Thom polynomials and around

By

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This is a quick review on Enumerative Geometry from the viewpoint of Singularity Theory. We shall feature Thom polynomial, the simplest universal cohomological obstruction for appearance of singular points of prescribed type in any given generic maps. The first main problem is to compute Thom polynomials for given singularity types. This had been thought for a bit long time as a technically difficult problem in algebraic geometry or representation theory, because traditional approaches require to construct an appropriate resolution of the orbit closure. Recently, however, a new effective method for computation has been brought from topology - classifying spaces, equivariant cohomology and cobordism theory. We work on complex analytic singularities, and $H_*$ stands for Borel-Moore homology.

§1. Enumerating Singularities: Examples

Example 1.1. (Riemann-Hurwitz formula)
Let $f : M \to N$ be a branched cover of a nonsingular complete curve; Set $\mu(f, x) = e_x - 1$, where $e_x$ is the branch index at $x \in M$. The classical Riemann-Hurwitz formula is rewritten as follows:

$$
\sum \mu(f, x) = \deg f \cdot \chi(N) - \chi(M) \\
= c_1(TN) \cap f_*[M] - c_1(TM) \cap [M] \\
= (c_1(f^*TN) - c_1(TM)) \cap [M] \\
= c_1(f^*TN - TM) \cap [M].
$$

If $f$ has only $A_1$-singularities (i.e., $\mu(f, x) = 1$, namely it is locally expressed by $z \mapsto z^2$), the formula counts the number of $A_1$-points: Actually in this case (equidimensional case), the Thom polynomial $tp(A_1)$ for $A_1$-singularity type is just the monomial $c_1 = c_1(f^*TN - TM)$. 

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Example 1.2. (Thom-Porteous formula)
Let $E \rightarrow M$ and $F \rightarrow M$ be vector bundles of rank $m$ and $n$, respectively, with the same smooth base $M$. Let $\varphi : E \rightarrow F$ be a vector bundle morphism, that is a section of the vector bundle $\text{Hom}(E, F) \rightarrow M$, which may also be regarded as a family of ‘equivalent classes’ of linear maps $\mathbb{C}^m \rightarrow \mathbb{C}^n$ parametrized by $M$. We are herewith interested in the loci corresponding to ‘singular linear maps’, that is, the degeneracy loci

$$\Sigma^k(\varphi) = \{ x \in M, \ \text{dim ker} \varphi_x = k \}.$$ 

Below we denote $k = \text{dim ker} \varphi_x$ and $l = \text{dim coker} \varphi_x = n - m + k$. The Giambelli-Thom-Porteous formula [30] states that for suitably generic $\varphi$, the dual to the closure of $\Sigma^k(\varphi)$ is expressed by

$$\text{Dual} [\overline{\Sigma^k(\varphi)}] = \Delta_{l^k}(c(F - E)) = |c_{l-1}c_{l}c_{l}c_{l+1}\cdots|$$

where $c_i$ is the $i$-th component of $c(F - E) = \frac{1 + c_1(F) + \cdots}{1 + c_1(E) + \cdots}$. The Schur function of type $l^k = (l, \cdots, l)$ in the Chern classes $c_i$ is the Thom polynomial $tp(\Sigma^k)$ for the singularity type $\Sigma^k$.

For maps $f : M^m \rightarrow N^n$, we denote by $\Sigma^k(f)$ the degeneracy loci of the differential $df : TM \rightarrow f^*TN$. Note that $tp(A_1) = tp(\Sigma^1) = c_1$ in Example 1.1.

Example 1.3. (Singularities of maps)
As an example of classification of map-germs $(\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ (mono-singularities), let us see the case of $m = n + 1$ and some generic types:

$$A_1 : (x, y, v) \mapsto (x^2 + y^2, v),$$

$$A_2 : (x, y, u, v) \mapsto (x^3 + ux + y^2, u, v),$$

$$A_3 : (x, y, u_1, u_2, v) \mapsto (x^4 + u_1x^2 + u_2x + y^2, u_1, u_2, v),$$

where $v$ stands for parameters of the trivial unfolding of some dimension.

