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Thom polynomials and around

By

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This is a quick review on Enumerative Geometry from the viewpoint of Singularity Theory. We shall feature *Thom polynomial*, the simplest universal cohomological obstruction for appearence of singular points of prescribed type in any given generic maps. The first main problem is to *compute* Thom polynomials for given singularity types. This had been thought for a bit long time as a technically difficult problem in algebraic geometry or representation theory, because traditional approaches require to construct an appropriate resolution of the orbit closure. Recently, however, a new effective method for computation has been brought from topology - classifying spaces, equivariant cohomology and cobordism theory. We work on complex analytic singularities, and H_* stands for Borel-Moore homology.

§ 1. Enumerating Singularities: Examples

Example 1.1. (**Riemann-Hurwitz formula**)
Let $f : M \to N$ be a branched cover of a nonsingular complete curve; Set $\mu(f, x) = e_x - 1$, where e_x is the branch index at $x \in M$. The *classical Riemann-Hurwitz formula* is rewritten as follows:

$$\begin{aligned}\sum \mu(f, x) &= \deg f \cdot \chi(N) - \chi(M)\\ &= c_1(TN) \cap f_*[M] - c_1(TM) \cap [M]\\ &= (c_1(f^*TN) - c_1(TM)) \cap [M]\\ &= c_1(f^*TN - TM) \cap [M].\end{aligned} \tag{1.1}$$

If f has only A_1-singularities (i.e., $\mu(f, x) = 1$, namely it is locally expressed by $z \mapsto z^2$), the formula counts the number of A_1-points: Actually in this case (equidimenional case), the Thom polynomial $tp(A_1)$ for A_1-singularity type is just the monomial $c_1 = c_1(f^*TN - TM)$.

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Example 1.2. (**Thom-Porteous formula**)
Let $E \to M$ and $F \to M$ be vector bundles of rank m and n, respectively, with the same smooth base M. Let $\varphi : E \to F$ be a vector bundle morphism, that is a section of the vector bundle $\mathrm{Hom}\,(E, F) \to M$, which may also be regarded as a family of 'equivalent classes' of linear maps $\mathbb{C}^m \to \mathbb{C}^n$ parametrized by M. We are herewith interested in the loci corresponding to 'singular linear maps', that is, the *degeneracy loci*

$$\Sigma^k(\varphi) = \{\ x \in M,\ \ \dim\ker\varphi_x = k\ \}.$$

Below we denote $k = \dim\ker\varphi_x$ and $l = \dim\operatorname{coker}\varphi_x = n - m + k$. The *Giambelli-Thom-Porteous formula* [30] states that for suitably generic φ [1], the dual to the closure of $\Sigma^k(\varphi)$ is expressed by

$$\mathrm{Dual}\,[\overline{\Sigma^k(\varphi)}] \;=\; \Delta_{l^k}(c(F-E)) \;=\; \overbrace{\begin{vmatrix} c_l & c_{l+1} & \cdots \\ c_{l-1} & c_l & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}}^{k} \tag{1.2}$$

where c_i is the i-th component of $c(F-E) = \dfrac{1 + c_1(F) + \cdots}{1 + c_1(E) + \cdots}$. The Schur function of type $l^k = (l, \cdots, l)$ in the Chern classes c_i is the Thom polynomial $tp(\Sigma^k)$ for the singularity type Σ^k.

For maps $f : M^m \to N^n$, we denote by $\Sigma^k(f)$ the degeneracy loci of the differential $df : TM \to f^*TN$. Note that $tp(A_1) = tp(\Sigma^1) = c_1$ in Example 1.1.

Example 1.3. (**Singularities of maps**)
As an example of classification of map-germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ (*mono-singularities*), let us see the case of $m = n + 1$ and some generic types:

$$\begin{aligned} &A_1 : (x, y, v) \mapsto (x^2 + y^2, v), \\ &A_2 : (x, y, u, v) \mapsto (x^3 + ux + y^2, u, v), \\ &A_3 : (x, y, u_1, u_2, v) \mapsto (x^4 + u_1x^2 + u_2x + y^2, u_1, u_2, v), \end{aligned} \tag{1.3}$$

where v stands for parameters of the trivial unfolding of some dimension.

