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Double Schubert polynomials of classical type and Excited Young diagrams

By

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Abstract

In this article we announce the main results of our forthcoming paper with L.Mihalcea [8] that introduces double Schubert polynomials for types B,C,D. Another aim is to show how "excited Young diagrams" introduced in [9] can be used in the equivariant Schubert calculus.

§ 1. Introduction

We studied previously the equivariant Schubert classes for maximal isotropic Grassmannians in [7],[9], and found that they are represented by factorial Schur P, Q-functions defined by V.N.Ivanov [7]. For type A Schubert calculus, there exist well known (double) Schubert polynomials defined by A.Lascoux and M.-P.Schützenberger [13],[14]. For Grassmannian permutations, these double Schubert polynomials coincide with factorial Schur functions $s_\lambda(x|a)$ and a geometric interpretation of these polynomials is given by L.Mihalcea in [17] using Kempf-Laksov's formula. In an attempt to extend this geometric interpretation to factorial Schur P, Q-functions, we found double Schubert polynomials for type B, C, D [8]. These are natural analogue of type A double Schubert polynomials so that they share many common properties. Most notable one is the stability. Also these polynomials are extensions of both factorial Schur P, Q-functions and Billey-Haiman's (single) Schubert polynomials [2]. They are polynomials in two series of infinite variables $z_1, z_2, ..., t_1, t_2, ...$ with coefficients in the ring Γ' of Schur P-functions and charactereized by left and right divided difference relations. While the existence and meaning of these polynomials are established in [8] within a geometric framework such as GKM-description of equivariant cohomology [6], here we treat these

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polynomials in another approach using the combinatorics of excited Young diagrams, which was first introduced in [9] for the Schubert calculas of isotropic Grassmannians.

In §2, we state the main results of [8] without using geometric language, so that we focus on combinatorial properties of the double Schubert polynomials. In §3, we give three effective ways to compute the double Schubert polynomials. The first two are supplementary to [8], while the third one is included in [8]. In §3.4, we will give a sketch of an alternative proof for Thm 3.3 — a key formula for the third way of computing DSPs — using EYDs. The details of this argument will be included in [10], where we will give a general framework for EYDs. In §4, we discuss specialization of our polynomials to equivariant cohomology classes for flag varieties. In §5, we discuss some related topics for future works.

§ 2. Notations and Definitions

§ 2.1. Expression of elements of Weyl groups

We realize the Weyl group of type C_n as a subgroup of the symmetric group S_{2n} of $2n$ alphabets $\{\bar{n},\cdots,\bar{1},1,\cdots,n\}$ with linear order $\bar{n}<\cdots<\bar{1}<1<\cdots<n$. Let $w_0=(\bar{1},1)(\bar{2},2)\cdots(\bar{n},n)\in S_{2n}$. Then

$$W(C_n)=\left\{v\in S_{2n}\,\middle|\;\; vw_0=w_0v\right\}$$

with Coxeter generators $s_0=(\bar{1},1), s_i=(i,i+1)(\bar{i},\overline{i+1})$, $1\le i\le n-1$. It is convenient for our purpose to write $w\in W(C_n)$ in one line notation $w=[w(1),w(2),...,w(n)]$. Then $w_0=w_{0,C}^{(n)}=[\bar{1},...,\bar{n}]$ is the longest element in $W(C_n)$ with length $\ell(w_0)=n^2$. A permutation $w\in W(C_n)$ is Grassmannian if $w(1)<w(2)<\cdots<w(n)$, Example. $[\bar{3},\bar{1},2,4]=s_2s_0s_1s_0$ is a Grassmannian permutation in $W(C_4)$. The Weyl group $W(D_n)$ of type D_n is a subgroup of index 2 in $W(C_n)$, consisting of permutations with even number of barred parts in one line notation. Coxeter generators of $W(D_n)$ are $s_{\hat{1}}=(\bar{1},2)(\bar{2},1)$, $s_i=(i,i+1)(\bar{i},\overline{i+1})$, $1\le i\le n-1$ and the longest element $w_{0,D}^{(n)}$ has length $n(n-1)$. Natural inclusion of $W(C_n)\subset W(C_{n+1})$ becomes inductive system of Coxeter groups and we can define the infinite hyperoctahedral group $W(C_\infty)=\bigcup_n W(C_n)$ with subgroup $W(D_\infty)=\bigcup_n W(D_n)$. Weyl group of type B_n is $W(B_n)=W(C_n)$. These groups contain as a subgroup $S_\infty=\bigcup_n S_n$ of infinite symmetric group generated by s_i, $i\ge 1$.

§ 2.2. Factorial Schur P,Q-functions

According to Ivanov [11], we define the factorial Schur P- or Q-functions as follows. Let $\mathcal{SP}:=\{\lambda=(\lambda_1,...,\lambda_r)\mid\lambda_1>\cdots>\lambda_r>0\}$ be the set of strict partitions. For an infinite sequence of parameters $a = (a_i)_{i=1}^{\infty}$, we define the factorial k-th power as $(y|a)^k := (y - a_1)(y - a_2) \cdots (y - a_k)$.

Definition 2.1. Let $\lambda = (\lambda_1, ..., \lambda_r) \in \mathcal{SP}$. Put

$$P_\lambda^{(n)}(x|a) = \frac{1}{(n-r)!} \sum_{w \in S_n} w \left((x_1|a)^{\lambda_1} \cdots (x_r|a)^{\lambda_r} \prod_{1 \le i \le r,\, i<j \le n} \frac{x_i + x_j}{x_i - x_j} \right), \tag{2.1}$$

where $w \in S_n$ acts as a permutation of variables $x_1, \ldots, x_n$. If $r > n$ then we set $P_\lambda^{(n)}(x|a) = 0$. We also put $Q_\lambda^{(n)}(x|a) = 2^r P_\lambda^{(n)}(x|a)$.

