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A Product formula for decomposition numbers of the cyclotomic q-Schur algebra and its analogue for the Fock space

By

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Abstract

Let $\mathscr{S}$ be the cyclotomic q-Schur algebra associated to the Ariki-Koike algebra. We describe a certain product formula for v-decomposition numbers of $\mathscr{S}$. Moreover, we also describe a product formula for entries of the transition matrix between two bases, the canonical basis and the standard basis, in the v-deformed Fock space. The formula for the Fock space is regarded as a counter part for the formula on v-decomposition numbers of $\mathscr{S}$ through the Yvonne's conjecture. This paper is a survey of the results in [SW1], [W] and [SW2].

§ 0. Introduction

Let $\mathscr{H}_{n,r}$ be the Ariki-Koike algebra associated to $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and $\mathscr{S}$ be the cyclotomic q-Schur algebra associated to $\mathscr{H}_{n,r}$ introduced by Dipper-James-Mathas [DJM]. In [DJM], it was shown that $\mathscr{S}$ is a cellular algebra in the sense of Graham-Lehrer [GL]. Thus, one of the fundamental problems is determining decomposition numbers, namely multiplicities of a simple $\mathscr{S}$-module in the composition factors of a Weyl (cell or standard) module.

In the case of $r = 1$ (namely $\mathscr{S}$ is the q-Schur algebra of type A_{n-1}), where the base field is $\mathbb{C}$ and q is a root of unity in $\mathbb{C}$, Varagnolo-Vasserot [VV] proved that decomposition numbers of $\mathscr{S}$ can be described by the transition matrix between the canonical basis and the standard basis of the v-deformed Fock space of level 1. One of

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the points of their proof is the Schur-Weyl duality between the Iwahori-Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ of type A_{n-1} and the quantum group $U_q(\mathfrak{gl}_m)$ ([Du]). As a consequence of the Schur-Weyl duality, $\mathscr{S}$ (in the case of type A_{n-1}) turns out to be a quotient of $U_q(\mathfrak{gl}_m)$. In the case of $r \geq 2$, Yvonne [Y] conjectured that decomposition numbers of $\mathscr{S}$ can be described by the transition matrix between the canonical basis and the standard basis of a v-deformed Fock space of level r (see the section 5 for more details). This conjecture is a generalization of the result of [VV], and still open.

On the other hand, in [SakS], Sakamoto-Shoji proved the Schur-Weyl duality, over $\mathbb{Q}(q)$ with an indeterminate q, between the Ariki-Koike algebra $\mathscr{H}_{n,r}$ and the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a certain Levi subalgebra of $\mathfrak{gl}_m$. In [SawS], Sawada-Shoji constructed the modified cyclotomic q-Schur algebra $\overline{\mathscr{S}}^0$ which is a quotient of $U_q(\mathfrak{g})$, where q is an element of a ground ring and $U_q(\mathfrak{g})$ is the specialized algebra obtained from Kostant-Lusztig's integral form by specializing the parameters at q. They studied the relationship with the original cyclotomic q-Schur algebra $\mathscr{S}$, and Sawada [Saw] proved the product formula for decomposition numbers of $\mathscr{S}$ by using such relationships and the structure of $\overline{\mathscr{S}}^0$. Sawada also showed that the relation between $\mathscr{S}$ and $\overline{\mathscr{S}}^0$ can be obtained from the cellular structure of $\mathscr{S}$ without using Sawada and Shoji's Schur-Weyl duality.

Fix $\mathbf{p} = (r_1, \cdots, r_g) \in \mathbb{Z}_{>0}^g$ such that $r_1 + \cdots + r_g = r$. In [SW1], we constructed a subalgebra $\mathscr{S}^{\mathbf{p}}$ of $\mathscr{S}$ and its quotient $\overline{\mathscr{S}}^{\mathbf{p}}$ associated with $\mathbf{p}$ by using the cellular structure of $\mathscr{S}$. Moreover we proved that $\overline{\mathscr{S}}^{\mathbf{p}}$ is isomorphic to a direct sum of tensor products of various cyclotomic q-Schur algebras with smaller rank than the original $\mathscr{S}$. Then we obtained a product formula for decomposition numbers of $\mathscr{S}$ by using the relation between $\mathscr{S}$ and $\overline{\mathscr{S}}^{\mathbf{p}}$ via $\mathscr{S}^{\mathbf{p}}$ and using the structure of $\overline{\mathscr{S}}^{\mathbf{p}}$. In the case of $\mathbf{p} = (1, \cdots, 1)$, $\overline{\mathscr{S}}^{\mathbf{p}}$ coincides with $\overline{\mathscr{S}}^0$ in [Saw], and the structure theorem for $\overline{\mathscr{S}}^0$ has been obtained in [SawS]. Thus these results are generalization of results in [SawS] and [Saw]. Note that the structure theorem of $\overline{\mathscr{S}}^0$ in [SawS] needs certain conditions on the parameters, but our structure theorem of $\overline{\mathscr{S}}^{\mathbf{p}}$ does not require such conditions. In §2 and §3, we shall review these results in [SW1].