For a given generic map $f : M^{n+1} \to N^n$ between manifolds, we denote by $A_k(f)$ the set of points x of M at which the germ of f is of type A_k, that is, the germ f at x is written in the normal form in suitable local coordinates of source and target. Then $A_k(f)$ is a locally closed submanifold of codimension $k + 1$ in M. We denote by $A_1^2(f)$

[1]This means that the section φ is transverse to the variety $\Sigma^s(E, F)$ (consisting of all linear maps with $\dim\ker = s$) for all $s \geq k$ in the total space of $\mathrm{Hom}\,(E, F)$. One can replace the LHS of (1.2) by a certain *localized class* ([10], [16], [19]) that makes the formula hold even for non-generic φ (cf. the Milnor number μ in Example 1.1).

the transverse self-intersection locus of $f(A_1(f))$ in N (i.e., the locus corresponding to the family of fibers having exactly two nodes), which has codimension 2 in N. One can consider many other combinations like as $A_1^k, A_1A_2, \cdots$, called *multi-singularities.*

The closures $\overline{A_k(f)}$ in M and $\overline{A_1^2(f)}$ in N become possibly singular closed subvarieties, and their Poincaré dual are universally expressed by *Thom polynomials*

$$(1.4) \qquad \begin{array}{l} tp(A_1) = c_1^2 - c_2, \quad tp(A_2) = 2c_1(c_1^2 - c_2), \\ tp(A_3) = 5c_1^4 - 4c_1^2c_2 - c_1c_3 \end{array} \in H^*(M)$$
$$n(A_1^2) = (s_2 - s_{01})^2 - s_{0001} + 8s_{001}s_{01} - 7s_3 \in H^*(N)$$

where c_i is the i-th component of $c(f) = c(f^*TN - TM)$ as same as before, and $s_{i_1i_2\cdots}$ are the *Landweber-Novikov classes* $s_I := f_*(c^I(f)) = f_*(c_1^{i_1}(f)\cdots c_k^{i_k}(f))$.

The existence theorem of such universal polynomials $tp(\eta)$ in variables c_i *for mono-singularities* η goes back to René Thom (e.g. [33] in 1950's), while *the existence theorem* of universal polynomials $n(\eta_{ml})$ in variables s_I *for multi-singularities* η_{ml} is rather quite new, that is due to M. Kazarian ([14] '03)[2]. Indeed it is quite recent that many computational results have been obtained (see [31], [14], [15] etc) by the method which will be explained in §4 below.

§ 2. Classification of Singularities

- 1^{st} year Linear Algebra:

  $G = GL(m) \times GL(n)$ acts on $V = \mathrm{Hom}\,(\mathbb{C}^m, \mathbb{C}^n)$ by $(A, B).H = BHA^{-1}$.
Orbits are determined by kernel dimension k (or rank), denoted by Σ^k.

- Classification of map/function-germs:

  Roughly, the classification of map-germs is the study of 'jet space representations' of automorphism groups of coordinate changes. Basic references are e.g. [22], [1], [20].

Let $\mathcal{V} = \{f : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0\} \simeq (\mathrm{m}_m)^n$, the space of map-germs preserving the origin, where m_m is the maximal ideal of the ring $\mathcal{O}_{\mathbb{C}^m,0}$ of regular function-germs. There are two basic equivalences (group action on $\mathcal{V}$), called $\mathcal{A}$ *and* $\mathcal{K}$*-equivalences*:

$$\mathcal{A} = Aut(\mathbb{C}^m, 0) \times Aut(\mathbb{C}^n, 0) \text{ acts on } \mathcal{V} \text{ by } \quad (\phi, \tau).f = \tau \circ f \circ \phi^{-1},$$
$$\mathcal{K} = Aut(\mathbb{C}^m, 0) \times \{B : \mathbb{C}^m, 0 \to GL(n)\} \text{ by } \ ((\phi, B).f)(x) = B(x)f(\phi^{-1}(x)).$$

$\mathcal{A}$-equivalent map-germs are also $\mathcal{K}$-equivalent [22]: the $\mathcal{K}$-equivalence is indeed the classification of variety-germs (=ideals) $f = 0$ under automorphisms of the source.