It is known that $P_\lambda^{(n)}(x|a)$ is supersymmetric in the sense that it is symmetric with respect to the variables $x_1, x_2, ..., x_n$, and $P_\lambda^{(n)}(x_1, ..., x_{n-2}, t, -t|a)$ does not depend on t. If $a_1 = 0$ then $P^{(n)}(x|a)$ has a stability i.e. $P^{(n+1)}(x_1, \ldots, x_n, 0|a) = P^{(n)}(x_1, \ldots, x_n|a)$. In general it does not hold. For example

$$P_1^{(n)}(x|a) = \begin{cases} x_1 + \cdots + x_n & \text{if } n \text{ is even,} \\ x_1 + \cdots + x_n - a_1 & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

But $P_\lambda^{(n)}(x|a)$ has mod 2 stability, i.e. $P_\lambda^{(n+2)}(x_1, \ldots, x_n, 0, 0|a) = P_\lambda^{(n)}(x_1, \ldots, x_n|a)$, so that we can define $P_\lambda(x|a) := \lim_{\substack{\longleftarrow \\ n:\text{ even}}} P_\lambda^{(n)}(x|a)$. For $Q_\lambda^{(n)}(x|a)$, we will always assume $a_1 = 0$ and omit it, i.e. $Q_\lambda(x|a) = Q_\lambda(x_1, x_2, ...|a_2, a_3, ...) := \lim_{\longleftarrow n} Q_\lambda^{(n)}(x|a)$. By specializing all parameters a_i to 0, we get usual Schur P- or Q- functions $P_\lambda(x) = P_\lambda(x|0)$ and $Q_\lambda(x) = Q_\lambda(x|0)$ cf.[16].

§ 2.3. The ring R_∞ and R'_∞ and divided difference operators

Let $P_\lambda(x), Q_\lambda(x)$ denote the Schur P, Q-functions and put

$$\Gamma = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z} Q_\lambda(x) = \mathbb{Z}[Q_1(x), Q_2(x), ...], \quad \Gamma' = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z} P_\lambda(x) = \mathbb{Z}[P_1(x), P_2(x), ...],$$

$$R_\infty := \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, t_2, \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[z_1, z_2, \ldots], \ R'_\infty := \Gamma' \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, t_2, \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[z_1, z_2, \ldots].$$

We define two kinds of actions ρ_z and ρ_t of $W(C_\infty)$ on R_∞ and R'_∞ as follows. For $i > 0$ let $\rho_z(s_i)$ interchange z_i and z_{i+1}, and fix other z_j's and let $\rho_z(s_0)$ replace z_1 and $-z_1$, and fix other z_j's. The ρ_z action is trivial on t_i's. The action on Γ (and on Γ') is defined from the rules

$$\rho_z(s_0) Q_k(x) = Q_k(x) + 2 \sum_{j=1}^{k} z_1^j Q_{k-j}(x) \ \text{ and } \ \rho_z(s_i) Q_k(x) = Q_k(x) \text{ for } i > 0.$$

Note that the action ρ_z on $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[z_1, z_2, ...]$ is identical to the action defined in [2]. The action ρ_t is defined by

$$\rho_t(w) = \omega\rho_z(w)\omega$$

where $\omega : R_\infty \to R_\infty$ is the involution defined by $\omega(z_i) = -t_i, \omega(t_i) = -z_i, \omega(Q_k(x)) = Q_k(x)$.

We define *divided difference operators* ∂_i, δ_i $(i = 0, 1, 2, \cdots)$, $\partial_{\hat{1}}, \delta_{\hat{1}}$ on R_∞ and R'_∞ by

$$\partial_0 f = \frac{f - \rho_z(s_0)f}{-2z_1}, \quad \partial_i f = \frac{f - \rho_z(s_i)f}{-(z_{i+1} - z_i)} \quad \text{for} \quad i > 0, \quad \partial_{\hat{1}} f = \frac{f - \rho_z(s_{\hat{1}})f}{-(z_2 + z_1)}$$

$$\delta_0 f = \frac{f - \rho_t(s_0)f}{2t_1}, \quad \delta_i f = \frac{f - \rho_t(s_i)f}{t_{i+1} - t_i} \quad \text{for} \quad i > 0, \quad \delta_{\hat{1}} f = \frac{f - \rho_t(s_{\hat{1}})f}{t_2 + t_1}.$$

§ 2.4. Definition and basic properties of double Schubert polynomials

We define double Schubert polynomials of type C (resp. D) as polynomials in the unique family in R_∞ (resp. R'_∞) satisfying the condition of the theorem below. Note that type B double Schubert polynomial can be defined from type C polynomials as in [2], i.e. $\mathfrak{B}_w(z, t; x) = 2^{-s(w)}\mathfrak{C}_w(z, t; x) \in R'$, where $s(w)$ is the number of s_0 in a reduced expression of w.

Theorem 2.2. *[8] There exists a unique family of elements $\{\mathfrak{C}_w\}_{w \in W(C_\infty)} \subset R_\infty$ satisfying the equations*

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if} \quad \ell(ws_i) < \ell(w) \\ 0 & \text{otherwise} \end{cases}, \quad \delta_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_i w} & \text{if} \quad \ell(s_i w) < \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

for all $i \geq 0$, together with the condition that the constant term of $\mathfrak{C}_w$ is zero except for $w = e$, and that $\mathfrak{C}_e = 1$.