For a given generic map $f : M^{n+1} \rightarrow N^n$ between manifolds, we denote by $A_k(f)$ the set of points $x$ of $M$ at which the germ of $f$ is of type $A_k$, that is, the germ $f$ at $x$ is written in the normal form in suitable local coordinates of source and target. Then $A_k(f)$ is a locally closed submanifold of codimension $k + 1$ in $M$. We denote by $A^2_1(f)$

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1This means that the section $\varphi$ is transverse to the variety $\Sigma^s(E, F)$ (consisting of all linear maps with $\text{dim ker} = s$) for all $s \geq k$ in the total space of $\text{Hom}(E, F)$. One can replace the LHS of (1.2) by a certain localized class ([10], [16], [19]) that makes the formula hold even for non-generic $\varphi$ (cf. the Milnor number $\mu$ in Example 1.1).
the transverse self-intersection locus of $f(A_1(f))$ in $N$ (i.e., the locus corresponding to
the family of fibers having exactly two nodes), which has codimension 2 in $N$. One can
consider many other combinations like as $A_1^k, A_1A_2, \ldots$, called multi-singularities.

The closures $\overline{A_k(f)}$ in $M$ and $\overline{A_1^2(f)}$ in $N$ become possibly singular closed subvarieties, and their Poincaré dual are universally expressed by Thom polynomials

\begin{align}
  tp(A_1) &= c_1^2 - c_2, \quad tp(A_2) = 2c_1(c_1^2 - c_2), \quad \in H^*(M) \\
  tp(A_3) &= 5c_1^4 - 4c_1^2c_2 - c_1c_3 \\
  n(A_1^2) &= (s_2 - s_{01})^2 - s_{0001} + 8s_{001}s_{01} - 7s_3 \in H^*(N)
\end{align}

where $c_i$ is the $i$-th component of $c(f) = c(f^*TN - TM)$ as same as before, and $s_{i_1i_2...}$
are the Landweber-Novikov classes $s_I := f_*(c^I(f)) = f_*(c_1^{i_1}(f) \cdots c_k^{i_k}(f))$.

The existence theorem of such universal polynomials $tp(\eta)$ in variables $c_i$ for monosingularities $\eta$ goes back to René Thom (e.g. [33] in 1950’s), while the existence theorem of universal polynomials $n(\eta_{ml})$ in variables $s_I$ for multi-singularities $\eta_{ml}$ is rather quite new, that is due to M. Kazarian ([14] ’03)². Indeed it is quite recent that many computational results have been obtained (see [31], [14], [15] etc) by the method which will be explained in §4 below.

§ 2. Classification of Singularities

- 1st year Linear Algebra:
  
  $G = GL(m) \times GL(n)$ acts on $V = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ by $(A, B).H = BHA^{-1}$.
  
  Orbits are determined by kernel dimension $k$ (or rank), denoted by $\Sigma^k$.

- Classification of map/function-germs:
  
  Roughly, the classification of map-germs is the study of ‘jet space representations’
of automorphism groups of coordinate changes. Basic references are e.g. [22], [1], [20].

  Let $\mathcal{V} = \{f : \mathbb{C}^m, 0 \rightarrow \mathbb{C}^n, 0\} \simeq (m_m)^n$, the space of map-germs preserving the
origin, where $m_m$ is the maximal ideal of the ring $\mathcal{O}_{\mathbb{C}^m, 0}$ of regular function-germs.
  
  There are two basic equivalences (group action on $\mathcal{V}$), called $\mathcal{A}$ and $\mathcal{K}$-equivalences:

  $\mathcal{A} = \text{Aut}(\mathbb{C}^m, 0) \times \text{Aut}(\mathbb{C}^n, 0)$ acts on $\mathcal{V}$ by $(\phi, \tau).f = \tau \circ f \circ \phi^{-1},$

  $\mathcal{K} = \text{Aut}(\mathbb{C}^m, 0) \times \{B : \mathbb{C}^m, 0 \rightarrow GL(n)\}$ by $((\phi, B).f)(x) = B(x)f(\phi^{-1}(x)).$

  $\mathcal{A}$-equivalent map-germs are also $\mathcal{K}$-equivalent [22]: the $\mathcal{K}$-equivalence is indeed the
classification of variety-germs (=ideals) $f = 0$ under automorphisms of the source.