[2]Note that $n(A_1^2)$ has the form that the square of $f_*(tp(A_1))$ plus some '*residual polynomial*'. The residual terms are much related to the geometry of the strata adjacent to the A_1-locus.

The infinitesimal structure of orbits looks as follows (in case of $\mathcal{K}$; we omit $\mathcal{A}$): For $f \in \mathcal{V}$, let $\theta(f)$ denote the $\mathcal{O}_{\mathbb{C}^m}$-module consisting of all germs of sections of $f^*T\mathbb{C}^n$ at $0 \in \mathbb{C}^m$, then the space of infinitesimal deformations of f in $\mathcal{V}$ is identified with $\mathrm{m}_m\theta(f)$. Denote the differential by $tf(\boldsymbol{v}) := df(\boldsymbol{v})$ for vector-field-germs $\boldsymbol{v} \in \theta(id_m)$, id_m being the identity of $\mathbb{C}^m$. The *tangent space* to the orbit $\mathcal{K}.f$, which consists of infinitesimal deformations of the form $\frac{d}{dt}(\phi_t, B_t).f|_{t=0}$, $(\phi_t, B_t) \in \mathcal{K}$, is the $\mathcal{O}_{\mathbb{C}^m}$-submodule given by

$$T\mathcal{K}.f = tf(\mathrm{m}_m\theta(id_m)) + f^*\mathrm{m}_n\,\theta(f) \quad \subset \ \mathrm{m}_m\theta(f) = T_f\mathcal{V}.$$

All groups and spaces described above are infinite dimensional, but our classification does essentially deal with some finite dimensional representatives: For $f \in \mathcal{V}$, *the k-jet* $j^kf(0)$ is defined to be the class of f modulo $\mathrm{m}_m^k\mathcal{V}$, i.e., the truncated Taylor expansion of f at 0 up to order k. Denote *the k-jet space* by $J^k\mathcal{V} = \mathcal{V}/\mathrm{m}_m^k\mathcal{V}$. A key notion is *finite determinacy* for $G = \mathcal{A}$ or $\mathcal{K}$, that is, $f \in \mathcal{V}$ is said to be *k-G-determined* if any $g \in \mathcal{V}$ with $j^kg(0) = j^kf(0)$ is G-equivalent to f (e.g. the Morse singularity (A_1) is 2-determined for both equivalences). Our classification process goes to list up k-determined germs from low order k to higher. It is known that

(a) f is finitely G-determined[3], (i.e. k-determined for some k) $\Longleftrightarrow$ the orbit $G.f$ in $\mathcal{V}$ has finite codimension, i.e., $\dim_{\mathbb{C}} T_f\mathcal{V}/TG.f < \infty$, [22], [1], [20];

(b) For finitely determined map-germs f, its stabilizer subgroup has maximal compact (reductive) subgroups conjugate to some subgroup of $GL(m) \times GL(n)$ $(= J^1\mathcal{A} = J^1\mathcal{K})$, see [35].

Example 2.1. (**Simple Curve Singularities**: Example 1.3)
A G-orbit η is called to be *simple* (G-simple) if there are only finitely many G-orbits which intersect with a sufficiently small neighborhood of η.

In the classification of functions $\mathbb{C}^2, 0 \to \mathbb{C}, 0$ and ICIS curve-germs $\mathbb{C}^{n+1}, 0 \to \mathbb{C}^n, 0$, the list of low codimensional $\mathcal{K}$-orbits are shown as follows (those are simple $\mathcal{K}$-orbits):

1-jet/codim	0	2	3	4	5	6	7 $\cdots$
Σ^1	A_0						$\cdots$
Σ^2		A_1	A_2	A_3	A_4	A_5	A_6 $\cdots$
					D_4	D_5	D_6 $\cdots$
							E_6 $\cdots$
Σ^3						S_5	S_6 $\cdots$

[3] In particular, in case of $m \geq n$, f is finitely $\mathcal{K}$-determined $\Longleftrightarrow$ $f = 0$ defines an *isolated* complete intersection singularity (ICIS) at 0.

