic $R$ matrix for the eight-vertex model[3], whereas the one of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ [17, 21, 22, 25] yields the face type elliptic Boltzmann weight of the SOS face model associated with $\mathfrak{g}$ [1, 16]. Since the latter Boltzmann weights are nothing but the elliptic dynamical $R$ matrices used in Felder’s elliptic quantum groups, we classify Felder’s ones as the face type. See also [4] for a universal formulation of the vertex-face correspondence in the quasi-Hopf algebra scheme.

Each approach has advantages and disadvantages. An advantage of the quasi-Hopf algebra is that each of $\mathcal{A}_{q,p}(\widehat{\mathfrak{s}\mathfrak{l}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ has an apparent connection to $U_q(\mathfrak{g})$ by the twist procedure. In particular, we can formulate both algebraic and representation theoretical objects of the quasi-Hopf algebra, such as the comultiplication, the universal dynamical $R$ matrices and the vertex operators, from the corresponding objects of $U_q(\mathfrak{g})$[17]. However a disadvantage is a complication of the coalgebra structure due to the twist procedure mentioned above, so that it is not suitable for a practical calculation.

In contrast, the coalgebra structure of the known $H$-Hopf algebroid is simple enough for practical use. In fact it was already applied to a study of tensor product representations[13] and of co-representations[19, 20]. In particular, by studying the co-representation, Koelink, van Norden and Rosengren have succeeded to derive the terminating very-well-poised balanced elliptic hypergeometric series $12V_{11}$ and their biorthogonality relations, which were introduced by Frenkel-Turaev[14] and developed by Spiridonov and Zhedanov[32, 33]. However a disadvantage is a lack of direct connection to $U_q(\mathfrak{g})$ even as an associative algebra. This defect seems to be an obstacle to extend the known $H$-Hopf algebroids associated with finite-dimensional simple Lie algebras to those associated with affine Lie algebras and to develop their representation theory in systematic way.

In this paper, we explain a new realization of the face type elliptic quantum group given by the elliptic algebra $U_{q,p}(\widehat{\mathfrak{s}\mathfrak{l}}_2)$[26, 27]. It is a realization by the Drinfeld generators of the quantum affine algebra $U_q(\widehat{\mathfrak{s}\mathfrak{l}}_2)$ and has an $H$-Hopf algebroid structure, so that it provides a complement to the above two approaches.
The elliptic algebra $U_{q,p}(\hat{\mathfrak{s}l}_2)$ was introduced in [24] as an elliptic analogue of the algebra of the Drinfeld currents for $U_q(\mathfrak{sl}_2)$. As an associative algebra, $U_{q,p}(\hat{\mathfrak{s}l}_2)$ is isomorphic to the tensor product of $U_q(\mathfrak{sl}_2)$ and the Heisenberg algebra $\{P, e^{Q}\}$ [18]. A similar algebra was studied in [35]. The generators of $U_{q,p}(\hat{\mathfrak{s}l}_2)$ are treated through the generating functions called the elliptic currents. In terms of the elliptic currents, we can construct the $L$ operator and derive the $RLL$ relation for $U_{q,p}(\hat{\mathfrak{s}l}_2)$ [18]. It turns out that the resultant $RLL$ relation is nothing but the one for Felder’s elliptic quantum group with a central extension. We hence formulate the $H$-Hopf algebroid structure on $U_{q,p}(\hat{\mathfrak{s}l}_2)$ in a way similar to [10, 20], but with a modification due to the existence of the non-zero central element.

Due to a direct connection to $U_q(\mathfrak{sl}_2)$, we can derive all representations of $U_{q,p}(\hat{\mathfrak{s}l}_2)$ constructively from those of $U_q(\mathfrak{sl}_2)$. This yields quite a parallel structure to $U_q(\mathfrak{sl}_2)$ for both finite and infinite-dimensional representations. In particular, we can state a criterion for the finiteness of irreducible pseudo-highest weight representations in terms of an elliptic analogue of the Drinfeld polynomials. This provides an elliptic analogue of the works by Drinfeld [7] and by Chari and Pressley [5]. As an example of the application of infinite-dimensional representations, we also report on a formulation and derivation of the type I and II vertex operators of $U_{q,p}(\hat{\mathfrak{s}l}_2)$ studied in [26].

This paper is organized as follows. In the next section, we review the elliptic algebra $U_{q,p}(\hat{\mathfrak{s}l}_2)$. The $L$ operator and the $RLL$ relation are also introduced. In Sect.3, we describe an $H$-Hopf algebroid structure of $U_{q,p}(\hat{\mathfrak{s}l}_2)$ following [26, 27]. In Sect.4 we summarize some basic facts on the dynamical representations. Theorem 4.2 is fundamental in a constructive derivation of the dynamical representations of $U_{q,p}(\hat{\mathfrak{s}l}_2)$. Sect.5 is devoted to a study of finite-dimensional irreducible pseudo-highest weight representations and includes an announcement of new results. In Sect.6, we report a result on the vertex operators of $U_{q,p}(\hat{\mathfrak{s}l}_2)$ following [26].

In the forthcoming paper [27], we plan to provide proofs of the statements in Sect. 3, 4, 5 and also report on a structure of the finite-dimensional tensor product representations as well as an alternative derivation of the $12V_{11}$.

\section{The Elliptic Algebra $U_{q,p}(\hat{\mathfrak{s}l}_2)$}

In this section we review a definition of the elliptic algebra $U_{q,p}(\hat{\mathfrak{s}l}_2)$ and its $RLL$ relation following [18, 26, 27].

\subsection{Definition of $U_{q,p}(\hat{\mathfrak{s}l}_2)$}

Let us fix a complex number $q$ such that $q \neq 0, |q| < 1$.

**Definition 2.1.** [8] For a field $\mathbb{K}(\supset \mathbb{C})$, the quantum affine algebra $\mathbb{K}[U_q(\mathfrak{sl}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the standard
Drinfeld generators \( a_n \ (n \in \mathbb{Z}_{\neq 0}) \), \( x_n^\pm \ (n \in \mathbb{Z}) \), \( h \), \( c \), \( d \). The defining relations are given as follows.

\[
c : \text{central},
\]
\[
[h,d] = 0, \quad [d,a_n] = na_n, \quad [d,x_n^\pm] = nx_n^\pm,
\]
\[
h,a_n = 0, \quad [h,x^\pm(z)] = \pm 2x^\pm(z),
\]
\[
[a_n,a_m] = \frac{2n}{[2n]_{q}}[cn]_{q} q^{-c|m|} \delta_{n+m,0},
\]
\[
[a_n,x^+(z)] = \frac{2n}{[2n]_{q}} q^{-c|n|} z^n x^+(z),
\]
\[
[a_n,x^-(z)] = -\frac{2n}{[2n]_{q}} q^{-c|n|} z^n x^-(z),
\]
\[
(z - q^{\pm 2}w)x^\pm(z)x^\pm(w) = (q^{\pm 2}z - w)x^\pm(w)x^\pm(z),
\]
\[
[x^+(z),x^-(w)] = \frac{1}{q-q^{-1}} \left( \delta(q^{-c} \frac{z}{w}) \psi(q^{c/2}w) - \delta(q^{c} \frac{z}{w}) \varphi(q^{-c/2}w) \right).
\]

where we use \([n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}, \delta(z) = \sum_{n \in \mathbb{Z}} z^n\) and the Drinfeld currents defined by

\[
x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n},
\]
\[
\psi(q^{c/2}z) = q^h \exp \left( (q - q^{-1}) \sum_{n > 0} a_n z^{-n} \right), \quad \varphi(q^{-c/2}z) = q^{-h} \exp \left( -(q - q^{-1}) \sum_{n > 0} a_n z^{n} \right).
\]

We also denote by \( \mathbb{K}[U_q'(\widehat{\mathfrak{sl}}_2)] \) the subalgebra of \( \mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)] \) generated by the same generators as \( \mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)] \) except \( d \).