In [W], the author proved a similar product formula for v-decomposition numbers of $\mathscr{S}$. A v-decomposition number is a v-analogue of the decomposition number defined by using Jantzen filtrations of a Weyl module of $\mathscr{S}$. The product formula for v-decomposition numbers is obtained by showing that the arguments in [SW1] are compatible with a Jantzen filtration of the Weyl module for each algebra. In §4, we shall review these results in [W].

In [SW2], we obtained a product formula for entries of the transition matrix between the canonical basis and the standard basis of the v-deformed Fock space. This formula is regarded as a counter-part for the formula for v-decomposition numbers of $\mathscr{S}$ under the Yvonne's conjecture. After giving a brief review on the Fock space and Yvonne's conjecture in §5, we shall explain our results [SW2] in §6.

In the body of the paper, we survey results in [SW1], [W] and [SW2]. The first and the third are joint work with T. Shoji.

§ 1. Cyclotomic q-Schur algebra associated to the Ariki-Koike algebra

1.1. Let $\mathscr{H} = \mathscr{H}_{n,r}$ be the Ariki-Koike algebra over an integral domain R associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ with parameters $q, Q_1, \cdots, Q_r \in R$ such that q is invertible in R, namely $\mathscr{H}$ is an associative algebra with generators $T_0, T_1, \cdots, T_{n-1}$ and relations

$$
\begin{aligned}
&(T_0 - Q_1)\cdots(T_0 - Q_r) = 0\,, \\
&(T_i - q)(T_i + 1) = 0 && (i \geq 1)\,, \\
&T_0T_1T_0T_1 = T_1T_0T_1T_0\,, \\
&T_iT_j = T_jT_i && (|i-j| \geq 2)\,, \\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (1 \leq i \leq n-2)\,.
\end{aligned}
$$

The subalgebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathscr{H}$ generated by $T_1, \cdots, T_{n-1}$ is isomporphic to the Iwahori-Hecke algebra associated to the symmeteic group $\mathfrak{S}_n$. Let $s_i = (i, i+1)$ for $i = 1, \cdots, n-1$ be adjacent transpositions, then $s_1, \cdots, s_{n-1}$ are the Coxeter generators of $\mathfrak{S}_n$. For $w \in \mathfrak{S}_n$, let $w = s_{i_1}\cdots s_{i_k}$ be a reduced expression of w, and set $T_w = T_{i_1}\cdots T_{i_k}$. Then T_w is determined independent of the choice of reduced expressions. It is known that $\{T_w \,|\, w \in \mathfrak{S}_n\}$ is a basis of $\mathscr{H}(\mathfrak{S}_n)$.

1.2. A composition $\mu = (\mu_1, \mu_2, \cdots)$ is a finite sequence of non-negative integers, the length of the sequence is called the number of parts of μ, and $|\mu| = \sum_i \mu_i$ is called the size of μ. If a composition λ is a weakly decreasing sequence, λ is called a partition. An r-tuple $\mu = (\mu^{(1)}, \cdots, \mu^{(r)})$ of compositions is called an r-composition, and the size $|\mu|$ of μ is defined by $\sum_{i=1}^r |\mu^{(i)}|$. In particular, if all $\mu^{(i)}$ are partitions, μ is called an r-partition. Fix an r-tuple $\mathbf{m} = (m_1, \cdots, m_r) \in \mathbb{Z}_{>0}^r$ such that $m_i \geq n$ for any $i = 1, \cdots, r$. We denote by $\Lambda = \Lambda_{n,r}(\mathbf{m})$ the set of r-compositions $\mu = (\mu^{(1)}, \cdots, \mu^{(r)})$ such that $|\mu| = n$ and that $\mu^{(k)}$ has m_k parts for $k = 1, \cdots, r$. We define $\Lambda^+ = \Lambda^+_{n,r}(\mathbf{m})$ as the subset of Λ consisting of r-partitions . By the condition $m_i \geq n$ for any $i = 1, \cdots, r$, Λ^+ is the set of all r-partitions of size n. We define a partial order, so-called the "dominance order", on Λ by $\mu \trianglerighteq \nu$ if and only if