Normal forms of those singularities are as follows:

$$A_\mu : x^{\mu+1} + y^2 \ (\mu \geq 1), \ \ D_\mu : x^2 y + y^{\mu-1} \ (\mu \geq 4), \ \ E_6 : x^3 + y^4, \cdots ,$$
$$S_\mu : (x^2 + y^2 + z^{\mu-3}, yz), \cdots .$$

For example, look at the case of $A_3 : (x,y) \mapsto z = f(x,y) = x^4 + y^2$. Set $\boldsymbol{e} = \frac{\partial}{\partial z}$ and then $\theta(f) = \mathcal{O}_{x,y}\, \boldsymbol{e}$. The normal vector space $T_f\mathcal{V}/T\mathcal{K}.f$ has a basis $\{x^3\boldsymbol{e}, y\boldsymbol{e}, x^2\boldsymbol{e}, x\boldsymbol{e}\}$, thus the codimension is 4. The first two vectors are parallel to $tf(\frac{\partial}{\partial x})$ and $tf(\frac{\partial}{\partial y})$. The last two vectors define a deformation of f by $f(x,y)+u_1x^2+u_2x$ with parameters u_1, u_2, which gives a *miniversal unfolding* of A_3-singularity (see e.g. [22], [1] for the detail). The obtained map-germ is actually the third one without trivial parameters v in (1.3). Note that the source space (x, y, u_1, u_2) of the miniversal unfolding of f is identified with the normal to the $\mathcal{K}$-orbit of f in $\mathcal{V}$.

§ 3. Classifying Spaces

Let G be a complex Lie group acting on a complex space V. In principle, any family φ of G-orbits in V parametrized by M is regarded as a certain morphism from M to the Borel construction (or the classifying stack $[V/G]$), called a *classifying morphism*[4],

$$\bar{\varphi} : M \longrightarrow B_G V := EG \times_G V$$

where $EG \to BG$ is the universal G-principal bundle and $EG \times_G V = (EG \times V)/G$ is the total space of the associated bundle with fiber V over BG. Any two classifying morphisms corresponding to the same family φ are isomorphic in a categorical sense. The G-equivariant cohomology of V (in the sense of Borel) is defined by $H^*_G(V) := H^*(B_G V)$. If V is an G-affine space, then $H^*_G(V) \simeq H^*_G(pt) \simeq H^*(BG)$.

Definition 3.1. For a G-orbit η in a G-affine space V, the *Thom polynomial* of η is the G-characteristic class which represents the equivariant fundamental class of the orbit closure $\bar{\eta}$ via the equivariant Poincaré dual[5]:

$$tp(\eta) := \mathrm{Dual}^G[\,\bar{\eta}\,]_G \in H^*_G(V) = H^*(BG).$$

Intuitively, $tp(\eta)$ is the dual to the 'finite codimensional subvariety' $B_G\bar{\eta} = EG \times_G \bar{\eta}$ in $B_G V$. An alternative description is given in the next section.

[4]For the construction in algebraic context, see [34].

[5]The G-versions of the fundamental class and the Poincaré dual are defined by the linductive limits using a finite dimensional approximation of the Borel construction (In algebraic context, see [34] and [4]).

Example 3.2. Let $G = GL(m) \times GL(n)$ and $V = \mathrm{Hom}\,(\mathbb{C}^m, \mathbb{C}^n)$. Note that $BG = BGL(m) \times BGL(n) = \mathrm{Grass}(m, \infty) \times \mathrm{Grass}(n, \infty)$, which is the classifying space for vector bundles $\mathrm{Hom}\,(E, F) \to M$. By the following diagram, the *classifying space for vector bundle morphisms* $\varphi : E \to F$ should be the total space $B_G V$ (the classifying morphism is given by $\bar{\varphi} = \bar{\rho} \circ \varphi$):

$$
\begin{array}{cccccc}
 & & & B_G(V \times V) & & \\
 & & \overset{\bar{\varphi}'}{\nearrow} & \Delta \uparrow & & \\
 & \mathrm{Hom}\,(E, F) & \overset{\bar{\rho}}{\rightarrow} & B_G V & = \bigsqcup B_G \Sigma^k \\
 & \varphi \uparrow & \overset{\bar{\varphi}}{\nearrow} & \downarrow & & \\
\bigsqcup \Sigma^k(\varphi) = & M & \overset{\rho}{\rightarrow} & BG & &
\end{array}
$$

The diagonal embedding $V \to V \times V$ defines the *universal section* Δ. By the tautological construction, φ is induced as the pullback of Δ via $\bar{\varphi}$.