²Note that $n(A_1^2)$ has the form that the square of $f_*(tp(A_1))$ plus some ‘residual polynomial’. The
residual terms are much related to the geometry of the strata adjacent to the $A_1$-locus.
The infinitesimal structure of orbits looks as follows (in case of $\mathcal{K}$; we omit $\mathcal{A}$): For $f \in \mathcal{V}$, let $\theta(f)$ denote the $\mathcal{O}_{\mathbb{C}^m}$-module consisting of all germs of sections of $f^*T\mathbb{C}^n$ at $0 \in \mathbb{C}^m$, then the space of infinitesimal deformations of $f$ in $\mathcal{V}$ is identified with $m_m\theta(f)$. Denote the differential by $t_f(v) := df(v)$ for vector-field-germs $v \in \theta(id_m)$, $id_m$ being the identity of $\mathbb{C}^m$. The tangent space to the orbit $\mathcal{K}.f$, which consists of infinitesimal deformations of the form $\frac{d}{dt}(\phi_t, B_t).f|_{t=0}$, $(\phi_t, B_t) \in \mathcal{K}$, is the $\mathcal{O}_{\mathbb{C}^m}$-submodule given by

$$T\mathcal{K}.f = t_f(m_m\theta(id_m)) + f^*m_n\theta(f) \subset m_m\theta(f) = T_f\mathcal{V}.$$  

All groups and spaces described above are infinite dimensional, but our classification does essentially deal with some finite dimensional representatives: For $f \in \mathcal{V}$, the $k$-jet $j^k f(0)$ is defined to be the class of $f$ modulo $m^k_m\mathcal{V}$, i.e., the truncated Taylor expansion of $f$ at 0 up to order $k$. Denote the $k$-jet space by $J^k\mathcal{V} = \mathcal{V}/m^k_m\mathcal{V}$. A key notion is finite determinacy for $G = \mathcal{A}$ or $\mathcal{K}$, that is, $f \in \mathcal{V}$ is said to be $k$-$G$-determined if any $g \in \mathcal{V}$ with $j^k g(0) = j^k f(0)$ is $G$-equivalent to $f$ (e.g. the Morse singularity $(A_1)$ is 2-determined for both equivalences). Our classification process goes to list up $k$-determined germs from low order $k$ to higher. It is known that

(a) $f$ is finitely $G$-determined, (i.e. $k$-determined for some $k$) $\iff$ the orbit $G.f$ in $\mathcal{V}$ has finite codimension, i.e., $\dim_{\mathbb{C}} T_f\mathcal{V}/TG.f < \infty$, [22], [1], [20];

(b) For finitely determined map-germs $f$, its stabilizer subgroup has maximal compact (reductive) subgroups conjugate to some subgroup of $GL(m) \times GL(n)$ ($= J^1 \mathcal{A} = J^1 \mathcal{K}$), see [35].

Example 2.1. (Simple Curve Singularities: Example 1.3) A $G$-orbit $\eta$ is called to be simple ($G$-simple) if there are only finitely many $G$-orbits which intersect with a sufficiently small neighborhood of $\eta$.

In the classification of functions $\mathbb{C}^2, 0 \to \mathbb{C}$, and ICIS curve-germs $\mathbb{C}^{n+1}, 0 \to \mathbb{C}^n, 0$, the list of low codimensional $\mathcal{K}$-orbits are shown as follows (those are simple $\mathcal{K}$-orbits):

| 1-jet/codim | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |...
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<tr>
<td>$\Sigma^1$</td>
<td>$A_0$</td>
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| $\Sigma^2$  | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $D_4$ | $D_5$ | $D_6$ | $E_6$ | $\cdots$
| $\Sigma^3$  | $S_5$ | $S_6$ | $\cdots$

$^3$In particular, in case of $m \geq n$, $f$ is finitely $\mathcal{K}$-determined $\iff f = 0$ defines an isolated complete intersection singularity (ICIS) at 0.
Normal forms of those singularities are as follows:

\[ A_{\mu} : x^{\mu+1} + y^2 (\mu \geq 1), \quad D_{\mu} : x^2 y + y^{\mu-1} (\mu \geq 4), \quad E_{6} : x^3 + y^4, \cdots , \]
\[ S_{\mu} : (x^2 + y^2 + z^{\mu-3}, yz), \cdots . \]

For example, look at the case of \( A_{3} : (x, y) \mapsto z = f(x, y) = x^4 + y^2 \). Set \( e = \frac{\partial}{\partial z} \) and then \( \theta(f) = \mathcal{O}_{x,y} e \).