$$
\sum_{i=1}^{l-1} |\mu^{(i)}| + \sum_{j=1}^{k} \mu_j^{(l)} \geq \sum_{i=1}^{l-1} |\nu^{(i)}| + \sum_{j=1}^{k} \nu_j^{(l)}
$$

for $1 \le l \le r$, $1 \le k \le m_l$. If $\mu \trianglerighteq \nu$ and $\mu \neq \nu$, we write it as $\mu \triangleright \nu$.

For $\mu \in \Lambda$, the diagram of μ is the set

$$[\mu] = \{(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \{1, \cdots, r\} \mid 1 \le j \le \mu_i^{(k)}\}.$$

A standard tableau $\mathfrak{t}$ of shape λ ($\lambda \in \Lambda^+$) is a bijection $\mathfrak{t} : [\lambda] \to \{1, \cdots, n\}$ such that $\mathfrak{t}(i_1, j_1, k) \le \mathfrak{t}(i_2, j_2, k)$ for any $i_1 \le i_2$, $j_1 \le j_2$, $1 \le k \le r$. For $\lambda \in \Lambda^+$, we denote by $\mathrm{Std}(\lambda)$ the set of standard tableaux of shape λ. A semistandard λ-tableau T of type μ ($\lambda \in \Lambda^+, \mu \in \Lambda$) is a map $T : [\lambda] \to \mathbb{N} \times \{1, \cdots, r\}$ such that $\sharp\{(i,j,k) \in [\lambda] \mid T(i,j,k) = (a,l)\} = \mu_a^{(l)}$, $k \le l$ if $T(i,j,k) = (a,l)$ for $(i,j,k) \in [\lambda]$, $T(i_1,j,k) \le T(i_2,j,k)$ for any $i_1 < i_2$, j, k and $T(i,j_1,k) < T(i,j_2,k)$ for any i, $j_1 < j_2$, k, where the order on $\mathbb{N} \times \{1, \cdots, r\}$ is defined by $(a,l) < (a',l')$ if $l < l'$ or if $a < a'$ and $l = l'$. For $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, we denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard λ-tableau of type μ. One can see that $\lambda \trianglerighteq \mu$ if $\mathcal{T}_0(\lambda,\mu) \neq \emptyset$ from the definition. Set $\mathcal{T}_0(\lambda) = \cup_{\mu \in \Lambda} \mathcal{T}_0(\lambda, \mu)$.

1.3. Set $L_1 = T_0$ and $L_i = q^{-1} T_{i-1} L_{i-1} T_{i-1}$ for $i = 2, \cdots, n$. For $\mu \in \Lambda$, set

$$x_\mu = \sum_{w \in \mathfrak{S}_\mu} T_w, \quad u_\mu^+ = \prod_{k=1}^{r} \prod_{i=1}^{a_k} (L_i - Q_k),$$

where $\mathfrak{S}_\mu$ is the Young subgroup of $\mathfrak{S}_n$ corresponding to μ, $a_1 = 0$ and $a_k = \sum_{j=1}^{k-1} |\mu^{(j)}|$. Then u_μ^+ commute with x_μ. Put $m_\mu = u_\mu^+ x_\mu$, and define a right $\mathscr{H}$-module M^μ by $M^\mu = m_\mu \mathscr{H}$. The cyclotomic q-Schur algebra $\mathscr{S}$ is defined by

$$\mathscr{S} = \mathscr{S}(\Lambda) = \mathrm{End}_{\mathscr{H}}\big(\bigoplus_{\mu \in \Lambda} M^\mu\big).$$