$B_G V$ is stratified via $B_G \Sigma^k = EG \times_G \Sigma^k$ ($\max(0, m-n) \le k \le m$), and the degeneracy locus $\Sigma^k(\varphi)$ is obtained as the intersection of the classifying morphism $\bar{\varphi}$ with the stratum $B_G \Sigma^k$. We say φ is generic if $\bar{\varphi}$ is transverse to those strata in $B_G V$. For generic φ it holds that $\mathrm{Dual}\,[\overline{\Sigma^k(\varphi)}] = \bar{\varphi}^* \mathrm{Dual}\,[\overline{B_G \Sigma^k}] = \rho^* tp(\Sigma^k)$.

Remark. The case of mono-germ classification is similar as Example 3.2; We just take the jet space $J^k \mathcal{V}$ and the group $J^k \mathcal{K}$ (or $J^k \mathcal{A}$) for some order k. However, the case of multi-singularities of maps is much harder (Kazarian [14]): it requires the classifying space of complex cobordism (or probably a glued space of classifying stacks).

§ 4. Actual Computations

We briefly review a new approach due to R. Rimányi [31], which is effective for computation (also see Kazarian [13], [15], Fehér-Rimányi [7]).

Let η be a G-orbit in a G-affine space V. Look at the exact sequence

$$\to H^*_G(V, V - \overline{\eta}) \xrightarrow{j} H^*_G(V) \xrightarrow{\alpha} H^*_G(V - \overline{\eta}) \to .$$

In $H^*_G(V)$, $\ker \alpha (= \mathrm{Im}\, j)$ is an ideal. If its non-trivial lowest cohomological degree part, say $H^{2s}_G(V) \cap \ker \alpha$, is isomorphic to $\mathbb{Z}$, we define $tp(\eta)$ to be the generator of the homogeneous part so that $tp(\eta)$ restricted to the orbit η itself equals the Euler class of the normal bundle ν_η of η. The last condition fixes the ambiguity in the choice of the generator, and this definition of $tp(\eta)$ is compatible[6] with Definition 3.1. Note that if $\overline{\eta}$

[6]In fact it holds that $H^i_G(V, V - \overline{\eta}) \simeq H^G_{\dim V - i}(\overline{\eta})$ (the Alexander duality, cf [4]), thus if $\overline{\eta}$ is of (complex) codimension s and irreducible, $H^i_G(V, V - \overline{\eta})$ is trivial for $i < 2s$ and is isomorphic to $\mathbb{Z}$ for $i = 2s$, the generator of which corresponds to the G-fundamental class of $\overline{\eta}$, cf. (equivariant version of) §19.1 in [10].

is smooth, $tp(\eta)$ is nothing but the j-image of the Thom class; but in general our $\overline{\eta}$ is singular.

If V admits a certainly nice G-invariant stratification, one can describe $tp(\eta)$ as a solution of some linear equations on equivariant cohomology classes:

> Assume that V consists of finitely many orbits ξ, and also that their normal Euler classes are non-zero: $c_{top}(\nu_\xi) \neq 0$ for each ξ. Let G_ξ be the stabilizer subgroup of ξ, and $\iota_\xi : G_\xi \subset G$ the inclusion. Then it turns out that
>
> $$\oplus \iota_\xi^* : H_G^*(V) = H^*(BG) \longrightarrow \oplus_\xi H^*(BG_\xi)$$
>
> is an isomorphism. In particular it holds that
>
> $$\iota_\xi^*\, tp(\eta) = 0 \quad (\text{for } \xi \not\subset \bar{\eta}), \quad \iota_\eta^*\, tp(\eta) = c_{top}(\nu_\eta) \tag{4.1}$$
>
> and the system of these equations has a unique solution.

This gives an effective method for computing the correct form of $tp(\eta)$ by solving equations (4.1), which works well[7] at least for *simple orbits* of (mono)singularities [31], [7], [15]. Also for multi-singularities, this method is used to determine the residual polynomials appearing in $n(\eta_{ml})$ [14], [15].