The normal vector space \( T_{f} \mathcal{V}/T\mathcal{K}.f \) has a basis \( \{ xe, ye, xe, xe \} \), thus the codimension is 4. The first two vectors are parallel to \( tf(\frac{\partial}{\partial x}) \) and \( tf(\frac{\partial}{\partial y}) \). The last two vectors define a deformation of \( f \) by \( f(x, y) + u_{1}x^{2} + u_{2}x \) with parameters \( u_{1}, u_{2} \), which gives a miniversal unfolding of \( A_{3} \)-singularity (see e.g. [22], [1] for the detail). The obtained map-germ is actually the third one without trivial parameters \( v \) in (1.3). Note that the source space \( (x, y, u_{1}, u_{2}) \) of the miniversal unfolding of \( f \) is identified with the normal to the \( \mathcal{K} \)-orbit of \( f \) in \( \mathcal{V} \).

### §3. Classifying Spaces

Let \( G \) be a complex Lie group acting on a complex space \( V \). In principle, any family \( \varphi \) of \( G \)-orbits in \( V \) parametrized by \( M \) is regarded as a certain morphism from \( M \) to the Borel construction (or the classifying stack \( [V/G] \)), called a classifying morphism\(^4\),

\[ \bar{\varphi} : M \longrightarrow B_{G}V := EG \times G V \]

where \( EG \to BG \) is the universal \( G \)-principal bundle and \( EG \times G V = (EG \times V)/G \) is the total space of the associated bundle with fiber \( V \) over \( BG \). Any two classifying morphisms corresponding to the same family \( \varphi \) are isomorphic in a categorical sense. The \( G \)-equivariant cohomology of \( V \) (in the sense of Borel) is defined by \( H_{G}^{*}(V) := H^{*}(B_{G}V) \). If \( V \) is an \( G \)-affine space, then \( H_{G}^{*}(V) \simeq H^{*}(BG) \).

**Definition 3.1.** For a \( G \)-orbit \( \eta \) in a \( G \)-affine space \( V \), the **Thom polynomial** of \( \eta \) is the \( G \)-characteristic class which represents the equivariant fundamental class of the orbit closure \( \overline{\eta} \) via the equivariant Poincaré dual\(^5\):

\[ tp(\eta) := \text{Dual}^{G}[ \overline{\eta} ]_{G} \in H_{G}^{*}(V) = H^{*}(BG). \]

Intuitively, \( tp(\eta) \) is the dual to the ‘finite codimensional subvariety’ \( B_{G}\overline{\eta} = EG \times G \overline{\eta} \) in \( B_{G}V \). An alternative description is given in the next section.

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\(^4\)For the construction in algebraic context, see [34].

\(^5\)The \( G \)-versions of the fundamental class and the Poincaré dual are defined by the inductive limits using a finite dimensional approximation of the Borel construction (In algebraic context, see [34] and [4]).
Example 3.2. Let $G = GL(m) \times GL(n)$ and $V = \text{Hom} \left( \mathbb{C}^m, \mathbb{C}^n \right)$. Note that $BG = BGL(m) \times BGL(n) = \text{Grass}(m, \infty) \times \text{Grass}(n, \infty)$, which is the classifying space for vector bundles $\text{Hom} \left( E, F \right) \to M$. By the following diagram, the classifying space for vector bundle morphisms $\varphi : E \to F$ should be the total space $B_G V$ (the classifying morphism is given by $\tilde{\varphi} = \tilde{\rho} \circ \varphi$):

\[
\begin{array}{cccccccc}
B_G (V \times V) \\
\nearrow \varphi' & \searrow \Delta & \nearrow \\
\text{Hom} \left( E, F \right) & \xrightarrow{\tilde{\rho}} & B_G V & = & \bigcup B_G \Sigma^k \\
\varphi & \uparrow & \downarrow \\
\bigcup \Sigma^k(\varphi) & \xrightarrow{\rho} & M & \xrightarrow{\rho} & BG
\end{array}
\]

The diagonal embedding $V \to V \times V$ defines the universal section $\Delta$. By the tautological construction, $\varphi$ is induced as the pullback of $\Delta$ via $\tilde{\varphi}$.