Theorem 2.3. *[8] There exists a unique family of elements $\{\mathfrak{D}_w\}_{w \in W(D_\infty)} \subset R'_\infty$ satisfying the equations*

$$\partial_i \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{ws_i} & \text{if} \quad \ell(ws_i) < \ell(w) \\ 0 & \text{otherwise} \end{cases}, \quad \delta_i \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{s_i w} & \text{if} \quad \ell(s_i w) < \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

for all $i \geq 1$ and $i = \hat{1}$, together with the condition that the constant term of $\mathfrak{D}_w$ is zero except for $w = e$, and that $\mathfrak{D}_e = 1$.

We collect here some basic properties of $\mathfrak{C}_w = \mathfrak{C}_w(z, t; x)$. The polynomials $\mathfrak{D}_w$ also have similar properties.

Theorem 2.4. *[8]*

1. *The double Schubert polynomials* $\{\mathfrak{C}_w\}_{w\in W(C_\infty)}$ *form a* $\mathbb{Z}[t]$*-basis of* R_∞.

2. *For all* $w \in W(C_\infty)$ *we have*

$$\mathfrak{C}_w(z,0;x) = \mathfrak{C}_w^{BH}(z;x),$$

where $\mathfrak{C}_w^{BH}(z;x)$ *is Billey-Haiman's Schubert polynomial([2]).*

3. *(Symmetry)* $\mathfrak{C}_w(-t,-z;x) = \mathfrak{C}_{w^{-1}}(z,t;x).$

4. *(Positivity) When we write*

$$\mathfrak{C}_w(z,t;x) = \sum_{\lambda\in\mathcal{SP}} f_{w,\lambda}(z,t)Q_\lambda(x),$$

we have $f_{w,\lambda}(z,t) \in \mathbb{N}[-t_1,\ldots,-t_{n-1},z_1,\ldots,z_{n-1}]$ *if* $w \in W(C_n)$.

There are similar results for type B and D (see[8]). For example both $\{\mathfrak{B}_w\}_{w\in W(B_\infty)}$ and $\{\mathfrak{D}_w\}_{w\in W(D_\infty)}$ form $\mathbb{Z}[t]$-basis of R'_∞.

For a Grassmannian permutation$w = [w_1, w_2, ..., w_n]$, if the barred part of w is $\overline{b_1}, ..., \overline{b_r}$, we set $\lambda_w = (b_1, ..., b_r)$ for type B_n and C_n, while $\lambda_w = (b_1 - 1, ..., b_r - 1)$ for type D_n.

Theorem 2.5. *[8] If* $w \in W(X_\infty)$ *is a Grassmannian permutation corresponding to strict partition* $\lambda = \lambda_w$ *in* $\mathcal{SP}$*, then*

$$\begin{aligned}
\mathfrak{B}_w(z,t;x) &= P_\lambda(x|0,t) \quad \text{for } X = B,\\
\mathfrak{C}_w(z,t;x) &= Q_\lambda(x|t) \quad\ \ \text{for } X = C,\\
\mathfrak{D}_w(z,t;x) &= P_\lambda(x|t) \quad\ \ \text{for } X = D.
\end{aligned}$$

Example.
$\mathfrak{B}_{[\bar{3},\bar{2},1]} = P_{3,2}(x|0,t) = P_{3,2} + P_{3,1}(-t_1) + P_{2,1}t_1^2,$
$\mathfrak{C}_{[\bar{3},\bar{2},1]} = Q_{3,2}(x|t) = Q_{3,2} + Q_{3,1}(-t_1) + Q_{2,1}t_1^2,$
$\mathfrak{D}_{[\bar{3},\bar{2},1]} = P_{2,1}(x|t) = P_{2,1} + P_2(-t_1) + P_1t_1^2.$

§ 3. How to calculate double Schubert polynomials

There are at least three ways to calculate double Schubert polynomials.

(1) use explicit form of single Schubert polynomials,

(2) use transition equation,

(3) use divided difference and an expression of the longest element w_0.

The most efficient way is (2), once we know that factorial Schur $P-$(or $Q-$) functions correspond to Grassmannian permutations.

§ 3.1. Double Schubert polynomials in terms of single ones

For $w \in S_n$, let $\mathfrak{S}_w(z)$ be the single Schubert polynomial of type A. For $w \in W(C_n)$ (resp. $w \in W(D_n)$), let $F_w(x)$ (resp. $E_w(x)$) be the Stanley symmetric function of type C (resp. type D), (cf. [2])

$$F_w(x) := \sum_{\mathbf{a}\in R(w)} \sum_{(i_1\le\cdots\le i_\ell)\in A(P(\mathbf{a}))} 2^{|\mathbf{i}|} x_{i_1}x_{i_2}\cdots x_{i_\ell},$$
$$E_w(x) := \sum_{\mathbf{a}\in R(w)} \sum_{(i_1\le\cdots\le i_\ell)\in A(P(\mathbf{a}))} 2^{|\mathbf{i}|-\mathbf{o}(\mathbf{a})} x_{i_1}x_{i_2}\cdots x_{i_\ell},$$

where $R(w)$ is the set of reduced expressions of w, $\mathbf{a} = a_1a_2\cdots a_\ell$ corresponds to the reduced expression $w = s_{a_1}s_{a_2}\cdots s_{a_\ell}$. The condition $(i_1 \le \cdots \le i_\ell) \in A(P(\mathbf{a}))$ means that we do not have $i_{j-1} = i_j = i_{j+1}$ if $a_{j-1} < a_j > a_{j+1}$. $|\mathbf{i}|$ is the number of distinct $i_j's$ in the sequence $\mathbf{i} = (i_1, i_2, ..., i_\ell)$ and $o(\mathbf{a})$ is the total numbers of 1's and $\hat{1}$'s in $\mathbf{a}$.