Example 4.1. (**Computing Tp for degeneracy loci (1.2)**)
For instance, let us see a very quick proof of the Gambielli-Thom-Porteous formula $tp(\Sigma^k) = \Delta_{l^k}(c)$ in Example 1.2, that should be compared with proofs using the embedded resolution of $\overline{\Sigma^k}$, see [30], [16], [19] (also [10] Chap. 14; [32], [5]). Let $G = GL(m) \times GL(n)$, $V = \mathrm{Hom}\,(\mathbb{C}^m, \mathbb{C}^n)$ (the space of complex $n \times m$ matricies) and $\eta = \Sigma^k$. We shall determine the universal form of $tp(\Sigma^k)$. This case, it is enough to use only the last equation in (4.1). Put $l = n - m + k$.

Take a representative h in Σ^k in a standard way, then the tangent space of the orbit at the point h consists of all matrices in the following form:

$$h = \left[\begin{array}{c|c} I_{m-k} & O \\ \hline O & O \end{array}\right] \in \Sigma^k, \quad T_h\Sigma^k = \left\{ \left[\begin{array}{c|c} * & * \\ \hline * & O \end{array}\right] \right\} \subset T_hV = V.$$

Note that the normal space is isomorphic to $\mathrm{Hom}\,(\ker h, \mathrm{coker}\, h)$. The stabilizer group of $h \in \Sigma^k$, denoted by G_k, consists of pairs of square matrices

$$\left(\left[\begin{array}{c|c} P & O \\ \hline * & A \end{array}\right], \left[\begin{array}{c|c} P & * \\ \hline O & B \end{array}\right] \right), \quad (A, B, P) \in GL(k) \times GL(l) \times GL(m-k).$$

[7] In general the space V contains continuous families of orbits (moduli of orbits). In that case, by dimensional reason, this method can determine $tp(\eta)$ for all the orbits η whose codimension is less than the codimension of moduli strata.

Let ρ_1, ρ_2, ρ_3 be the representations of G_k on $\ker h$, $\operatorname{coker} h$, $\operatorname{Im} h$, respectively, and put $c(\rho_1) = \prod_{i=1}^k (1+a_i)$ and $c(\rho_2) = \prod_{j=1}^l (1+b_j)$. Then using (4.1) we have

$$\iota_k^*\, tp(\Sigma^k) = c_{top}(\nu_k) = c_{kl}(\rho_1^* \otimes \rho_2) = \prod (b_j - a_i) = \Delta_{l^k}(c(\rho_2 - \rho_1))$$

(the last expression of the resultant is classical).

Denote by c_i' and c_j'' universal Chern classes for $GL(m)$ and $GL(n)$, respectively, and then $H^*(BG) = \mathbb{Z}[c_1', \cdots, c_m', c_1'', \cdots, c_n'']$. The representation of G_k on the source and the target are $\lambda_1 = \rho_1 \oplus \rho_3$ and $\lambda_2 = \rho_2 \oplus \rho_3$. Thus $\iota_k^* : H^*(BG) \to H^*(BG_k)$ is determined by $c' \mapsto c(\lambda_1)$ and $c'' \mapsto c(\lambda_2)$, and in particular, it sends

$$1 + c_1 + c_2 + \cdots := \frac{1 + c_1'' + \cdots + c_n''}{1 + c_1' + \cdots + c_m'} \longmapsto \frac{c(\lambda_2)}{c(\lambda_1)} = 1 + c_1(\rho_2 - \rho_1) + \cdots .$$

Hence $\iota_k^* \Delta_{l^k}(c) = \Delta_{l^k}(c(\rho_2 - \rho_1))$. Finally it is easily checked that ι_k^* is *injective* for degree $\leq kl$, thus it follows that $tp(\Sigma^k) = \Delta_{l^k}(c)$. $\square$

Example 4.2. (**Computing Tp for simple ICI singularities**) Let ξ be a $\mathcal{K}$-orbit listed in Example (2.1) of ICIS curve-germs $\mathbb{C}^{n+1}, 0 \to \mathbb{C}^n, 0$. It is finitely determined, so the maximal reductive stabilizer group G_ξ can be taken in $J^1\mathcal{K} = GL(n+1) \times GL(n)$ as mentioned in §2 (b). Furthermore the normal form is quasi-homogeneous, thus we may assume[8] $G_\xi \simeq \mathbb{C}^*$. Let α denote the canonical 1-dimensional representation of $\mathbb{C}^*$ and put $a = c_1(\alpha) \in H^2(B\mathbb{C}^*)$.