By [DJM], it is known that $\mathscr{S}$ is a cellular algebra in the sense of [GL] with a cellular basis

$$\mathcal{C}(\Lambda) = \{\varphi_{ST} \mid S, T \in \mathcal{T}_0(\lambda) \text{ for some } \lambda \in \Lambda^+\},$$

where φ_{ST} is a certain homomorphism of $\mathscr{H}$-modules from M^ν to M^μ if $S \in \mathcal{T}_0(\lambda,\mu), T \in \mathcal{T}_0(\lambda,\nu)$ and is zero on M^τ for $\tau \in \Lambda$ such that $\tau \neq \nu$. In particular, for $\lambda \in \Lambda^+$, $\varphi_{T^\lambda T^\lambda}$ is the identity map on M^λ, where T^λ is the unique element of $\mathcal{T}_0(\lambda,\lambda)$. For $\mu \in \Lambda$, we denote by $\varphi_\mu \in \mathscr{S}$ the identity on M^μ and zero on M^τ for $\tau \in \Lambda$ such that $\tau \neq \mu$. Then we have $\varphi_\lambda = \varphi_{T^\lambda T^\lambda}$ for $\lambda \in \Lambda^+$.

1.4. By a general theory of the cellular algebra, we have the following. For $S, T \in \mathcal{T}_0(\lambda)$ and $\varphi \in \mathscr{S}$, we have

$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in \mathcal{T}_0(\lambda)} r_{T'}^{(T,\varphi)} \varphi_{ST'} \mod \mathscr{S}^{\vee\lambda}, \tag{1.1}$$

where $r_{T'}^{(T,\varphi)} \in R$ does not depend on $S \in \mathcal{T}_0(\lambda)$, and $\mathscr{S}^{\vee\lambda}$ is an R-submodule of $\mathscr{S}$ spanned by $\{\varphi_{UV} \,|\, U, V \in \mathcal{T}(\lambda')$ for some $\lambda' \in \Lambda^+$ such that $\lambda' \rhd \lambda\}$. It is known that $\mathscr{S}^{\vee\lambda}$ is a two-sided ideal of $\mathscr{S}$.

For $\lambda \in \Lambda^+$, let W^λ be a free R-module with a basis $\{\varphi_T \,|\, T \in \mathcal{T}_0(\lambda)\}$, and define a right action of $\mathscr{S}$ on W^λ by

$$\varphi_T \cdot \varphi = \sum_{T' \in \mathcal{T}_0(\lambda)} r_{T'}^{(T,\varphi)} \varphi_{T'} \qquad (T \in \mathcal{T}_0(\lambda),\, \varphi \in \mathscr{S}),$$

where $r_{T'}^{(T,\varphi)} \in R$ are as in (1.1). We can define the bilinear form $\langle\, ,\, \rangle$ on W^λ by

$$\langle \varphi_S, \varphi_T \rangle \varphi_{UV} \equiv \varphi_{US}\varphi_{TV} \mod \mathscr{S}^{\vee\lambda}.$$

Set $\operatorname{rad} W^\lambda = \{x \in W^\lambda \,|\, \langle x, y\rangle = 0$ for any $y \in W^\lambda\}$, then $\operatorname{rad} W^\lambda$ is an $\mathscr{S}$-submodule of W^λ. Thus we can define the quotient $\mathscr{S}$-module $L^\lambda = W^\lambda / \operatorname{rad} W^\lambda$. Since one can easily see that $\langle \varphi_{T^\lambda}, \varphi_{T^\lambda} \rangle = 1$, we have $L^\lambda \neq 0$ for any $\lambda \in \Lambda^+$. Thus we have the followimg theorem.

Theorem 1.5 ([DJM, Theorem 6.16]). *Suppose that R is a field. Then $\{L^\lambda \,|\, \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic (right) simple $\mathscr{S}$-modules.*

§ 2. A parabolic type subalgebra of $\mathscr{S}$ and its quotient algebra

In this section, we shall construct a parabolic type subalgebra $\mathscr{S}^{\mathbf{p}}$ of $\mathscr{S}$ and its quotient algebra $\overline{\mathscr{S}}^{\mathbf{p}}$. These constructions are carried out by using cellular structures of $\mathscr{S}$ only. We shall show that $\overline{\mathscr{S}}^{\mathbf{p}}$ decomposes into a direct sum of tensor products of various cyclotomic q-Schur algebras with smaller rank than the original $\mathscr{S}$. The details of the results in this and next section are given in [SW1].