$B_G V$ is stratified via $B_G \Sigma^k = EG \times_G \Sigma^k$ ($\max(0, m-n) \leq k \leq m$), and the degeneracy locus $\Sigma^k(\varphi)$ is obtained as the intersection of the classifying morphism $\tilde{\varphi}$ with the stratum $B_G \Sigma^k$. We say $\varphi$ is generic if $\tilde{\varphi}$ is transverse to those strata in $B_G V$. For generic $\varphi$ it holds that $\text{Dual} \left[ \Sigma^k(\varphi) \right] = \tilde{\varphi}^* \text{Dual} \left[ B_G \Sigma^k \right] = \rho^* tp(\Sigma^k)$.

Remark. The case of mono-germ classification is similar as Example 3.2; We just take the jet space $J^k \mathcal{V}$ and the group $J^k \mathcal{K}$ (or $J^k \mathcal{A}$) for some order $k$. However, the case of multi-singularities of maps is much harder (Kazarian [14]): it requires the classifying space of complex cobordism (or probably a glued space of classifying stacks).

§ 4. Actual Computations

We briefly review a new approach due to R. Rimányi [31], which is effective for computation (also see Kazarian [13], [15], Fehér-Rimányi [7]).

Let $\eta$ be a $G$-orbit in a $G$-affine space $V$. Look at the exact sequence

\[
\to H^*_G(V, V - \overline{\eta}) \xrightarrow{i} H^*_G(V) \xrightarrow{\alpha} H^*_G(V - \overline{\eta}) \to .
\]

In $H^*_G(V)$, $\ker \alpha (= \text{Im} \ j)$ is an ideal. If its non-trivial lowest cohomological degree part, say $H^*_G(V) \cap \ker \alpha$, is isomorphic to $\mathbb{Z}$, we define $tp(\eta)$ to be the generator of the homogeneous part so that $tp(\eta)$ restricted to the orbit $\eta$ itself equals the Euler class of the normal bundle $\nu_\eta$ of $\eta$. The last condition fixes the ambiguity in the choice of the generator, and this definition of $tp(\eta)$ is compatible\textsuperscript{6} with Definition 3.1. Note that if $\overline{\eta}$

\textsuperscript{6}In fact it holds that $H^i_G(V, V - \overline{\eta}) \simeq H^i_G(\text{dim}_V V - i)(\overline{\eta})$ (the Alexander duality, cf [4]), thus if $\overline{\eta}$ is of (complex) codimension $s$ and irreducible, $H^i_G(V, V - \overline{\eta})$ is trivial for $i < 2s$ and is isomorphic to $\mathbb{Z}$ for $i = 2s$, the generator of which corresponds to the $G$-fundamental class of $\overline{\eta}$, cf. (equivariant version of) §19.1 in [10].
is smooth, $tp(\eta)$ is nothing but the $j$-image of the Thom class; but in general our $\overline{\eta}$ is singular.

If $V$ admits a certainly nice $G$-invariant stratification, one can describe $tp(\eta)$ as a solution of some linear equations on equivariant cohomology classes:

Assume that $V$ consists of finitely many orbits $\xi$, and also that their normal Euler classes are non-zero: $c_{top}(\nu_\xi) \neq 0$ for each $\xi$. Let $G_\xi$ be the stabilizer subgroup of $\xi$, and $\iota_\xi : G_\xi \subset G$ the inclusion. Then it turns out that

$$\oplus \iota_\xi^* : H^*_G(V) = H^*(BG) \longrightarrow \oplus \xi H^*(BG_\xi)$$

is an isomorphism. In particular it holds that

$$\tag{4.1} \iota_\xi^* tp(\eta) = 0 \quad (\text{for } \xi \not\subset \overline{\eta}), \quad \iota_\eta^* tp(\eta) = c_{top}(\nu_\eta)$$

and the system of these equations has a unique solution.

This gives an effective method for computing the correct form of $tp(\eta)$ by solving equations (4.1), which works well\textsuperscript{7} at least for simple orbits of (mono)singularities [31], [7], [15]. Also for multi-singularities, this method is used to determine the residual polynomials appearing in $n(\eta_{ml})$ [14], [15].