Remark. We follows the conventions in [18]. In particular, we occasionally treat \( c \) as a complex number on the understanding that we make a specialization each time.

Let \( r \) be a generic complex number. We set \( r^* = r - c \), \( p = q^{2r} \) and \( p^* = q^{2r^*} \). We define the Jacobi theta functions \([u]\) and \([u]^*\) by

\[
[u] = \frac{q^{\frac{u^2}{r}}}{(p;p)^3_\infty} \Theta_p(q^{2u}), \quad [u]^* = \frac{q^{\frac{u^2}{r^*}}}{(p^*;p^*)^3_\infty} \Theta_{p^*}(q^{2u}),
\]
\[
\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty,
\]

where

\[
(z; p_1, p_2, \cdots, p_m)_\infty = \prod_{n_1, n_2, \cdots, n_m = 0}^{\infty} (1 - z p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}).
\]

Setting \( p = e^{-\frac{2\pi i}{r}} \), \([u]\) satisfies the quasi-periodicity \([u+r] = -[u]\), \([u+r \tau] = -e^{-\pi i(2u/r+\tau)}[u]\).
Let \( \{P, e^Q\} \) be a Heisenberg algebra commuting with \( \mathbb{C}[U_q(\hat{sl}_2)] \) and satisfying

\begin{equation}
[P, e^Q] = -e^Q.
\end{equation}

We take the realization \( Q = \frac{\partial}{\partial P} \). We set \( H = \mathbb{C}P \oplus \mathbb{C}r^* \) and \( H^* = \mathbb{C}Q \oplus \mathbb{C}\frac{\partial}{\partial r^*} \) with the pairing \( \langle , , \rangle \). \n
\[ \langle Q, P \rangle = 1 = \langle \frac{\partial}{\partial r^*}, r^* \rangle, \]

the others are zero. We also consider the abelian group \( \hat{H}^* = \mathbb{Z}Q \). We denote by \( \mathbb{C}[\hat{H}^*] \) the group algebra over \( \mathbb{C} \) of \( \hat{H}^* \). We denote by \( e^\alpha \) the element of \( \mathbb{C}[\hat{H}^*] \) corresponding to \( \alpha \in \hat{H}^* \). These \( e^\alpha \) satisfy \( e^\alpha e^\beta = e^{\alpha + \beta} \) and \( (e^\alpha)^{-1} = e^{-\alpha} \). In particular, \( e^0 = 1 \) is the identity element.

Let \( M_{H^*} \) be the field of meromorphic functions on \( H^* \). We regard a meromorphic function \( \hat{f} = f(P, r^*) \) of \( P \) and \( r^* \) as an element of \( M_{H^*} \) by \( \hat{f}(\mu) = f(<\mu, P>, <\mu, r^*>) \mu \in H^* \).

Now we take the field \( \mathbb{F} = M_{H^*} \) as \( K \) and consider the semi-direct product \( \mathbb{C}\)-algebra \( U_{q,p}(\hat{sl}_2) = \mathbb{F}[U_q(\hat{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\overline{H}^*] \), whose multiplication is defined by

\[
(f(P, r^*)a \otimes e^\alpha) (g(P, r^*)b \otimes e^\beta) = f(P, r^*)g(P+<\alpha, P>, r^*)ab \otimes e^{\alpha+\beta},
\]

\( a, b \in \mathbb{C}[U_q(\hat{sl}_2)], f(P, r^*), g(P, r^*) \in \mathbb{F}, \alpha, \beta \in \overline{H}^* \).

The following automorphism \( \phi_r \) of \( \mathbb{C}[U_q(\hat{sl}_2)] \) is the key to our “elliptic deformation” [18].

\[
c \mapsto c, \quad h \mapsto h, \quad d \mapsto d,
\]

\[
x^+(z) \mapsto u^+(z, p)x^+(z), \quad x^-(z) \mapsto x^-(z)u^-(z, p),
\]

\[
\psi(z) \mapsto u^+(q^{c/2}z, p)\psi(z)u^-(q^{-c/2}z, p),
\]

\[
\varphi(z) \mapsto u^+(q^{-c/2}z, p)\varphi(z)u^-(q^{c/2}z, p).
\]

Here we set

\[
u^+(z, p) = \exp \left( \sum_{n>0} \frac{1}{[r^*n]_q} a_n(q^r z)^n \right), \quad u^-(z, p) = \exp \left( -\sum_{n>0} \frac{1}{[rn]_q} a_n(q^{-r} z)^{-n} \right).
\]

**Definition 2.2.** We define the elliptic currents \( E(u), F(u), K(u) \in U_{q,p}(\hat{sl}_2)[[u]] \)
and $\hat{d}$ by

\[ E(u) = \phi_r(x^+(z))e^{2Q}z^{-\frac{P-1}{r^*}}, \]
\[ F(u) = \phi_r(x^-(z))z^{\frac{P+h-1}{r}}, \]
\[ K(u) = \exp\left(\sum_{n>0} \frac{[n]_q}{[2n]_q[r*n]_q}a_{-n}(q^c z)^n\right) \exp\left(-\sum_{n>0} \frac{[n]_q}{[2n]_q[r*n]_q}a_n z^{-n}\right) \times e^{Q}z^{-\frac{c}{4r^*}(2P-1)+\frac{1}{2r}h}, \]
\[ \hat{d} = d - \frac{1}{4r^*}(P-1)(P+1) + \frac{1}{4r}(P+h-1)(P+h+1), \]

where we set $z = q^{2u}$.

From Definition 2.1 and (2.1), we can derive the following relations.

**Proposition 2.3.**

- $c$: central,
  \[ [h, a_n] = 0, \quad [h, E(u)] = 2E(u), \quad [h, F(u)] = -2F(u), \]
  \[ [\hat{d}, h] = 0, \quad [\hat{d}, a_n] = n a_n, \]
  \[ [\hat{d}, E(u)] = (-z \frac{\partial}{\partial z} - \frac{1}{r^*})E(u), \quad [\hat{d}, F(u)] = (-z \frac{\partial}{\partial z} - \frac{1}{r})F(u), \]
  \[ [a_n, a_m] = \frac{[2n]_q}{[2n]_q}[cn]_q q^{-c|n|}\delta_{n+m,0}, \]
  \[ [a_n, E(u)] = \frac{[2n]_q}{[2n]_q}q^{-c|n|}z^n E(u), \]
  \[ [a_n, F(u)] = -\frac{[2n]_q}{[2n]_q}z^n F(u), \]

- \[ E(u)E(v) = \frac{[u-v+1]^*}{[u-v-1]^*}E(v)E(u), \]
- \[ F(u)F(v) = \frac{[u-v-1]}{[u-v+1]}F(v)F(u), \]
- \[ [E(u), F(v)] = \frac{1}{q - q^{-1}} \left( \delta \left(q^{-c}\frac{z}{w}\right) H^+(q^{c/2}w) - \delta \left(q^{c}\frac{z}{w}\right) H^-(q^{-c/2}w) \right), \]

where $z = q^{2u}$, $w = q^{2v}$,

\[ H^\pm(z) = \kappa K \left( u \pm \frac{1}{2}(r - \frac{c}{2}) + \frac{1}{2} \right) K \left( u \pm \frac{1}{2}(r - \frac{c}{2}) - \frac{1}{2} \right), \]
\[ \kappa = \lim_{z \to q^{-2}} \frac{\xi(z; p^*, q)}{\xi(z; p, q)}, \quad \xi(z; p, q) = \frac{(q^2 z; p, q^4)_\infty (pq^2 z; p, q^4)_\infty}{(p z; p, q^4)_\infty (p q z; p, q^4)_\infty}. \]
Definition 2.4. We call a pair \((\mathbb{F}[U_q(\widehat{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\hat{H}^*], \phi_r)\) the elliptic algebra \(U_{q,p}(\widehat{sl}_2)\). We also denote by \(U_{q,p}^\prime(\widehat{sl}_2)\) the subalgebra \(\mathbb{F}[U_q^\prime(\widehat{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\hat{H}^*]\) of \(U_{q,p}(\widehat{sl}_2)\).