It is easy to write down in terms of weights of the normal forms the induced homomorphism $\iota_\xi^* : H^*(B\mathcal{K}) = H^*(BJ^1\mathcal{K}) \to H^*(BG_\xi)$ and the normal Euler class $c_{top}(\nu_\xi) \in H^*(BG_\xi)$. For example, let ξ be $A_3 : (x, y) \mapsto x^4 + y^2$. The representations of $G_{A_3} = \mathbb{C}^*$ on the source and target are $\lambda_1 = \alpha \oplus \alpha^2$ and $\lambda_2 = \alpha^4$. In particular $\iota_{A_3}^*$ sends

$$1 + c_1 + c_2 + \cdots := \frac{c''}{c'} \longmapsto \frac{c(\lambda_2)}{c(\lambda_1)} = \frac{1 + 4a}{(1+a)(1+2a)} = 1 + a - 5a^2 + \cdots .$$

The group G_{A_3} acts also on the normal to the orbit in $\mathcal{V}$, such as $\tilde{\lambda}_1 = \alpha \oplus \alpha^2 \oplus \alpha^2 \oplus \alpha^3$ on parameters (w_1, w_2, u_1, u_2) where one expresses the normal vectors by $w_1 x^3 \boldsymbol{e} + w_2 y \boldsymbol{e} + u_1 x^2 \boldsymbol{e} + u_2 x \boldsymbol{e}$ as mentioned at the end of §2 (or equivalently, $\tilde{\lambda}_1 = \lambda_1 \oplus \alpha^2 \oplus \alpha^3$ on the source space (x, y, u_1, u_2) of the miniversal unfolding of A_3 in (1.3), since G_{A_3} is also the stabilizer of the unfolding). Thus $c_{top}(\nu_{A_3}) = c_{top}(\tilde{\lambda}_1) = 12a^4$.

By using these data and (4.1), Thom polynomials $tp(\eta)$ for simple ICIS can be computed. Let $\eta = D_5$, for instance. Since it has codimension 6, we may set $tp(D_5) = \sum a_I c^I$ of degree 6 with *unknown coefficients* a_I, thanks to the existence theorem of tp. The equations (4.1) are $\iota_\xi^*\, tp(D_5) = 0$ for $\xi = A_0, \ldots, A_5, S_5$ and $\iota_{D_5}^*\, tp(D_5) = c_6(\nu_{D_5})$.

[8] An exceptional case is $A_1 : (x, y) \mapsto xy$, which has two obvious symmetries $\mathbb{C}^* \times \mathbb{C}^*$.

These form a system of linear equations of unknowns a_I (e.g., in case $\xi = A_3$, substitute $c_1 = a, c_2 = -5a^2, \cdots$ in $tp(D_5) = \sum a_I c^I$, then the resulting coefficient of a^6 must be zero). Solving the linear equations provides the correct answer:

$$\begin{aligned} tp(D_5) &= 4c_1^6 - 2c_1^4c_2 - 18c_1^3c_3 - 6c_1^2c_4 + 12c_1^2c_2^2 \\ &\qquad +2c_1c_5 + 12c_1c_2c_3 - 4c_2^3 - 4c_3^2 + 4c_2c_4 \\ &= 24\Delta_{42} + 12\Delta_{411} + 24\Delta_{33} + 64\Delta_{321} + 26\Delta_{3111} \\ &\qquad +24\Delta_{222} + 42\Delta_{2211} + 4\Delta_{111111}. \end{aligned}$$

The last expansion is with respect to the Schur function basis, and it is observed that all the coefficients are non negative. In fact the *positivity* holds in general, see Pragacz-Weber [29].

We remark that it must be hard to find the precise forms of $tp(A_4), tp(A_5), tp(D_5)$ etc by the traditional approach using embedded resolutions.