Theorem 3.1. *[8] (for single Schubert polynomials [2] Theorem 3A,4A)*

$$\mathfrak{C}_w(z,t;x) = \sum_{\substack{v_1uv_2=w\\ \ell(v_1)+\ell(u)+\ell(v_2)=\ell(w)\\ v_1,v_2\in S_n, u\in W(C_n)}} \mathfrak{S}_{v_1^{-1}}(-t)F_u(x)\mathfrak{S}_{v_2}(z),$$

$$\mathfrak{D}_w(z,t;x) = \sum_{\substack{v_1uv_2=w\\ \ell(v_1)+\ell(u)+\ell(v_2)=\ell(w)\\ v_1,v_2\in S_n, u\in W(D_n)}} \mathfrak{S}_{v_1^{-1}}(-t)E_u(x)\mathfrak{S}_{v_2}(z).$$

If we set $t = 0$ then these formulas become the original formulas [2].

More generally, we have

$$\mathfrak{C}_w(z,t;x) = \sum_{\substack{vu=w\\ \ell(v)+\ell(u)=\ell(w)}} \mathfrak{S}_v(y,t)\mathfrak{C}_u(z,y;x),$$
$$\mathfrak{D}_w(z,t;x) = \sum_{\substack{vu=w\\ \ell(v)+\ell(u)=\ell(w)}} \mathfrak{S}_v(y,t)\mathfrak{D}_u(z,y;x).$$

Example.

$\mathfrak{C}_{s_0}(z,t;x) = Q_1(x)$,

$\mathfrak{C}_{s_i}(z,t;x) = Q_1(x) + (z_1 + \cdots + z_i - t_1 - \cdots - t_i)$ for $i \ge 1$

$\mathfrak{C}_{s_1s_0}(z,t;x) = \mathfrak{C}_{s_1s_0}(z;x) + \mathfrak{S}_{s_1}(-t)\mathfrak{C}_{s_0}(z;x) = Q_2(x) + (-t_1)Q_1(x)$

$$\begin{aligned}&\mathfrak{C}_{s_1s_2s_0s_1}(z,t;x)=\mathfrak{C}_{s_1s_2s_0s_1}(z;x)+\mathfrak{S}_{s_1}(-t)\mathfrak{C}_{s_2s_0s_1}(z;x)+\mathfrak{S}_{s_2s_1}(-t)\mathfrak{C}_{s_0s_1}(z;x)\\&=(Q_{3,1}+z_1Q_3+z_1Q_{2,1}+z_1^2Q_2)+(-t_1)(Q_3+Q_{2,1}+2z_1Q_2+z_1^2Q_1)+t_1^2(Q_2+z_1Q_1)\\&=Q_{3,1}+(z_1-t_1)Q_3+(z_1-t_1)Q_{2,1}+(z_1-t_1)^2Q_2+(-z_1t_1)(z_1-t_1)Q_1.\end{aligned}$$

§ 3.2. Transition equation

For type A Schubert polynomials, it is known that there exists a recurrence relation called transition equation [13]. S. Billey [1] extended it to other classical types. We extend Billey's results to our double Schubert polynomials. The reflections in $W(C_\infty)$ are of the form $t_{ir}=(i,r)(\bar{i},\bar{r}), s_{ir}=(i,\bar{r})(\bar{i},r)$, and $s_{rr}=(r,\bar{r})$. Using equivariant Chevalley formula, we get the following recurrence formula called transition equation. (for single Schubert polynomials cf.[1])

Proposition 3.2. *For a permutation $w=[w(1),w(2),...,w(n),...]$ of type B,C,D, let r be the last descent of w i.e. the largest r such that $w(r)>w(r+1)$, and s be the largest index such that $s>r$ and $w(s)<w(r)$. Put $v=wt_{rs}$. $\mathfrak{X}$ represents one of $\mathfrak{B},\mathfrak{C},\mathfrak{D}$. Then*

$$\mathfrak{X}_w(z,t;x)=(z_r-v(t_r))\,\mathfrak{X}_v(z,t;x)+\sum_{1\le i<r}\mathfrak{X}^*_{vt_{ir}}(z,t;x)+\sum_{i\neq r}\mathfrak{X}^*_{vs_{ir}}(z,t;x)+\chi\mathfrak{X}^*_{vs_{rr}}(z,t;x)$$

where $\mathfrak{X}^*_u=\mathfrak{X}_u$ if $\ell(u)=\ell(w)$ and 0 otherwise. $\chi=2,1,0$ according to type B,C,D.

Using this equation recursively, we can calculate double Schubert polynomials as a linear combination of factorial Schur $P-$ or $Q-$ functions.

Example 1.
$w=[3,\bar{1},2]\in W(C_3)$. In this case $r=1,s=3$ and $v=[2,\bar{1},3]$. Then

$$\mathfrak{C}_{[3,\bar{1},2]}(z,t;x)=(z_1-t_2)\mathfrak{C}_{[2,\bar{1},3]}(z,t;x)+\mathfrak{C}_{[1,\bar{2},3]}(z,t;x)+\mathfrak{C}_{[\bar{2},\bar{1},3]}(z,t;x).$$

The term $\mathfrak{C}_{[2,\bar{1},3]}(z,t;x)$ and $\mathfrak{C}_{[1,\bar{2},3]}(z,t;x)$ can be also rewritten as

$$\mathfrak{C}_{[2,\bar{1},3]}=(z_1+t_1)\mathfrak{C}_{[\bar{1},2,3]}+\mathfrak{C}_{[\bar{2},1,3]},\ \mathfrak{C}_{[1,\bar{2},3]}=(z_1+t_2)\mathfrak{C}_{[\bar{2},1,3]}+\mathfrak{C}_{[\bar{3},1,2]}.$$