**Example 4.1. (Computing $Tp$ for degeneracy loci (1.2))**

For instance, let us see a very quick proof of the Gambielli-Thom-Porteous formula $tp(\Sigma^k) = \Delta_k(c)$ in Example 1.2, that should be compared with proofs using the embedded resolution of $\overline{\Sigma}^k$, see [30], [16], [19] (also [10] Chap. 14; [32], [5]). Let $G = GL(m) \times GL(n)$, $V = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ (the space of complex $n \times m$ matrices) and $\eta = \Sigma^k$. We shall determine the universal form of $tp(\Sigma^k)$. This case, it is enough to use only the last equation in (4.1). Put $l = n - m + k$.

Take a representative $h$ in $\Sigma^k$ in a standard way, then the tangent space of the orbit at the point $h$ consists of all matrices in the following form:

$$h = \begin{bmatrix} I_{m-k} & O \\ O & O \end{bmatrix} \in \Sigma^k, \quad T_h \Sigma^k = \left\{ \begin{bmatrix} * & * \\ * & O \end{bmatrix} \right\} \subset T_h V = V.$$

Note that the normal space is isomorphic to $\text{Hom}(\ker h, \text{coker} h)$. The stabilizer group of $h \in \Sigma^k$, denoted by $G_k$, consists of pairs of square matrices

$$\begin{bmatrix} P & O \\ * & A \end{bmatrix}, \begin{bmatrix} P & * \\ O & B \end{bmatrix}, \quad (A, B, P) \in GL(k) \times GL(l) \times GL(m-k).$$

\textsuperscript{7}In general the space $V$ contains continuous families of orbits (moduli of orbits). In that case, by dimensional reason, this method can determine $tp(\eta)$ for all the orbits $\eta$ whose codimension is less than the codimension of moduli strata.
Let $\rho_1, \rho_2, \rho_3$ be the representations of $G_k$ on $\ker h$, $\coker h$, $\text{Im} h$, respectively, and put $c(\rho_1) = \prod_{i=1}^{k} (1 + a_i)$ and $c(\rho_2) = \prod_{j=1}^{l} (1 + b_j)$. Then using (4.1) we have
\[
\iota_k^* \text{tp}(\Sigma^k) = c_{\text{top}}(v_k) = c_{kl}(\rho_1^* \otimes \rho_2) = \prod (b_j - a_i) = \Delta_\nu(c(\rho_2 - \rho_1))
\] (the last expression of the resultant is classical).

Denote by $c'_i$ and $c''_i$ universal Chern classes for $GL(m)$ and $GL(n)$, respectively, and then $H^*(BG) = Z[c'_1, \ldots, c'_m, c''_1, \ldots, c''_n]$. The representation of $G_k$ on the source and the target are $\lambda_1 = \rho_1 \oplus \rho_3$ and $\lambda_2 = \rho_2 \oplus \rho_3$. Thus $\iota_k^*: H^*(BG) \to H^*(BG_k)$ is determined by $c' \mapsto c(\lambda_1)$ and $c'' \mapsto c(\lambda_2)$, and in particular, it sends
\[
1 + c_1 + c_2 + \cdots := \frac{1 + c''_1 + \cdots + c''_n}{1 + c'_1 + \cdots + c'_m} \mapsto \frac{c(\lambda_2)}{c(\lambda_1)} = 1 + c_1(\rho_2 - \rho_1) + \cdots.
\]
Hence $\iota_k^* \Delta_\nu(c) = \Delta_\nu(c(\rho_2 - \rho_1))$. Finally it is easily checked that $\iota_k^*$ is injective for degree $\leq kl$, thus it follows that $\text{tp}(\Sigma^k) = \Delta_\nu(c)$. $\square$

**Example 4.2.** (Computing $\text{Tp}$ for simple ICI singularities)

Let $\xi$ be a $K$-orbit listed in Example (2.1) of ICIS curve-germs $\mathbb{C}^{n+1}, 0 \to \mathbb{C}^n, 0$. It is finitely determined, so the maximal reductive stabilizer group $G_{\xi}$ can be taken in $J^1\mathcal{K} = GL(n+1) \times GL(n)$ as mentioned in §2 (b). Furthermore the normal form is quasi-homogeneous, thus we may assume $G_{\xi} \simeq \mathbb{C}^*$. Let $\alpha$ denote the canonical 1-dimensional representation of $\mathbb{C}^*$ and put $a = c_1(\alpha) \in H^2(BC^*)$.