Example 4.3. (**Schubert varieties and Tp**)
Let $m \le n$ and $G = GL(m) \times GL(n)$, $V = \mathrm{Hom}\,(\mathbb{C}^m, \mathbb{C}^n)$. Note that each linear map in the G-orbit Σ^{m-r} determines a r-dimensional subspace in $\mathbb{C}^n$, and that any linear maps having the same image can be translated each other by the source changes $GL(m)$. Now fix a complete flag in the target linear space $\mathbb{C}^n$ by using coordinates and put $G^{\triangle} := GL(m) \times \{$ lower triangular matrices in $GL(n)$ $\}$, the subgroup of G which preserves the flag. Then the G-orbit Σ^{m-r} breaks into several $G^{\triangle}$-orbits, which enjoy the one-to-one correspondence

$$G^{\triangle}\text{-orbits in } \Sigma^{m-r} \stackrel{1:1}{\longleftrightarrow} \text{Schubert cells in } \mathrm{Grass}(r, \mathbb{C}^n).$$

We may write $H^*_{G^{\triangle}}(V) = \mathbb{Z}[c'_1, \cdots, c'_m, b_1, \cdots, b_n]$, where b_i are Chern roots (corresponding to the maximal torus of $GL(n)$). In [9] it is shown that the Thom polynomial of a $G^{\triangle}$-orbit (in $H^*_{G^{\triangle}}(V)$) coincides with a *double Schur polynomial* in the sense of [16], and that the specialization under all $b_j = 0$ gives a *Schur polynomial* $\Delta_\lambda(c')$. This picture should be related to equivariant Schubert classes in $H^*_T(\mathrm{Grass}(r, \mathbb{C}^n))$ and their localization [18] (also [24]; see [11], [12] for other classical groups).

§ 5. Current interests

Finally I comment about a few topics in current interests working in progress.

1. *Computing Tp for moduli strata of $\mathcal{K}$-orbits*: As seen above, to compute tp for simple $\mathcal{K}$-orbits (up to certain codimension) is basically possible. The next objects are moduli (continous familes) of $\mathcal{K}$-orbits, such as unimodular singularities, Thom-Boardmann strata $\Sigma^{i,j,k,\cdots}$ etc, for definition, see [1], [23]. Although $tp(\Sigma^{i,j})$ have

already been studied in [32] by the desingularization method (for tp of some $\Sigma^{i,j,k}$, see [5]), the restriction method approach to $tp(\Sigma^{i,j})$ dealt in[6] gives a new insight.

2. *Generating functions of Tp* : It is natural to ask what each coefficient arising in $tp(\eta)$ (with respect to monomial basis or Schur function basis) does mean, or ask whether there is a universal rule for the appearance of such numbers. On one hand, there are several 'stems' in classifications of mono-singularities $\mathbb{C}^m, 0 \to \mathbb{C}^{m+\ell}, 0$ such as A_μ, D_μ, ... or multi-singularities A_1^s $(s = 1, 2, \cdots)$ etc. So it is reasonable to think of generating functions of tp's for such series of singularities where μ, ℓ or s is regarded as a parameter, see [3], [9], [15].

3. *Equivariant Chern classes for singularities* (*higher degree generalization of Tp*): In general, singular varieties (such as orbit closures $\overline{\eta}$) admit several variants of 'Chern (homology) classes', and usually the top term of such classes is the fundamental class of the variety. The most useful functorial Chern class theory is *the Chern-Schwartz-MacPherson class* [21], and the G-equivariant version of CSM classes has been established in [25]. Applying this theory to the orbit closure $\overline{\eta}$ in a G-affine space V, we obtain a non-homogeneous series in $H^*(BG)$ whose lowest degree homogeneous term is just $tp(\eta)$, since $tp(\eta)$ corresponds to the equivariant fundamental class of the orbit closure. Such a 'total class version' of tp is a meaningful class in $\ker \alpha$ mentioned in §3 (i.e., the ideal of G-characteristic classes having supports on $\overline{\eta}$), which has information about the combinatorial structure of adjacencies of G-orbits and several invariants, cf. [25], [26], [27]. On one hand there are some examples of computational aspects: Chern-Schwartz-MacPherson classes of degeneracy loci $\overline{\Sigma^k}$ and of Schubert varieties have already been studied in [28] and [2], respectively.

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