Therefore we get

$$\begin{aligned}\mathfrak{C}_{[3,\bar{1},2]}&=Q_3(x|t)+Q_{2,1}(x|t)+2z_1Q_2(x|t)+(z_1-t_2)(z_1+t_1)Q_1(x|t)\\&=Q_3+Q_{2,1}+Q_2(2z_1-t_1-t_2)+Q_1z_1(z_1-t_1-t_2).\end{aligned}$$

Example 2.
$w=[\bar{3},1,\bar{2}]\in D_3$. In this case $r=2,s=3$ and $v=[\bar{3},\bar{2},1]$. Then
$\mathfrak{D}_{[\bar{3},1,\bar{2}]}=(z_2+t_2)\mathfrak{D}_{[\bar{3},\bar{2},1]}+\mathfrak{D}_{[\bar{2},\bar{3},1]}$ and $\mathfrak{D}_{[\bar{2},\bar{3},1]}=(z_1+t_3)\mathfrak{D}_{[\bar{3},\bar{2},1]}+\mathfrak{D}_{[\bar{4},\bar{3},1,2]}$.
Therefore we get

$$\begin{aligned}\mathfrak{D}_{[\bar{3},1,\bar{2}]} &= P_{3,1}(x|t) + (z_1+z_2+t_2+t_3)P_{2,1}(x|t)\\ &= P_{3,1} + P_{2,1}(z_1+z_2-t_1) + P_3(-t_1) + P_2 t_1(t_1-z_1-z_2) + P_1 t_1^2(z_1+z_2).\end{aligned}$$

§ 3.3. The Double Schubert polynomials for the longest elements

The main object of this subsection is to express double Schubert polynomials for the longest element w_0 in terms of factorial Schur P- or Q- functions. For the proof we use excited Young diagrams.

Theorem 3.3. *[8]*
$B_n : \mathfrak{B}_{w_{0,B}^{(n)}}(z,t;x) = P_{\rho_n+\rho_{n-1}}(x \mid 0, -z_1, t_1, -z_2, t_2, ...)$,
$C_n : \mathfrak{C}_{w_{0,C}^{(n)}}(z,t;x) = Q_{\rho_n+\rho_{n-1}}(x \mid -z_1, t_1, -z_2, t_2, ...)$,
$D_n : \mathfrak{D}_{w_{0,D}^{(n)}}(z,t;x) = P_{\rho_{n-1}+\rho_{n-1}}(x \mid -z_1, t_1, -z_2, t_2, ...)$,
where $\rho_n = (n, n-1, \cdots, 1)$.

Example.

$$\begin{aligned}&\mathfrak{C}_{w_{0,C}^{(3)}}(x,z;t) = Q_{5,3,1}(x \mid -z_1, t_1, -z_2, t_2)\\ &= Q_{5,3,1} + Q_{4,3,1}(z_1+z_2-t_1-t_2) + Q_{5,2,1}(z_1-t_1) + Q_{4,2,1}(z_1-t_1)(z_1+z_2-t_1-t_2)\\ &+ Q_{3,2,1}((z_1^2-z_1z_2+z_2^2)(z_2-t_2) + z_1(-t_1)(z_1-t_1))\end{aligned}$$

$$\begin{aligned}&\mathfrak{D}_{w_{0,D}^{(3)}}(z,t;x) = P_{4,2}(x \mid -z_1, t_1, -z_2, t_2)\\ &= P_{4,2} + P_{3,2}(z_1+z_2-t_1-t_2) + P_{4,1}(z_1-t_1) + P_{3,1}(z_1-t_1)(z_1+z_2-t_1-t_2)\\ &+ P_{2,1}(z_1^2z_2 - t_1^2t_2 + z_1t_1t_2 - z_1z_2t_1 + z_1^2(-t_1-t_2) + t_1^2(z_1+z_2)) + P_4(-z_1t_1)\\ &+ P_3(-z_1t_1)(z_1+z_2-t_1-t_2) + P_2(-z_1t_1)(-z_1t_1 - z_2t_1 - z_1t_2 + z_1z_2 + t_1t_2)\\ &+ P_1(z_1^2t_1^2)(z_2-t_2).\end{aligned}$$

The above theorem follows from the next proposition. We only indicate for the case of type C_n.

Proposition 3.4. *([8])*

$$\begin{aligned}&\delta_n\cdots\delta_1\delta_0\delta_1\cdots\delta_n Q_{\rho_{n+1}+\rho_n}(x|-z_1,t_1,-z_2,t_2,\cdots,-z_n,t_n)\\ &= Q_{\rho_n+\rho_{n-1}}(x|-z_1,t_1,-z_2,t_2,\cdots,-z_{n-1},t_{n-1}),\end{aligned}$$

and for each step the polynomial has an expression in terms of a factorial Schur Q-function. More precisely,

$$\begin{aligned}&\delta_{i+1}\cdots\delta_n Q_{\rho_{n+1}+\rho_n}(x|-z_1,t_1,\cdots,-z_n,t_n)\\ &= Q_{\rho_{n+1}+\rho_n-1^{n-i}}(x|-z_1,t_1,\cdots,-z_i,t_i,\underline{t_{i+1},-z_{i+1}},t_{i+2},-z_{i+2},\cdots,t_n,-z_n),\end{aligned}$$

$$\begin{aligned}&\delta_i\cdots\delta_1\delta_0\delta_1\cdots\delta_n Q_{\rho_{n+1}+\rho_n}(x|-z_1,t_1,\cdots,-z_n,t_n)\\ &= Q_{\rho_n+\rho_{n-1}+1^{n-i}}(x|-z_1,t_1,\cdots,-z_i,t_i,\underline{t_{i+1},-z_{i+1}},t_{i+2},-z_{i+2},\cdots,t_n,-z_n).\end{aligned}$$

Remark: By the duality $\mathfrak{C}_w(z,t;x) = \mathfrak{C}_{w^{-1}}(-t,-z;x)$ and $(w_0^{(n+1)})^{-1} = w_0^{(n+1)}$, the sequence $\partial_n \cdots \partial_1 \partial_0 \partial_1 \cdots \partial_n \mathfrak{C}_{w_0^{(n+1)}}(z,t;x)$ also has the above property.