The following relations are crucial in the formulation of the \(H\)-Hopf algebroid structure on \(U_{q,p}(\widehat{sl}_2)\). See Sect.3.2.

Proposition 2.5.

\[
[K(u), P] = K(u), \quad [E(u), P] = 2E(u), \quad [F(u), P] = 0,
\]
\[
[K(u), P + h] = K(u), \quad [E(u), P + h] = 0, \quad [F(u), P + h] = 2F(u).
\]

§ 2.2. The RLL-relation for \(U_{q,p}(\widehat{sl}_2)\)

In order to define the \(L\) operator, we need the half currents defined by the following formulae.

Definition 2.6.

\[
K^+(u) = K(u + \frac{r+1}{2}),
\]
\[
E^+(u) = a^* \oint_{C^*} E(u') \frac{[u-u'+c/2-P+1]^* [1]^*}{[u-u+c/2][P-1]^*} \frac{dz'}{2\pi iz'},
\]
\[
F^+(u) = a \oint_{C} F(u') \frac{[u-u'+P+h-1][1]}{[u-u][P+h-1]} \frac{dz'}{2\pi iz'}.
\]

Here \(z' = q^{2u}\) and \(z = q^{2u}\). The contours \(C^*\) and \(C\) are chosen in such a way that \(C^*\) encircles \(zq^c p^n (n \geq 1)\) but not \(zq^c p^n (n \leq 0)\), \(C\) encircles \(zp^n (n \geq 1)\) but not \(zp^n (n \leq 0)\), respectively. The constants \(a, a^*\) are chosen to satisfy \(a^*a[1]^* \kappa / q-q^{-1} = 1\).

Then we define the operator \(\hat{L}^+(u) \in \text{End}_{\mathbb{C}} V \otimes U_{q,p}(\widehat{sl}_2)\) with \(V = \mathbb{C}^2\) as follows.

Definition 2.7.

\[
\hat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u-1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}.
\]

From the relations in Proposition 2.3, we obtain the RLL relation for \(U_{q,p}(\widehat{sl}_2)\).

Proposition 2.8. The operator \(\hat{L}^+(u)\) satisfies the following RLL relation.

\[
R^{+(12)}(u_1 - u_2, P + h) \hat{L}^{+(1)}(u_1) \hat{L}^{+(2)}(u_2) = \hat{L}^{+(2)}(u_2) \hat{L}^{+(1)}(u_1) R^{+(12)}(u_1 - u_2, P),
\]
where \( R^+(u, P + h) \) denotes the elliptic dynamical \( R \) matrices given by

\[
R^+(u, s) = \rho^+(u) \begin{pmatrix}
1 \\
b(u, s) c(u, s) \\
c(u, s) b(u, s)
\end{pmatrix}
\]

with

\[
\rho^+(u) = z^{\frac{1}{2r}} \frac{(pq^2z)^2}{(pz)(pq^4z)} \frac{(z^{-1})(q^4z^{-1})}{(q^2z^{-1})^2}, \quad \{z\} = (z;p, q^4)_{\infty},
\]
\[
b(u, s) = \frac{[s+1][s-1]}{[s]^2} \frac{[u]}{[1+u]}, \quad c(u, s) = \frac{[1]}{[s]} \frac{[s+u]}{[1+u]},
\]
\[
\overline{c}(u, s) = \frac{[1]}{[s]} \frac{[s-u]}{[1+u]}, \quad \overline{b}(u, s) = \frac{[u]}{[1+u]},
\]

and \( R^{+*}(u, P) \) denotes the \( R \) matrix obtained from \( R^+(u, P) \) by the replacements \( r \rightarrow r^*, p \rightarrow p^* \) and \([ \cdot ] \rightarrow [ \cdot ]^*\).

It is also worth while noting the following proposition immediately obtained from Proposition 2.8, which indicates a connection between \( U_{q,p}(\hat{\mathfrak{s}1}_2) \) and the quasi-Hopf algebra \( B_{q, \lambda}(\hat{\mathfrak{s}1}_2) \).

**Proposition 2.9.** [18] Let us set \( L^+(u, P) = \hat{L}^+(u)e^{-h \otimes Q}, \quad h = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \). Then \( L^+(u, P) \) is independent of \( Q \) and satisfies the following dynamical RLL relation.

\[
R^{+(12)}(u_1-u_2, P+h)L^{+(1)}(u_1, P)L^{+(2)}(u_2, P+h^{(1)}) = L^{+(2)}(u_2, P)L^{+(1)}(u_1, P+h^{(2)})R^{+*+(12)}(u_1-u_2, P).
\]

This is the same dynamical RLL relation that characterizes the quasi-Hopf algebra \( B_{q, \lambda}(\hat{\mathfrak{s}1}_2) \) with the parametrization \( \lambda = (r^* + 2)\Lambda_0 + (P + 1)\Lambda_1[18] \). In fact, under this parametrization, the vector representation of the universal dynamical \( R \) matrix \( R(\lambda) \) of \( B_{q, \lambda}(\hat{\mathfrak{s}1}_2) \) coincides with \( R^{+*}(u, P) \) in (2.3). Recalling also that by definition \( B_{q, \lambda}(\hat{\mathfrak{s}1}_2) \) with the usual parameter \( \lambda \in \mathfrak{h} \) is isomorphic to \( \mathbb{C}[U_q(\hat{\mathfrak{s}1}_2)] \) as an associative algebra, we have the isomorphism from \( B_{q, \lambda}(\hat{\mathfrak{s}1}_2) \) with \( \lambda = (r^* + 2)\Lambda_0 + (P + 1)\Lambda_1 \) to \( \mathbb{C}[U_q(\hat{\mathfrak{s}1}_2)] \). Combining these facts, we have the isomorphism \( U_{q,p}(\hat{\mathfrak{s}1}_2) \cong B_{q, \lambda}(\hat{\mathfrak{s}1}_2) \otimes \mathbb{C}[\hat{H}^*] \) with \( \lambda = (r^* + 2)\Lambda_0 + (P + 1)\Lambda_1 \) as a semi-direct product algebra.

Note also that the \( c = 0 \) case of (2.4) is the dynamical RLL relation studied by Felder [12, 9], whereas the \( c = 0 \) case of (2.2) is the RLL relation studied in [10, 11, 19] for the trigonometric \( R \), and in [20] for the elliptic \( R \).
§ 3. H-Hopf Algebroid Structure

This section is an exposition of the H-Hopf algebroid structure on $U_{q,p}(\hat{\mathfrak{s\mathfrak{l}}}_2)$[26, 27]. Our H-Hopf algebroid is an extension of the one studied in [10, 11] and [19, 20] to the one with the central element $c$.

§ 3.1. Definition of the H-Hopf Algebroid[10, 11, 19]

Let $A$ be a complex associative algebra, $H$ be a finite dimensional commutative subalgebra of $A$, and $M_{H^*}$ be the field of meromorphic functions on $H^*$ the dual space of $H$.

**Definition 3.1.** An $H$-algebra is a complex associative algebra $A$ with 1, which is bigraded over $H^*$, $A = \bigoplus_{\alpha, \beta \in H^*} A_{\alpha \beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : M_{H^*} \rightarrow A_{00}$ (the left and right moment maps), such that

$$\mu_l(f)a = a\mu_l(T_\alpha f), \quad \mu_r(f)a = a\mu_r(T_\beta f), \quad a \in A_{\alpha \beta}, \quad f \in M_{H^*},$$

where $T_\alpha$ denotes the automorphism $(T_\alpha f)(\lambda) = f(\lambda + \alpha)$ of $M_{H^*}$.

**Definition 3.2.** An $H$-algebra homomorphism is an algebra homomorphism $\pi : A \rightarrow B$ between two $H$-algebras $A$ and $B$ preserving the bigrading and the moment maps, i.e. $\pi(A_{\alpha \beta}) \subseteq B_{\alpha \beta}$ for all $\hat{\alpha}, \hat{\beta} \in H^*$ and $\pi(\mu^A_{l}(f)) = \mu^B_{l}(f), \pi(\mu^A_{r}(f)) = \mu^B_{r}(f)$.