2.1. We fix $\mathbf{p} = (r_1, \cdots, r_g) \in \mathbb{Z}_{>0}^g$ such that $r_1 + \cdots + r_g = r$. Set $p_k = \sum_{i=1}^{k-1} r_i$ for $k = 1, \cdots, g$ with $p_1 = 0$. For $\mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \in \Lambda$, we define g-tuples $\alpha_{\mathbf{p}}(\mu) = (n_1, \cdots, n_g)$ and $\mathbf{a}_{\mathbf{p}}(\mu) = (a_1, \cdots, a_g)$ by $n_k = \sum_{i=1}^{r_k} |\mu^{(p_k+i)}|$ and $a_k = \sum_{i=1}^{k-1} n_i$ for $k = 1, \cdots, g$ with $a_1 = 0$. We define a partial order on $\mathbb{Z}_{\geq 0}^g$ by $\mathbf{a} = (a_1, \cdots, a_g) \geq \mathbf{b} = (b_1, \cdots, b_g)$ if $a_k \geq b_k$ for $k = 1, \cdots, g$. One can easily see that

(i). $\mathbf{a}_{\mathbf{p}}(\mu) = \mathbf{a}_{\mathbf{p}}(\nu)$ if and only if $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu) \quad (\mu, \nu \in \Lambda)$.

(ii). If $\mu \trianglerighteq \nu$ then $\mathbf{a}_{\mathbf{p}}(\mu) \geq \mathbf{a}_{\mathbf{p}}(\nu) \quad (\mu, \nu \in \Lambda)$. In particular, If $\mathcal{T}_0(\lambda, \mu) \neq \emptyset$ then $\lambda \trianglerighteq \mu$, so that $\mathbf{a}_{\mathbf{p}}(\lambda) \geq \mathbf{a}_{\mathbf{p}}(\mu) \quad (\lambda \in \Lambda^+, \mu \in \Lambda)$.

For $\lambda \in \Lambda^+$, we set

$$\mathcal{T}_0^{\mathbf{p}}(\lambda) = \bigcup_{\substack{\mu \in \Lambda \\ \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)}} \mathcal{T}_0(\lambda, \mu).$$

Moreover, for $(\lambda,\varepsilon)\in\Lambda^+\times\{0,1\}$ set

$$I(\lambda,\varepsilon)=\begin{cases}\mathcal{T}_0^{\mathbf{p}}(\lambda) & \text{if } \varepsilon=0,\\[2ex] \displaystyle\bigcup_{\substack{\mu\in\Lambda\\ \mathbf{a}_{\mathbf{p}}(\lambda)>\mathbf{a}_{\mathbf{p}}(\mu)}}\mathcal{T}_0(\lambda,\mu) & \text{if } \varepsilon=1,\end{cases}\qquad J(\lambda,\varepsilon)=\begin{cases}\mathcal{T}_0^{\mathbf{p}}(\lambda) & \text{if } \varepsilon=0,\\[2ex] \mathcal{T}_0(\lambda) & \text{if } \varepsilon=1.\end{cases}$$

Let $\Sigma^{\mathbf{p}}=(\Lambda^+\times\{0,1\})\setminus\{(\lambda,1)\,|\,I(\lambda,1)=\emptyset\}$, then $I(\lambda,\varepsilon)$ and $J(\lambda,\varepsilon)$ are not empty for any $(\lambda,\varepsilon)\in\Sigma^{\mathbf{p}}$. (Note that $T^\lambda\in\mathcal{T}_0(\lambda,\lambda)\subset\mathcal{T}_0^{\mathbf{p}}(\lambda)\subset\mathcal{T}_0(\lambda)$ for any $\lambda\in\Lambda^+$.) We define a partial order $\geq$ on $\Sigma^{\mathbf{p}}$ by $(\lambda_1,\varepsilon_1)>(\lambda_2,\varepsilon_2)$ if $\lambda_1\rhd\lambda_2$ or if $\lambda_1=\lambda_2$ and $\varepsilon_1>\varepsilon_2$. We set

$$\mathcal{C}^{\mathbf{p}}(\Lambda)=\{\varphi_{ST}\in\mathcal{C}(\Lambda)\,|\,(S,T)\in I(\lambda,\varepsilon)\times J(\lambda,\varepsilon)\text{ for some }(\lambda,\varepsilon)\in\Sigma^{\mathbf{p}}\}.$$

Let $\mathscr{S}^{\mathbf{p}}$ be the R-submodule of $\mathscr{S}$ spanned by $\mathcal{C}^{\mathbf{p}}(\Lambda)$. Then we have the following theorem.