It is easy to write down in terms of weights of the normal forms the induced homomorphism $\iota_{\xi}^*: H^*(B\mathcal{K}) = H^*(BJ^1\mathcal{K}) \to H^*(BG_{\xi})$ and the normal Euler class $c_{\text{top}}(v_\xi) \in H^*(BG_{\xi})$. For example, let $\xi$ be $A_3: (x, y) \mapsto x^4 + y^2$. The representations of $G_{A_3} = \mathbb{C}^*$ on the source and target are $\lambda_1 = \alpha \oplus \alpha^2$ and $\lambda_2 = \alpha^4$. In particular $\iota_{A_3}^*$ sends
\[
1 + c_1 + c_2 + \cdots := \frac{c''}{c'} \mapsto \frac{c(\lambda_2)}{c(\lambda_1)} = \frac{1 + 4a}{(1 + a)(1 + 2a)} = 1 + a - 5a^2 + \cdots.
\]
The group $G_{A_3}$ acts also on the normal to the orbit in $\mathcal{V}$, such as $\tilde{\lambda}_1 = \alpha \oplus \alpha^2 \oplus \alpha^2 \oplus \alpha^3$ on parameters $(w_1, w_2, u_1, u_2)$ where one expresses the normal vectors by $w_1 x^3 e + w_2 ye + u_1 x^2 e + u_2 xe$ as mentioned at the end of §2 (or equivalently, $\tilde{\lambda}_1 = \lambda_1 \oplus \alpha^2 \oplus \alpha^3$ on the source space $(x, y, u_1, u_2)$ of the miniversal unfolding of $A_3$ in (1.3), since $G_{A_3}$ is also the stabilizer of the unfolding). Thus $c_{\text{top}}(v_{A_3}) = c_{\text{top}}(\tilde{\lambda}_1) = 12a^4$.

By using these data and (4.1), Thom polynomials $\text{tp}(\eta)$ for simple ICIS can be computed. Let $\eta = D_5$, for instance. Since it has codimension 6, we may set $\text{tp}(D_5) = \sum a_I c^I$ of degree 6 with unknown coefficients $a_I$, thanks to the existence theorem of $\text{tp}$. The equations (4.1) are $\iota_\xi^* \text{tp}(D_5) = 0$ for $\xi = A_0, \ldots, A_5, S_5$ and $\iota_{D_5}^* \text{tp}(D_5) = c_6(\nu_{D_5})$.

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8An exceptional case is $A_1: (x, y) \mapsto xy$, which has two obvious symmetries $\mathbb{C}^* \times \mathbb{C}^*$. 

These form a system of linear equations of unknowns $a_I$ (e.g., in case $\xi = A_3$, substitute $c_1 = a, c_2 = -5a^2, \cdots$ in $tp(D_5) = \sum a_I c^I$, then the resulting coefficient of $a^6$ must be zero). Solving the linear equations provides the correct answer:

$$tp(D_5) = 4c_1^6 - 2c_1^4c_2 - 18c_1^3c_3 - 6c_1^2c_4 + 12c_1^2c_2^2 + 2c_1c_5 + 12c_1c_2c_3 - 4c_2^3 - 4c_3^2 + 4c_2c_4 = 24\Delta_{42} + 12\Delta_{411} + 24\Delta_{33} + 64\Delta_{321} + 26\Delta_{3111} + 24\Delta_{222} + 42\Delta_{2211} + 4\Delta_{11111}.$$  

The last expansion is with respect to the Schur function basis, and it is observed that all the coefficients are non-negative. In fact the positivity holds in general, see Pragacz-Weber [29].

We remark that it must be hard to find the precise forms of $tp(A_4), tp(A_5), tp(D)$ etc by the traditional approach using embedded resolutions.