To prove proposition, we use excited Young diagram (EYD). These are defined in [9] for describing equivariant multiplicity of the Schubert class for Grassmannians. But EYD can also describe Schur functions and factorial Schur functions [10]. Here we briefly explain this for the case of factorial Schur Q-functions.

§ 3.4. Factorial Q-function in terms of Excited Young Diagrams.

In order to manipulate infinite variables $x_1, x_2, \ldots$, we need infinite rows numbered $1, 2, \ldots$ from bottom to top. We also need infinite columns numbered $\ldots, -2, -1, 0, 1, 2, \ldots$. Let $D = \{(i,j) \mid i > 0, i, j \in \mathbb{Z}\}$ be the set of cells. At the cell (row-i,column-j) we put weight $wt(i,j)$ as follows.

$wt(i,j) = x_i + x_{-j}$ if $j < 0$, and $wt(i,j) = x_i - a_{j+2}$ if $j \geq 0$.

For a strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{SP}$, let $\mathcal{E}(\lambda)$ be the set of excited Young diagrams (EYD's for short) corresponding to λ, i.e. all the diagrams C' obtained from forward and backward elementary excitations $C(\lambda) \to \cdots \to C'$, where

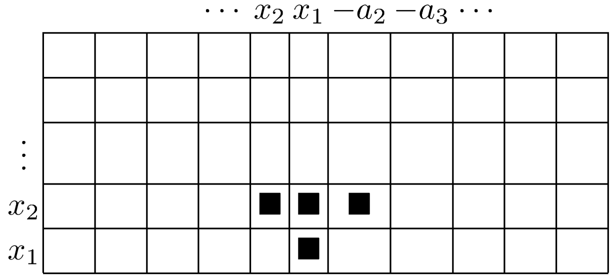
$$C(\lambda) := \{\ (r+1-i, i+j-r-2) \mid 1 \leq i \leq r, 1 \leq j \leq \lambda_i\} \subset D.$$

Forward *elementary excitation* is as defined as follows ([9]). If a box $(i,j) \in C$ s.t. $(i,j+1), (i-1,j), (i-1,j+1) \notin C$, we move the box to the position $(i-1,j+1)$ to make $C' = (C \backslash (i,j)) \cup \{(i-1,j+1)\}$. We call this move $C \to C'$ a forward elementary excitation. Backward elementary excitation is the reverse move. Then we have

Proposition 3.5. *([10])*

$$Q_\lambda(x|a) = \sum_{C \in \mathcal{E}(\lambda)} \prod_{(i,j) \in C} wt(i,j).$$

Example. for $\lambda = (3,1)$ the diagram $C(\lambda)$ is indicated by black boxes.



$$\begin{aligned} Q_{3,1}(x|a) &= 2x_1 2x_2(x_2+x_1)(x_2-a_2) + 2x_1 2x_2(x_2+x_1)(x_1-a_3) \\ &\quad + 2x_1 2x_3(x_2+x_1)(x_2-a_2) + 2x_1 2x_3(x_3+x_2)(x_2-a_2) + \cdots \end{aligned}$$

Remark. In [11] there exists another tabelaux sum formula for $Q_\lambda(x|a)$. The relation between two different formulas can be explained by excited Young diagram arguments cf. [10].

Lemma 3.6. *(Local change of weights for Excited Young Diagrams)*
For some column j and $j+1$, $j \geq 0$ in the diagram D, if there is a weight pattern of the form of left hand side of the diagram below, one can replace the weights as shown in the right hand side without changing the value of weight sum of EYD's.

$$\begin{array}{|c|c|} \hline \beta_{h+1}+t & 0 \\ \hline \beta_h + t & \beta_h \\ \hline \vdots & \vdots \\ \hline \beta_2 + t & \beta_2 \\ \hline \beta_1 + t & \beta_1 \\ \hline 0 & \beta_0 \\ \hline \end{array} \iff \begin{array}{|c|c|} \hline \beta_{h+1} & 0 \\ \hline \beta_h & \beta_h + t \\ \hline \vdots & \vdots \\ \hline \beta_2 & \beta_2 + t \\ \hline \beta_1 & \beta_1 + t \\ \hline 0 & \beta_0 + t \\ \hline \end{array}$$

Proof) Using a determinantal formula for weight sum, it is enough to show the invariance for the case of one cell and two cells , cf. [10].