Let $A$ and $B$ be two $H$-algebras. The tensor product $A \tilde{\otimes} B$ is the $H^*$-bigraded vector space with

$$(A \tilde{\otimes} B)_{\alpha \beta} = \bigoplus_{\gamma \in H^*} (A_{\alpha \gamma} \otimes_{M_{H^*}} B_{\gamma \beta}),$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the following relation.

$$\mu^A_{r}(f)a \otimes b = a \otimes \mu^B_{l}(f)b, \quad a \in A, b \in B, f \in M_{H^*}.$$

The tensor product $A \tilde{\otimes} B$ is again an $H$-algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps

$$\mu^A_{l} \tilde{\otimes} B = \mu^A_{l} \otimes 1, \quad \mu^A_{r} \tilde{\otimes} B = 1 \otimes \mu^B_{r}.$$

Let $D$ be the algebra of automorphisms $M_{H^*} \rightarrow M_{H^*}$

$$D = \{ \sum_i \hat{f}_i T_{\beta_i} | \hat{f}_i \in M_{H^*}, \beta_i \in H^* \}.$$
Equipped with the bigrading $\mathcal{D}_{\alpha \alpha} = \{ \hat{T}_{- \alpha} | \hat{T} \in \mathcal{M}_{H^{*}}, \ \alpha \in H^{*} \}$, $\mathcal{D}_{\alpha \beta} = 0$ ($\alpha \neq \beta$), and the moment maps $\mu_{l}^{p}, \mu_{r}^{p} : M_{H^{*}} \to \mathcal{D}_{00}$ defined by $\mu_{l}^{p}(\hat{T}) = \mu_{r}^{p}(\hat{T}) = \hat{T}_{0}$, $\mathcal{D}$ is an $H$-algebra. For any $H$-algebra $A$, we have the canonical isomorphism as an $H$-algebra

$$A \cong A \overline{\otimes} \mathcal{D} \cong \mathcal{D} \overline{\otimes} A$$

by $a \cong a \overline{\otimes} \hat{T}_{- \beta} \cong \hat{T}_{- \alpha} \overline{\otimes} a$ for all $a \in A_{\alpha \beta}$. Hence $\mathcal{D}$ plays the role of unit object in the category of $H$-algebras.

**Definition 3.3.** An $H$-bialgebroid is an $H$-algebra $A$ equipped with two $H$-algebra homomorphisms $\Delta : A \to A \overline{\otimes} A$ (the comultiplication) and $\varepsilon : A \to \mathcal{D}$ (the counit) such that

$$(\Delta \overline{\otimes} \mathrm{id}) \circ \Delta = (\mathrm{id} \overline{\otimes} \Delta) \circ \Delta,$$

$$(\varepsilon \overline{\otimes} \mathrm{id}) \circ \Delta = (\mathrm{id} \overline{\otimes} \varepsilon) \circ \Delta,$$

under the identification (3.2).

**Definition 3.4.** An $H$-Hopf algebroid is an $H$-bialgebroid $A$ equipped with a $\mathbb{C}$-linear map $S : A \to A$ (the antipode), such that

$$S(\mu_{r}(\hat{T})a) = S(a)\mu_{l}(\hat{T}), \quad S(a \mu_{l}(\hat{T})) = \mu_{r}(\hat{T})S(a), \quad \forall a \in A, \hat{T} \in M_{H^{*}},$$

$$m \circ (\mathrm{id} \overline{\otimes} S) \circ \Delta(a) = \mu_{l}(\varepsilon(a)1), \quad \forall a \in A,$$

$$m \circ (S \overline{\otimes} \mathrm{id}) \circ \Delta(a) = \mu_{r}(T_{\alpha}(\varepsilon(a)1)), \quad \forall a \in A_{\alpha \beta},$$

where $m : A \overline{\otimes} A \to A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{H^{*}}$.

§3.2. $H$-Hopf Algebroid Structure on $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_{2})$

Now let us consider the elliptic algebra $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_{2})$. The commutative subalgebra $H$ is given in §2.1. Let $\hat{\mathfrak{h}} = \mathbb{C}h$ be the Cartan subalgebra, $\alpha_{1}$ the simple root and $\hat{\mathfrak{h}}_{1}$ the fundamental weight of $\mathfrak{sl}(2, \mathbb{C})$. We set $Q = \mathbb{Z}\alpha_{1}$ and $\hat{\mathfrak{h}}^{*} = \mathbb{C}\hat{\Lambda}_{1}$. Let $\langle , \rangle$ be the standard paring of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^{*}$. Using the isomorphism $\phi : Q \to H^{*}$ by $n\alpha_{1} \mapsto nQ$, we define the $\hat{H}^{*}$-bigrading of $U_{q,p} = U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_{2})$ by

$$U_{q,p} = \bigoplus_{\alpha, \beta \in H^{*}} (U_{q,p})_{\alpha \beta},$$

$$(U_{q,p})_{\alpha \beta} = \left\{ x \in U_{q,p} \bigg| q^{\langle \phi^{-1}(\alpha-\beta), h \rangle} x = q^{\langle \phi^{-1}(\alpha-\beta), h \rangle} x, \right. \left. q^{\langle p, x \rangle} = q^{\langle p, x \rangle} x \right\}.$$
Let \( M_H \) be the field of meromorphic functions given in \( \S 2.1 \). We define two moment maps \( \mu_l, \mu_r : M_H \rightarrow (U_{q,p})_{00} \) as follows

\[
(3.4) \quad \mu_l(\hat{f}) = f(P + h, r^* + c), \quad \mu_r(\hat{f}) = f(P, r^*).
\]

From (3.3), one finds for \( x \in (U_{q,p})_{\alpha \beta} \)

\[
\mu_l(\hat{f})x = f(P + h, r^* + c)x = xf(P + h + < \alpha, P >, r^* + c) = x\mu_l(T_\alpha \hat{f}),
\]

\[
\mu_r(\hat{f})x = f(P, r^*)x = xf(P + < \beta, P >, r^*) = x\mu_r(T_\beta \hat{f}),
\]

where \( T_\alpha = e^\alpha \) denotes a shift operator \( M_H \rightarrow M_H \) defined by

\[
(T_\alpha \hat{f}) = e^\alpha f(P, r^*)e^{-\alpha} = f(P + < \alpha, P >, r^*).
\]

Equipped with the bigrading structure (3.3) and the two moment maps (3.4), the elliptic algebra \( U_{q,p}(sl_2) \) is an \( H \)-algebra.

We also have the \( H \)-algebra \( \mathcal{D} \) of the shift operators on \( M_H \)

\[
\mathcal{D} = \{ \sum_i \hat{f}_i T_{\alpha_i} \mid \hat{f}_i \in M_H, \alpha_i \in \overline{H}^* \}
\]

whose bigrading structure and moment maps are given as in \( \S 3.1 \).

The tensor product among the \( H \)-algebras \( U_{q,p} \) and \( \mathcal{D} \) is defined as in \( \S 3.1 \). In particular, we have the \( H \)-algebra isomorphism \( U_{q,p} \otimes \mathcal{D} \cong U_{q,p} \cong \mathcal{D} \otimes U_{q,p} \) by \( x \otimes T_{-\beta} = x = T_{-\alpha} \otimes x \) for \( x \in (U_{q,p})_{\alpha \beta} \).