**Example 4.3. (Schubert varieties and Tp)**

Let $m \leq n$ and $G = GL(m) \times GL(n), V = Hom(\mathbb{C}^m, \mathbb{C}^n)$. Note that each linear map in the $G$-orbit $\Sigma^{m-r}$ determines a $r$-dimensional subspace in $\mathbb{C}^n$, and that any linear maps having the same image can be translated each other by the source changes $GL(m)$. Now fix a complete flag in the target linear space $\mathbb{C}^n$ by using coordinates and put $G^{\Delta} := GL(m) \times \{ \text{lower triangular matrices in } GL(n) \}$, the subgroup of $G$ which preserves the flag. Then the $G$-orbit $\Sigma^{m-r}$ breaks into several $G^{\Delta}$-orbits, which enjoy the one-to-one correspondence

$$G^{\Delta}-\text{orbits in } \Sigma^{m-r} \overset{1:1}{\longleftrightarrow} \text{Schubert cells in } \text{Grass}(r, \mathbb{C}^n).$$

We may write $H^*_{G^{\Delta}}(V) = \mathbb{Z}[c_1', \cdots, c_m', b_1, \cdots, b_n]$, where $b_i$ are Chern roots (corresponding to the maximal torus of $GL(n)$). In [9] it is shown that the Thom polynomial of a $G^{\Delta}$-orbit (in $H^*_{G^{\Delta}}(V)$) coincides with a double Schur polynomial in the sense of [16], and that the specialization under all $b_j = 0$ gives a Schur polynomial $\Delta_\lambda(c')$. This picture should be related to equivariant Schubert classes in $H^*_T(\text{Grass}(r, \mathbb{C}^n))$ and their localization [18] (also [24]; see [11], [12] for other classical groups).

§ 5. Current interests

Finally I comment about a few topics in current interests working in progress.

1. **Computing $Tp$ for moduli strata of $\mathcal{K}$-orbits**: As seen above, to compute $tp$ for simple $\mathcal{K}$-orbits (up to certain codimension) is basically possible. The next objects are moduli (continuous families) of $\mathcal{K}$-orbits, such as unimodular singularities, Thom-Boardmann strata $\Sigma^{i,j,k,\cdots}$ etc, for definition, see [1], [23]. Although $tp(\Sigma^{i,j})$ have
already been studied in [32] by the desingularization method (for \( tp \) of some \( \Sigma^{i,j,k} \), see [5]), the restriction method approach to \( tp(\Sigma^{i,j}) \) dealt in[6] gives a new insight.

2. **Generating functions of \( Tp \)**: It is natural to ask what each coefficient arising in \( tp(\eta) \) (with respect to monomial basis or Schur function basis) does mean, or ask whether there is a universal rule for the appearance of such numbers. On one hand, there are several ‘stems’ in classifications of mono-singularities \( \mathbb{C}^{m}, 0 \to \mathbb{C}^{m+\ell}, 0 \) such as \( A_{\mu}, D_{\mu}, \ldots \) or multi-singularities \( A_{s}^{\alpha} \) \( (s = 1, 2, \cdots) \) etc. So it is reasonable to think of generating functions of \( tp \)'s for such series of singularities where \( \mu, \ell \) or \( s \) is regarded as a parameter, see [3], [9], [15].

3. **Equivariant Chern classes for singularities (higher degree generalization of \( Tp \))**: In general, singular varieties (such as orbit closures \( \bar{\eta} \)) admit several variants of ‘Chern (homology) classes’, and usually the top term of such classes is the fundamental class of the variety. The most useful functorial Chern class theory is the *Chern-Schwartz-MacPherson class* [21], and the \( G \)-equivariant version of CSM classes has been established in [25]. Applying this theory to the orbit closure \( \bar{\eta} \) in a \( G \)-affine space \( V \), we obtain a non-homogeneous series in \( H^{*}(BG) \) whose lowest degree homogeneous term is just \( tp(\eta) \), since \( tp(\eta) \) corresponds to the equivariant fundamental class of the orbit closure. Such a ‘total class version’ of \( tp \) is a meaningful class in \( \ker \alpha \) mentioned in §3 (i.e., the ideal of \( G \)-characteristic classes having supports on \( \bar{\eta} \)), which has information about the combinatorial structure of adjacencies of \( G \)-orbits and several invariants, cf. [25], [26], [27]. On one hand there are some examples of computational aspects: Chern-Schwartz-MacPherson classes of degeneracy loci \( \bar{\Sigma}^{k} \) and of Schubert varieties have already been studied in [28] and [2], respectively.

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**References**