Corollary 3.7. *For $Q_\lambda(x|a)$, fix $i \geq 2$ and assume that there is no $C \in \mathcal{E}(\lambda)$ such that $(1, i-2) \in C$ and $(1, i-1) \notin C$, then $Q_\lambda(x|a)$ is symmetric for the variables a_i and a_{i+1}.*

Proof) By the condition of λ, we can replace the weight $x_1 - a_i$ of the left diagram to zero, and apply the lemma above. After that we can replace the weight 0 to $x_1 - a_{i+1}$ without changing the weight sum.

$$\begin{array}{|c|c|} \vdots & \vdots \\ \hline x_3 - a_i & x_3 - a_{i+1} \\ \hline x_2 - a_i & x_2 - a_{i+1} \\ \hline x_1 - a_i & x_1 - a_{i+1} \\ \hline \end{array} \iff \begin{array}{|c|c|} \vdots & \vdots \\ \hline x_3 - a_{i+1} & x_3 - a_i \\ \hline x_2 - a_{i+1} & x_2 - a_i \\ \hline x_1 - a_{i+1} & x_1 - a_i \\ \hline \end{array}$$

Lemma 3.8. *(Divided difference and excited Young diagram)*
1) If a box $\blacksquare$ of $C(\lambda)$ in the position (p, q) is a corner box, i.e. $(p-1, q), (p, q+1) \notin C(\lambda)$,and $a_{p+q+1} = t_i$, $a_{p+q+2} = t_{i+1}$ for some $i > 0$, then

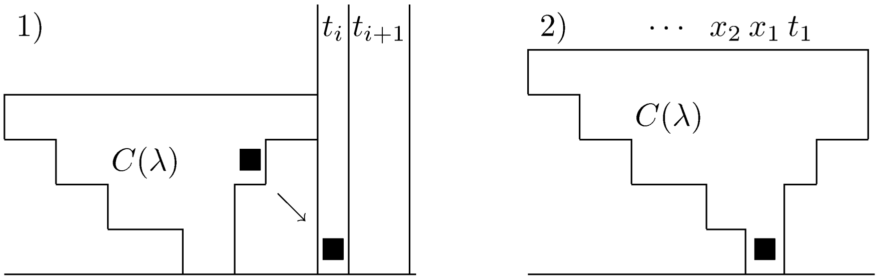
$$\delta_i(Q_\lambda(x|a_2, \ldots, a_{p+q}, t_i, t_{i+1}, a_{p+q+3}, \ldots)) = Q_{\overline{\lambda}}(x|a_2, \ldots, a_{p+q}, t_i, t_{i+1}, a_{p+q+3}, \ldots),$$

where $\overline{\lambda}$ is the shifted Young diagram obtained from λ by removing $\blacksquare$.

2) If the position $(1, -1)$ of $C(\lambda)$ is a corner box $\blacksquare$ and $a_2 = t_1$, then

$$\delta_0(Q_\lambda(x|t_1, a_3, a_4, \ldots) = Q_{\overline{\lambda}}(x|t_1, a_3, a_4, \ldots),$$

where $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{r-1})$.



Using above Corollary 3.7 and Lemma 3.8 we get Proposition 3.4 and Theorem 3.3.
Example.

$$
\begin{aligned}
& Q_{5,3,1}(x|-z_1,t_1,-z_2,t_2) = Q_{5,3,1}(x|-z_1,t_1,-z_2,t_2,t_3)\\
\xrightarrow{\delta_2}\ & Q_{4,3,1}(x|-z_1,t_1,\underline{-z_2,t_2}) = Q_{4,3,1}(x|-z_1,t_1,\underline{t_2,-z_2})\\
\xrightarrow{\delta_1}\ & Q_{4,2,1}(x|\underline{-z_1,t_1},t_2,-z_2) \overset{*}{=} Q_{4,2,1}(x|\underline{t_1,-z_1},t_2,-z_2)\\
\xrightarrow{\delta_0}\ & Q_{4,2}(x|t_1,\underline{-z_1,t_2},-z_2) = Q_{4,2}(x|t_1,\underline{t_2,-z_1},-z_2)\\
\xrightarrow{\delta_1}\ & Q_{4,1}(x|t_1,\underline{t_2,-z_1},-z_2) = Q_{4,1}(x|t_1,\underline{-z_1,t_2},-z_2)\\
& = Q_{4,1}(x|\underline{t_1,-z_1},t_2,-z_2) = Q_{4,1}(x|\underline{-z_1,t_1},t_2,-z_2) = Q_{4,1}(x|-z_1,t_1,t_2,t_3)\\
\xrightarrow{\delta_2}\ & Q_{3,1}(x|-z_1,t_1,t_2,t_3) = Q_{3,1}(x|-z_1,t_1).
\end{aligned}
$$

$$
\begin{array}{|c|c|c|c|llll}
\hline
2x_3 & x_3+x_2 & x_3+x_1 & x_3+z_1 & \multicolumn{1}{l}{x_3-t_1} & x_3-t_2 & x_3+z_2 & \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{|c|}{2x_2} & x_2+x_1 & \multicolumn{1}{l}{x_2+z_1} & x_2-t_1 & x_2-t_2 & x_2+z_2 & = Q_{4,2,1}(x|\underline{-z_1,t_1},t_2,-z_2)\\
\cline{2-3}
\multicolumn{2}{c}{} & \multicolumn{1}{|c|}{2x_1} & \multicolumn{1}{l}{x_1+z_1} & x_1-t_1 & x_1-t_2 & x_1+z_2 & \\
\cline{3-3}
\end{array}
$$

$$* \|$$

$$
\begin{array}{|c|c|c|c|llll}
\hline
2x_3 & x_3+x_2 & x_3+x_1 & x_3-t_1 & \multicolumn{1}{l}{x_3+z_1} & x_3-t_2 & x_3+z_2 & \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{|c|}{2x_2} & x_2+x_1 & \multicolumn{1}{l}{x_2-t_1} & x_2+z_1 & x_2-t_2 & x_2+z_2 & = Q_{4,2,1}(x|\underline{t_1,-z_1},t_2,-z_2)\\
\cline{2-3}
\multicolumn{2}{c}{} & \multicolumn{1}{|c|}{2x_1} & \multicolumn{1}{l}{x_1-t_1} & x_1+z_1 & x_1-t_2 & x_1+z_2 & \\
\cline{3-3}
\end{array}
$$

§ 4. Equivariant Schubert calculus for classical flag varieties

The double Schubert polynomials $\mathfrak{B}_w, \mathfrak{C}_w, \mathfrak{D}_w$ represent the torus equivariant Schubert classes $\sigma_T^w \in H_T^{2\ell(w)}(G/B)$ of classical full flag varieties G/B of type B, C, D.