Now let us consider the coalgebra structure of \( U_{q,p} \). Let \( \hat{L}^+(u) \) be the \( L \) operator introduced in \( \S 2.2 \). We write the entries of \( \hat{L}^+(u) \) as

\[
\hat{L}^+(u) = \begin{pmatrix}
\hat{L}_{++}^+(u) & \hat{L}_{+-}^+(u) \\
\hat{L}_{-+}^+(u) & \hat{L}_{--}^+(u)
\end{pmatrix}.
\]

From Proposition 2.5 and Definition 2.7, one finds

\[
\hat{L}_{\varepsilon_1 \varepsilon_2}^+(u) \in (U_{q,p})_{-\varepsilon_1 Q, -\varepsilon_2 Q}.
\]

It is also easy to check the relations

\[
f(P + h, r^* + c)\hat{L}_{\varepsilon_1 \varepsilon_2}^+(u) = \hat{L}_{\varepsilon_1 \varepsilon_2}^+(u)f(P + h - \varepsilon_1, r^* + c),
\]

\[
f(P, r^*)\hat{L}_{\varepsilon_1 \varepsilon_2}^+(u) = \hat{L}_{\varepsilon_1 \varepsilon_2}^+(u)f(P - \varepsilon_2, r^*).
\]

**Definition 3.5.** We define \( H \)-algebra homomorphisms, \( \varepsilon : U_{q,p} \rightarrow \mathcal{D} \) and \( \Delta : \)
\[ U_{q,p} \rightarrow U_{q,p} \otimes U_{q,p} \text{ by} \]
\[
\begin{align*}
\varepsilon(\hat{L}_{\varepsilon_1 \varepsilon_2}^+(u)) &= \delta_{\varepsilon_1, \varepsilon_2} T_{-\varepsilon_2 Q}, \\
\varepsilon(e^Q) &= e^Q, \quad \varepsilon(\mu_l(\hat{f})) = \varepsilon(\mu_r(\hat{f})) = f(P, r^*) T_0, \\
\Delta(\hat{L}_{\varepsilon_1 \varepsilon_2}^+(u)) &= \sum_{\varepsilon'} \hat{L}_{\varepsilon_1 \varepsilon'}(u) \otimes \hat{L}_{\varepsilon' \varepsilon_2}^+(u), \\
\Delta(e^Q) &= e^Q \otimes e^Q, \\
\Delta(\mu_l(\hat{f})) &= \mu_l(\hat{f}) \otimes 1, \quad \Delta(\mu_r(\hat{f})) = 1 \otimes \mu_r(\hat{f}).
\end{align*}
\]

We also define an \( H \)-algebra anti-homomorphism \( S : U_{q,p} \rightarrow U_{q,p} \) by
\[
\begin{align*}
S(\hat{L}_{++}^+(u)) &= \hat{L}_{--}^+(u-1), \\
S(\hat{L}_{+-}^+(u)) &= -\frac{[P+h+1]}{[P+h]} \hat{L}_{+-}^+(u-1), \\
S(\hat{L}_{-+}^+(u)) &= -\frac{[P]^*}{[P+1]^*} \hat{L}_{-+}^+(u-1), \\
S(e^Q) &= e^{-Q}, \quad S(\mu_r(\hat{f})) = \mu_l(\hat{f}), \quad S(\mu_l(\hat{f})) = \mu_r(\hat{f}).
\end{align*}
\]

One can check that \( \Delta \) and \( S \) preserve the RLL relation (2.2). Furthermore one finds that \( \varepsilon, \Delta \) and \( S \) are the counit, the comultiplication and the antipode satisfying the following relations.

**Proposition 3.6.** [27] The maps \( \varepsilon, \Delta \) and \( S \) satisfy
\[
\begin{align*}
(\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta, \\
(\varepsilon \otimes \text{id}) \circ \Delta &= \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta.
\end{align*}
\]

Equipped with \( (\Delta, \varepsilon, S) \), the \( H \)-algebra \( U_{q,p}(\widehat{\mathfrak{sl}_2}) \) is an \( H \)-Hopf algebroid.

**Definition 3.7.** We call the \( H \)-Hopf algebroid \( (U_{q,p}(\widehat{\mathfrak{sl}_2}), H, M_{H*}, \mu_l, \mu_r, \Delta, \varepsilon, S) \) the elliptic quantum group \( U_{q,p}(\widehat{\mathfrak{sl}_2}) \).

§ 4. Representations

In this section, we summarize some basic facts on the dynamical representations of \( H \)-algebras [10, 19] and their application to \( U'_{q,p}(\widehat{\mathfrak{sl}_2}) \).

Let us consider a vector space \( \hat{V} \) over \( \mathbb{F} \), which is \( \mathfrak{h} \)-diagonalizable,
\[
\hat{V} = \bigoplus_{\mu \in \mathfrak{h}^*} \hat{V}_\mu, \quad \hat{V}_\mu = \{ v \in \hat{V} \mid q^h v = q^\mu v \quad (h \in \mathfrak{h}) \}.
\]
Let us define the $H$-algebra $\mathcal{D}_{H,\hat{\mathcal{V}}}$ of the $\mathbb{C}$-linear operators on $\hat{\mathcal{V}}$ by

$$
\mathcal{D}_{H,\hat{\mathcal{V}}} = \bigoplus_{\alpha, \beta \in H^*} \mathcal{D}_{H,\hat{\mathcal{V}}}_{\alpha \beta},
$$

$$
(\mathcal{D}_{H,\hat{\mathcal{V}}})_{\alpha \beta} = \left\{ X \in \text{End}_{\mathbb{C}} \hat{\mathcal{V}} \mid \begin{aligned}
X(f(P, r^*)v) &= f(P - < \beta, P >, r^*)X(v), \quad v \in \hat{\mathcal{V}}, \\
f(P, r^*) &\in \mathbb{F}, \quad X(\hat{V}_\mu) \subseteq \hat{V}_{\mu + \phi^{-1}(\alpha - \beta)},
\end{aligned} \right\},
$$

$$
\mu_l^{D_{H,\hat{\mathcal{V}}}}(\hat{f})v = f(P + \mu, r^* + c)v, \quad \mu_r^{D_{H,\hat{\mathcal{V}}}}(\hat{f})v = f(P, r^*)v, \quad \hat{f} \in M_{H^*}, \quad v \in \hat{V}_\mu.
$$

Note that the subspace $(\mathcal{D}_{H,\hat{\mathcal{V}}})_{\alpha \beta}$ consists of the $\mathbb{C}$-linear operators on $\hat{\mathcal{V}}$ of the form $x \overline{\otimes} e^{-\beta}$, where $x$ is a $\mathbb{C}$-linear operator carrying the weight $\phi^{-1}(\alpha - \beta)$.

**Definition 4.1.** [10, 19] A dynamical representation of $U'_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$ on $\hat{\mathcal{V}}$ is an $H$-algebra homomorphism $\pi : U'_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2) \rightarrow \mathcal{D}_{H,\hat{\mathcal{V}}}$.

Let $(\tilde{\pi}_V, \hat{\mathcal{V}}), (\tilde{\pi}_W, \hat{\mathcal{W}})$ be two dynamical representations of $U'_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$. We define the tensor product $\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}$ by

$$
\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}} = \bigoplus_{\mu \in \hat{\mathfrak{h}}^*} (\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}})_\mu, \quad (\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}})_\mu = \bigoplus_{\nu \in \hat{\mathfrak{h}}^*} \hat{V}_\nu \otimes_{M_{H^*}} \hat{W}_{\mu - \nu},
$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the relation

$$
f(P, r^*)v \otimes w = v \otimes f(P + \nu, r^* + c)w
$$

for $w \in \hat{W}_\nu$. The action of scalars $f(P, r^*) \in \mathbb{F}$ on the tensor space $\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}$ is defined by

$$
f(P, r^*). (v \overline{\otimes} w) = \Delta(\mu_r(\hat{f}))(v \overline{\otimes} w) = v \overline{\otimes} f(P, r^*)w.
$$

Then one finds a natural $H$-algebra embedding $\theta_{\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}} : \mathcal{D}_{H,\hat{\mathcal{V}}} \overline{\otimes} \mathcal{D}_{H,\hat{\mathcal{W}}} \rightarrow \mathcal{D}_{H,\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}}$ by

$$
X_{\hat{\mathcal{V}}} \overline{\otimes} X_{\hat{\mathcal{W}}} \in \mathcal{D}_{H,\hat{\mathcal{V}}} \overline{\otimes} \mathcal{D}_{H,\hat{\mathcal{W}}}, \quad \alpha \gamma, (\mathcal{D}_{H,\hat{\mathcal{V}}} \overline{\otimes} \mathcal{D}_{H,\hat{\mathcal{W}}})_{\alpha \beta} \Rightarrow X_{\hat{\mathcal{V}}} \overline{\otimes} X_{\hat{\mathcal{W}}} \in \mathcal{D}_{H,\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}}_{\alpha \beta}.
$$

Hence the map $\theta_{\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}} \circ (\tilde{\pi}_V \overline{\otimes} \tilde{\pi}_W) \circ \Delta : U'_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2) \rightarrow \mathcal{D}_{H,\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}}$ gives a dynamical representation of $U'_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$ on $\hat{\mathcal{V}} \overline{\otimes} \hat{\mathcal{W}}$.

Now let us consider a construction of dynamical representations of $U'_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}_2)$. Let $V$ be an $\hat{\mathfrak{h}}^*$-diagonalizable vector space over $\mathbb{F}$. Let $V_Q$ be a vector space over $\mathbb{C}$, and assume that an action of $e^Q$ on $V_Q$ is defined appropriately. Two important examples of $V_Q$ are $V_Q = \mathbb{C}1$ and $V_Q = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e^{nQ}$, where $1$ denotes the vacuum state satisfying $e^Q.1 = 1$. We then consider the vector space $\hat{\mathcal{V}} = V \otimes_{\mathbb{C}} V_Q$ over $\mathbb{F}$, where the actions of $f(P, r^*) \in \mathbb{F}$ and $e^Q$ on $\hat{\mathcal{V}}$ are defined by

$$
f(P, r^*)(v \otimes \xi) = f(P, r^*)v \otimes \xi,
$$

$$
e^Q(f(P, r^*)v \otimes \xi) = f(P + 1, r^*)v \otimes e^Q\xi
$$
for \( v \otimes \xi \in V \otimes V_Q \).

Then the following theorem is fundamental in our construction of dynamical representations.

**Theorem 4.2.** [27] Let \( \hat{V} \) be as above and \( \pi_V : \mathbb{F}[U'_q(\mathfrak{sl}_2)] \to \text{End}_\mathbb{F} V \) be an algebra homomorphism. Define a map \( \hat{\pi}_V = \pi_V \otimes \text{id} : U'_{q,p}(\mathfrak{sl}_2) = \mathbb{F}[U'_q(\mathfrak{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\overline{H}^*] \to \text{End}_\mathbb{C} \hat{V} \) by

\[
\hat{\pi}_V(E(u)) = \pi_V(\phi_r(x^+(z)))e^{2Q}z^{-\frac{P-1}{r^{*}}};
\]

\[
\hat{\pi}_V(F(u)) = \pi_V(\phi_r(x^-(z)))z^\frac{P+\pi_V(h)-1}{r};
\]

\[
\hat{\pi}_V(K(u)) = \exp\left(\sum_{n>0} \frac{[n]_q}{[2n]_q[r^{*}n]_q} \pi_V(a_{-n})(q^{c}z)^n\right) \exp\left(-\sum_{n>0} \frac{[n]_q}{[2n]_q[rn]_q} \pi_V(a_n)z^{-n}\right) \times e^{Q}z^{-\frac{c}{4rr^{*}}(2P-1)+\frac{1}{2r} \pi_V(h)}.
\]

Then \( (\hat{\pi}_V, \hat{V}) \) is a dynamical representation of \( U'_{q,p}(\mathfrak{sl}_2) \) on \( \hat{V} \).

Through this paper we consider dynamical representations obtained in this way. Let us also state the Poincaré–Birkhoff–Witt theorem for \( U'_{q,p}(\mathfrak{sl}_2) \).

**Definition 4.3.** Let \( \mathcal{H} \) (resp. \( \mathcal{N}_\pm \)) be the subalgebras of \( \mathbb{F}[U'_q(\mathfrak{sl}_2)] \) generated by \( c, h \) and \( a_k (k \in \mathbb{Z}_{\neq 0}) \) (resp. by \( x^\pm_n (n \in \mathbb{Z}) \)).

**Theorem 4.4.** [27]

\[
U'_{q,p}(\mathfrak{sl}_2) = (\mathcal{N}_- \otimes \mathcal{H} \otimes \mathcal{N}_+) \otimes \mathbb{C}[\overline{H}^*].
\]

Here the last \( \otimes \) should be understood as a semi-direct product.

### § 5. Finite-Dimensional Representations

This section is an announcement of some new results on the finite-dimensional dynamical representations of \( U'_{q,p}(\mathfrak{sl}_2) \). Detailed discussion will be published elsewhere [27].

#### § 5.1. Pseudo-highest Weight Representations

We begin by stating a characteristic feature of the finite-dimensional irreducible dynamical representations.

**Theorem 5.1.** Every finite-dimensional irreducible dynamical representation \( (\hat{\pi}_V, \hat{V} = V \otimes_{\mathbb{C}} V_Q) \) of \( U'_{q,p}(\mathfrak{sl}_2) \) contains a non-zero vector of the form \( \hat{\Omega} = \Omega \otimes 1, \Omega \in V \) such
that

\begin{align*}
1) \ x_n^+ \cdot \hat{\Omega} &= 0 \quad \forall n \in \mathbb{Z}, \\
2) \ \psi_n \cdot \hat{\Omega} &= d_n^+ \cdot \hat{\Omega}, \quad \phi_{-n} \cdot \hat{\Omega} &= d_{-n}^- \cdot \hat{\Omega} \quad \forall n \in \mathbb{Z}_{\geq 0}, \\
3) \ e^Q \cdot \hat{\Omega} &= \hat{\Omega}, \\
4) \ \hat{V} &= U_{q,p}' \cdot \hat{\Omega}.
\end{align*}

with some complex numbers \( d_{\pm n}^\pm, d_0^+ d_0^- = 1 \). Furthermore \( q^c \) acts as \( 1 \) or \(-1\) on \( \hat{V} \).

**Definition 5.2.** We define a pseudo-highest weight representation, a pseudo-highest weight vector and a pseudo-highest weight to be a dynamical representation (not necessarily irreducible) \((\hat{\pi}_V, \hat{V})\), a vector \( \hat{\Omega} \in \hat{V} \) and a set of complex numbers \( d = \{d_{\pm n}^\pm\}_{n \in \mathbb{Z}_{\geq 0}} \) satisfying the conditions 1) – 4) in Theorem 5.1, respectively.

The following theorem is useful.

**Theorem 5.3.** For a vector \( \hat{\Omega} \in \hat{V} \) satisfying \( e^Q \cdot \hat{\Omega} = \hat{\Omega} \), the conditions 1) and 2) in Definition 5.2 are equivalent to the following.

\begin{align*}
i) \ \hat{L}_{++}(u) \cdot \hat{\Omega} &= 0 \quad \forall u,
\end{align*}

\begin{align*}
ii) \ q^h \cdot \hat{\Omega} = q^\lambda \hat{\Omega} \quad \exists \lambda \in \mathbb{C}, \\
\hat{L}_{++}(u) \cdot \hat{\Omega} &= A(u) \hat{\Omega}, \quad \hat{L}_{--}(u) \cdot \hat{\Omega} = D(u) \hat{\Omega}
\end{align*}

with some meromorphic functions \( A(u) \) and \( D(u) \) satisfying \( D(u-1)^{-1} = A(u) \) and

\begin{equation}
A(u) = z^\lambda r \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}} A_{m,n} z^m p^n \quad A_{m,n} \in \mathbb{C}, z = q^{2u}, p = q^{2r}.
\end{equation}

\( U_{q,p}'(\hat{\mathfrak{sl}}_2) \) admits the universal pseudo-highest weight representation defined as follows.