§ 4.1. Specialization

Here we explain for the case of type C_n. In this case it is known that the T-equivariant cohomology ring $H_T^*(Sp_{2n}(\mathbb{C})/B)$ has a presentation $\mathbb{Z}[z_1,...,z_n,t_1,...,t_n]/I_n$, where the ideal I_n is generated by homogeneous parts of $\prod_{i=1}^n(1-t_i^2) - \prod_{i=1}^n(1-z_i^2)$.

There are at least three types of specialization $R_\infty \to H_T^*(Sp_{2n}(\mathbb{C})/B)$ by which $\mathfrak{C}_w$ goes to σ_T^w for $w \in W(C_n)$.

t_i and z_i become zero for $i > n$. $Q_\lambda(x)$ is specialized to $q_\lambda^{(a)}$ which is given as follows.

type 1) $\frac{\prod_{i=1}^n(1+t_i)}{\prod_{i=1}^n(1+z_i)} = 1 + \sum_{k=1}^\infty q_k^{(1)}$.
type 2) $\frac{\prod_{i=1}^n(1-z_i)}{\prod_{i=1}^n(1-t_i)} = 1 + \sum_{k=1}^\infty q_k^{(2)}$.
type 3) $\sqrt{\frac{\prod_{i=1}^n(1+t_i)\prod_{i=1}^n(1-z_i)}{\prod_{i=1}^n(1-t_i)\prod_{i=1}^n(1+z_i)}} = 1 + \sum_{k=1}^\infty q_k^{(3)}$.

For $i > j$, $q_{i,j}^{(a)} = q_i^{(a)} q_j^{(a)} + 2\sum_{k=1}^{j}(-1)^k q_{i+k}^{(a)} q_{j-k}^{(a)}$, $q_\lambda^{(a)} = \mathrm{Pf}(q_{\lambda_i,\lambda_j}^{(a)})$.

In type 3 case, $\mathfrak{C}_w(z,t;q^{(3)}) \in \mathbb{Q}[z_1,...,z_n,t_1,...,t_n]$.

Example. $n = 3$
$\mathfrak{C}_{s_0}(z,t;q^{(a)}) = t_1 + t_2 + t_3 - z_1 - z_2 - z_3$,
$\mathfrak{C}_{s_1}(z,t;q^{(a)}) = t_2 + t_3 - z_2 - z_3$, $\mathfrak{C}_{s_2}(z,t;q^{(a)}) = t_3 - z_3$ $(a = 1,2,3)$.

$\mathfrak{C}_{s_1 s_0}(z,t;q^{(1)}) = z_1^2 + z_2^2 + z_3^2 - t_1^2 + t_2t_3 - (t_2+t_3)(z_1+z_2+z_3) + z_1z_2 + z_1z_3 + z_2z_3$,
$\mathfrak{C}_{s_1 s_0}(z,t;q^{(2)}) = t_2^2 + t_2t_3 + t_3^2 - (t_2+t_3)(z_1+z_2+z_3) + z_1z_2 + z_1z_3 + z_2z_3$,
$\mathfrak{C}_{s_1 s_0}(z,t;q^{(3)}) = (t_1+t_2+t_3-z_1-z_2-z_3)(-t_1+t_2+t_3-z_1-z_2-z_3)/2$.

Remark. The type 3 specialization gives Fomin-Kirillov's combinatorial double Schubert polynomials of second kind [4].

§ 4.2. Schubert calculus

There are product formulas for Schur Q-functions, such as by Stembridge [19] or by Shimozono [18]. Therefore in principle we can calculate a product of Schubert polynomials, but it is rather hard to calculate in general.

Example. $\mathfrak{C}_{s_0s_1}\mathfrak{C}_{s_0} = (Q_2(x) + z_1Q_1(x)) \times Q_1(x) = Q_{2,1}(x) + 2Q_3(x) + 2z_1Q_2(x)$
$= \mathfrak{C}_{s_0s_1s_0} + 2\mathfrak{C}_{s_1s_0s_1} + 2t_1\mathfrak{C}_{s_0s_1}$.

§ 5. Future works, problems and comments

There are diverse directions related to double Schubert polynomials. We only point some of them relating to excited Young diagrams (EYD).

1. One of the most desirable thing to know is a combinatorial description of product formula for double Schubert polynomials. For the special case of Grassmannians, the product formulas of $Q_\lambda(x)$ given by Stembridge [19] or Shimozono [18] may be extended to that of factorial Schur $Q-$ functions. These product formulas, if exist, will serve for describing further general Littlewood-Richardson rule for double Schubert polynomials.

2. In many cases the double Schubert polynomials can be expressed using some variants of excited Young diagram. For example of type C_3 case, all but two elements $([\bar{1},3,2],[\bar{3},2,\bar{1}])$ have double Schubert polynomials expressed in terms of EYD. It is

hoped that vexillary elements of type B, C, D (cf. [3]) as well as type A vexillary elements (cf. [15]) have this property.

3. Using a generalization of excited Young diagrams we have a candidate of double Grothendieck polynomials to describe equivariant K-theory. The details will be explained elsewhere.

4. The expression of double Schubert polynomials for w_0 (Theorem 3.3) was found for the first time using excited Young diagrams as described here. But after that we get another proof not using EYD, that will be included in [8]. This expression means a degeneracy loci formula, different from Fulton's [5] and Kresch-Tamvakis's [12], and has a Pfaffian formula.

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