**Definition 5.4.** Let \( d = \{d_{\pm n}^\pm\}_{n \in \mathbb{Z}_{\geq 0}} \) be any sequence of complex numbers. The Verma module \( M(d) \) is the quotient of \( U_{q,p}'(\hat{\mathfrak{sl}}_2) \) by the two sided ideal generated by \( q^c - 1 \) and the left ideal generated by \( \{x_k^+ \ (k \in \mathbb{Z}), \ \psi_n - d_n^+ \cdot 1, \phi_{-n} - d_{-n}^- \cdot 1 \ (n \in \mathbb{Z}_{\geq 0}), \ e^Q - 1\} \).

**Proposition 5.5.** The Verma module \( M(d) \) is pseudo-highest weight with pseudo-highest weight \( d \). Every pseudo-highest weight representation with pseudo-highest weight \( d \) is isomorphic to a quotient of \( M(d) \). Moreover \( M(d) \) has a unique irreducible pseudo-highest weight module.

§ 5.2. Elliptic analogue of the Drinfeld Polynomials

We state a necessary and sufficient condition for an irreducible dynamical representations of \( U_{q,p}'(\hat{\mathfrak{sl}}_2) \) to be finite-dimensional. We introduce a natural elliptic analogue of the Drinfeld polynomials.
Theorem 5.6. The irreducible pseudo-highest weight representation \((\hat{\pi}_{V}, \hat{V})\) of \(U_{q,p}(\hat{\mathfrak{sl}}_{2})\) is finite-dimensional if and only if there exists an entire and quasi-periodic function \(P_{V}(u)\) such that

\[
H^{\pm}(u)\hat{\Omega} = c_{V} P_{V}(u+1) \frac{P_{V}(u+1)}{P_{V}(u)} \hat{\Omega},
\]

where \(P_{V}(u)\) is the entire quasi-periodic function associated to \(\hat{V}\), and \(\tau = -\frac{2\pi i}{\log p}\). The symbol \(c_{V}\) denotes a constant given by

\[
c_{V} = q^{\frac{r-1}{r} \deg P} \prod_{j=1}^{\deg P} a_{j}^{\frac{1}{r}},
\]

where \(\deg P\) is the number of zeros of \(P_{V}(u)\) in the period parallelogram \((1, \tau)\) (= the degree of the Drinfeld polynomial \(P(z) = \lim_{r \to \infty} P_{V}(u), z = q^{2u}\), and \(a_{j} = q^{2\alpha_{j}}\) with \(\alpha_{j}\) being a zero of \(P_{V}(u)\) in the period parallelogram. The function \(P_{V}(u)\) is unique up to a scalar multiple.

Remark. We can take \(c_{V} = 1\) by the gauge transformation given by (2.11) in [18]. An example is given in Theorem 5.8.

The following Proposition is a direct consequence of the comultiplication formula for \(\hat{L}^{+}(u)\) and Definition 2.7.

Proposition 5.7. Let \(\hat{V}\) and \(\hat{W}\) be finite dimensional dynamical representations of \(U_{q,p}'(\hat{\mathfrak{sl}}_{2})\) and assume that the tensor product \(\hat{V} \otimes \hat{W}\) is irreducible. Let \(P_{V}(u), P_{W}(u)\) and \(P_{V \otimes W}(u)\) be the entire quasi-periodic function associated to \(\hat{V}, \hat{W}\) and \(\hat{V} \otimes \hat{W}\) in Theorem 5.6. Then

\[
P_{V \otimes W}(u) = P_{V}(u) P_{W}(u).
\]

§ 5.3. Evaluation Representations

An important example of finite-dimensional irreducible dynamical representations of \(U_{q,p}'(\hat{\mathfrak{sl}}_{2})\) is the evaluation representation. We here give a summary on the \(l+1\)-dimensional evaluation representation obtained from the one of \(\mathbb{F}[U_{q}'(\hat{\mathfrak{sl}}_{2})]\).

Let \(V^{(l)} = \bigoplus_{m=0}^{l} \mathbb{F}v_{m}^{l}\), \(V_{w}^{(l)} = V^{(l)} \otimes \mathbb{C}[w, w^{-1}]\), and consider the operators \(h, S^{\pm}\) on \(V^{(l)}\) defined by

\[
h v_{m}^{l} = (l-2m)v_{m}^{l}, \quad S^{\pm} v_{m}^{l} = v_{m \mp 1}^{l}, \quad v_{m}^{l} = 0 \text{ for } m < 0, \ m > l.
\]
In terms of the Drinfeld generators, the \( l + 1 \)-dimensional evaluation representation \((\pi_{l,w}, V^{(l)}_{w})\) of \( \mathbb{F}[U_{q}'(\hat{\mathfrak{s}\mathfrak{l}}_{2})] \) is given by

\[
\pi_{l,w}(q^{e}) = 1,
\]

\[
\pi_{l,w}(a_{n}) = \frac{w^{n}}{n} \frac{1}{q-q^{-1}} ((q^{n} + q^{-n})q^{nh} - (q^{(l+1)n} + q^{-(l+1)n})),
\]

\[
\pi_{l,w}(x^{\pm}(z)) = S^{\pm} \left[ \frac{\pm h + l + 2}{2} \right]_{q} \delta(q^{h\pm 1}\frac{w}{z}).
\] (5.2)

Now let us consider the vector space \( \hat{\mathfrak{V}}^{(l)}(w) = V^{(l)}(w) \otimes \mathbb{C}1 \) and the map \( \hat{\pi}_{l,w} = \pi_{l,w} \otimes \text{id} \) on \( U_{q,p}'(\hat{\mathfrak{s}\mathfrak{l}}_{2}) \cong \mathbb{F}[U_{q}'(\hat{\mathfrak{s}\mathfrak{l}}_{2})] \otimes_{\mathbb{C}} \mathbb{C}[H^{*}] \). Applying (5.2) to Theorem 4.2 and noting Definitions 2.2 and 2.6, we can derive the evaluation representation of \( U_{q,p}'(\mathfrak{s}\mathfrak{l}_{2}) \) as follows.

**Theorem 5.8.** \((\hat{\pi}_{l,w}, \hat{\mathfrak{V}}^{(l)}(w))\) is the \( l + 1 \)-dimensional irreducible dynamical representation of \( U_{q,p}'(\mathfrak{s}\mathfrak{l}_{2}) \) with the pseudo-highest weight vector \( v^{0}_{l} \otimes 1 \). In particular, the images of the matrix elements of \( \hat{\mathcal{L}}^{+}(u) \) by the map \( \hat{\pi}_{l,w} \) are given, up to fractional powers of \( z, w \) and \( q \), by

\[
\hat{\pi}_{l,w}(\hat{\mathcal{L}}_{++}^{+}(u)) = -\frac{[u-v+\frac{h+1}{2}][l-h+2][l+h+2]}{\varphi(p)[P+p+1]} e^{Q},
\]

\[
\hat{\pi}_{l,w}(\hat{\mathcal{L}}_{+-}^{+}(u)) = S^{-} \frac{[u-v+\frac{h-1}{2}+P][\frac{l-h+2}{2}]}{\varphi(p)[P+h-1]} e^{-Q},
\]

\[
\hat{\pi}_{l,w}(\hat{\mathcal{L}}_{-+}^{+}(u)) = S^{+} \frac{[u-v-\frac{h+1}{2}-P][\frac{l+h+2}{2}]}{\varphi(p)[P]} e^{Q},
\]

\[
\hat{\pi}_{l,w}(\hat{\mathcal{L}}_{--}^{+}(u)) = -\frac{[u-v-\frac{h-1}{2}]}{\varphi(p)[P]} e^{-Q},
\]

where \( z = q^{2u}, w = q^{2v} \), and

\[
\varphi(p) = -z^{-\frac{l}{2}} \rho^{+}_{kl}(z, p)^{-1}[u + \frac{l+1}{2}],
\]

\[
\rho^{+}_{kl}(z, p) = q^{\frac{kl}{2}} \frac{pq^{-l+2}z}{pq^{k+1/2}} \frac{pq^{-k+l+2}z}{pq^{-k+1/2}z} \frac{pq^{-k-l+2}z}{pq^{k-l+2}z} \frac{pq^{-k-l+2}z}{pq^{k-l+2}z}.
\]

**Corollary 5.9.** The elliptic analogue of the Drinfeld polynomial associated to \( \hat{\mathfrak{V}}^{(l)}(q^{2v}) \) is given by

\[
P_{l,v}(u) = [u-v-\frac{l-1}{2}][u-v-\frac{l-1}{2}+1] \cdots [u-v+\frac{l-1}{2}].
\]

Obviously the zeros of \( P_{l,v}(u) \) modulo \( \mathbb{Z}r + \mathbb{Z}r\tau \) coincide with those of the Drinfeld polynomial corresponding to the evaluation representation \( V^{(l)}(q^{2v}) \) of \( U_{q}'(\mathfrak{s}\mathfrak{l}_{2}) \).
The following Proposition indicates a consistency of our construction of $\hat{\pi}_{l,w}$ and the standard fusion construction of the dynamical $R$ matrices (=face type Boltzmann weights).

**Proposition 5.10.** Let us define the matrix elements of $\hat{\pi}_{l,w}(\hat{L}_{\epsilon_{1}\epsilon_{2}}^{+}(u))$ by

$$
\hat{\pi}_{l,w}(\hat{L}_{\epsilon_{1}\epsilon_{2}}^{+}(u))v_{m}^{l} = \sum_{m'=0}^{l} (\hat{L}_{\epsilon_{1}\epsilon_{2}}^{+}(u))_{\mu_{m'},\mu_{m}}v_{m'}^{l},
$$

where $\mu_{m} = l - 2m$. Then we have

$$
(\hat{L}_{\epsilon_{1}\epsilon_{2}}^{+}(u))_{\mu_{m},\mu_{m}} = R_{1l}^{+}(u-v, P)_{\epsilon_{1}^{2}\mu_{m}^{m}}^{\epsilon\mu},
$$

Here $R_{1l}^{+}(u-v, P)$ is the $R$ matrix from (C.17) in [18]. In the case $l = 1$, $R_{11}^{+}(u-v, P)$ coincides with the image $(\pi_{1,z} \otimes \pi_{1,w})$ of the universal $R$ matrix $\mathcal{R}^{+}(\lambda)$[17] given in (2.3). In the case $l > 1$, $R_{1l}^{+}(u-v, P)$ coincides with the $R$ matrix obtained by the standard fusion procedure from $R_{11}^{+}(u-v, P)$. In particular the matrix element $R_{1l}^{+}(u-v, P)_{\epsilon\mu}^{\epsilon'\mu'}$ is gauge equivalent to the fusion face weight $W_{l1}(P+\epsilon', P+\epsilon'+\mu', P+\mu, Pu-v)$ from (4) in [6].

§ 6. Infinite-dimensional Representations and Vertex Operators

Theorem 4.2 is valid also for infinite-dimensional representations. Let $(\pi, V(\lambda_{l}))$ be the level-$k$ $(c = k)$ irreducible highest weight representation of $\mathbb{F}[U_{q}(\mathfrak{sl}_{2})]$ with the highest weight $\lambda_{l} = (k-l)\Lambda_{0} + l\Lambda_{1}$ $(0 \leq l \leq k)$. Here $\Lambda_{i}$ $(i = 0, 1)$ denote the fundamental weights of $\mathfrak{sl}(2, \mathbb{C})$. Then $(\hat{\pi} = \pi \otimes \text{id}, \hat{V}(\lambda_{l}) = \bigoplus_{m \in \mathbb{Z}} V(\lambda_{l}) \otimes \mathbb{C}e^{-mQ})$ is the level-$k$ highest weight irreducible dynamical representation of $U_{q,p}(\mathfrak{sl}_{2})$.

A realization of $\hat{V}(\lambda_{l})$ in terms of the Drinfeld generators $a_{n}$ $(n \in \mathbb{Z}_{\neq 0})$ and the $q$-deformed $\mathbb{Z}_{k}$-parafermion algebra was given in [23, 26]. In [24, 18], we also studied the Wakimoto representation of $U_{q,p}(\mathfrak{sl}_{2})$ labeled by an integer $J$ $(0 \leq J \leq k)$, which is nothing but the dynamical representation $\hat{V}(\lambda_{J})$.

The $H$-Hopf algebroid structure allows us to define the vertex operators of $U_{q,p}(\mathfrak{sl}_{2})$ as follows.

**Definition 6.1.** The type I and II vertex operators of spin $n/2$ are the intertwiners of the $U_{q,p}$-modules of the form

$$
\hat{\Phi}(u) : \hat{V}(\lambda) \to \hat{V}_{z}^{(n)} \otimes \hat{V}(\nu), \quad \hat{\Psi}^{*}(u) : \hat{V}(\lambda) \otimes \hat{V}_{z}^{(n)} \to \hat{V}(\nu),
$$

where $z = q^{\Delta_{u}}$, and $\hat{V}(\lambda)$ and $\hat{V}(\nu)$ denote the level-$k$ highest weight $U_{q,p}$-modules of highest weights $\lambda$ and $\nu$, respectively. They satisfy the intertwining relations with respect to the comultiplication $\Delta$ in Definition 3.5.

$$
(6.1) \quad \Delta(x)\hat{\Phi}(u) = \hat{\Phi}(u)x, \quad x\hat{\Psi}^{*}(u) = \hat{\Psi}^{*}(u)\Delta(x) \quad \forall x \in U_{q,p}.
$$
The physically interesting cases are \(n = k, \lambda = \lambda_l, \nu = \lambda_{k-l}\) for the type I and \(n = 1, \lambda = \lambda_l, \nu = \lambda_{l+1}\) for the type II vertex operators. See for example [23].

Let us define the components of the vertex operators as follows.

\[
\hat{\Phi}(v - \frac{1}{2}) = \sum_{m=0}^{n} v_m^n \otimes \Phi_m(v), \quad \hat{\Psi}^*(v - \frac{c+1}{2}) (\otimes v_m^n) = \Psi_m^*(v).
\]

By using the comultiplication formula for \(\hat{L}^+(u)\) in Definition 3.5 and Proposition 5.10, we obtain the following theorem.

**Theorem 6.2.** [26] The vertex operators satisfy the following linear equations.

\[
\hat{\Phi}(u) \hat{L}^+(v) = R_{1n}^{+(12)}(v-u, P+h) \hat{L}^+(v) \hat{\Phi}(u),
\]

\[
\hat{L}^+(v) \hat{\Psi}^*(u) = \hat{\Psi}^*(u) \hat{L}^+(v) R_{1n}^{+(13)}(v-u, P-h^{(1)}-h^{(3)}).
\]

The relation (6.3) should be understood on \(\hat{V}_w^{(1)} \otimes \hat{V}(\lambda)\), whereas (6.4) on \(\hat{V}_w^{(1)} \otimes \hat{V}(\lambda) \otimes \hat{V}_z^{(n)}\).

Equations (6.3) and (6.4) coincide with (5.3) and (5.4) in [18], respectively. In [18], those equations were derived by using the quasi-Hopf algebra structure on \(B_{q,\lambda}(\hat{\mathfrak{sl}}_2)\) and the isomorphism \(U_{q,p}(\hat{\mathfrak{sl}}_2) \cong B_{q,\lambda}(\hat{\mathfrak{sl}}_2) \otimes \mathbb{C}[H^*]\) as a semi-direct product algebra. Under certain analyticity conditions, these equations determine the vertex operators uniquely up to normalization. Combining the results obtained here and those in [29, 24, 18, 28, 23], we have established the algebraic analysis scheme for the fusion RSOS models as well as for the fusion eight-vertex models on the basis of the elliptic quantum group \(U_{q,p}(\hat{\mathfrak{sl}}_2)\).

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