Theorem 2.2 ([SW1, Theorem 2.6]). *$\mathscr{S}^{\mathbf{p}}$ is a subalgebra of $\mathscr{S}$ containing the unit element $1_{\mathscr{S}}$ of $\mathscr{S}$. Moreover, $\mathscr{S}^{\mathbf{p}}$ is a standardly based algebra with standard basis $\mathcal{C}^{\mathbf{p}}(\Lambda)$ in the sense of [DR].*

2.3. A standardly based algebra has a standard basis which has a similar property as a cellular basis, but is different from a cellular algebra in the following points. A cellular algebra has the canonical algebra anti-automorphism. Then the left standard module is obtained from the right standard module by applying the algebra anti-automorphism. But a standardly based algebra does not have such an algebra anti-automorphism. Thus, in the case of standardly based algebras, we need to consider the left standard modules in addition to the right standard modules.

By general theory of standardly based algebras, for each $(\lambda,\varepsilon)\in\Sigma^{\mathbf{p}}$, one can define the right standard $\mathscr{S}^{\mathbf{p}}$-module $Z^{(\lambda,\varepsilon)}$ with an R-free basis $\{\varphi_T^{(\lambda,\varepsilon)}\,|\,T\in J(\lambda,\varepsilon)\}$, and the left standard module ${}^\diamond Z^{(\lambda,\varepsilon)}$ with an R-free basis $\{{}^\diamond\varphi_S^{(\lambda,\varepsilon)}\,|\,S\in I(\lambda,\varepsilon)\}$. There exists a canonical bilinear form $\beta_{(\lambda,\varepsilon)}:{}^\diamond Z^{(\lambda,\varepsilon)}\times Z^{(\lambda,\varepsilon)}\to R$. Set $\operatorname{rad}Z^{(\lambda,\varepsilon)}=\{x\in Z^{(\lambda,\varepsilon)}\,|\,\beta_{(\lambda,\varepsilon)}(y,x)=0\text{ for any }y\in{}^\diamond Z^{(\lambda,\varepsilon)}\}$, then $\operatorname{rad}Z^{(\lambda,\varepsilon)}$ is a $\mathscr{S}^{\mathbf{p}}$-submodule of $Z^{(\lambda,\varepsilon)}$, and $L^{(\lambda,\varepsilon)}=Z^{(\lambda,\varepsilon)}/\operatorname{rad}Z^{(\lambda,\varepsilon)}$ is an absolutely irreducible $\mathscr{S}^{\mathbf{p}}$-module or zero. Note that $L^{(\lambda,0)}\neq 0$ for any $\lambda\in\Lambda^+$ since $T^\lambda\in\mathcal{T}_0^{\mathbf{p}}(\lambda)$ by a similar reason as in the case of W^λ. But we can not see whether $L^{(\lambda,1)}$ is zero or not. By a general theory of standardly based algebras, we have the following corollary.

Corollary 2.4. *Suppose that R is a field. Then $\{L^{(\lambda,\varepsilon)}\neq 0\,|\,(\lambda,\varepsilon)\in\Sigma^{\mathbf{p}}\}$ is a complete set of pairwise non-isomorphic (right) simple $\mathscr{S}^{\mathbf{p}}$-modules.*

2.5. We now construct a quotient algebra $\overline{\mathscr{S}}^{\mathbf{p}}$ of $\mathscr{S}^{\mathbf{p}}$. Let

$$\widehat{\mathcal{C}}^{\mathbf{p}} = \{\varphi_{ST} \,|\, (S,T) \in I(\lambda,1) \times J(\lambda,1) \text{ for some } (\lambda,1) \in \Sigma^{\mathbf{p}}\},$$

and let $\widehat{\mathscr{S}}^{\mathbf{p}}$ be the R-submodule of $\mathscr{S}^{\mathbf{p}}$ spanned by $\widehat{\mathcal{C}}^{\mathbf{p}}$. One can see that $\widehat{\mathscr{S}}^{\mathbf{p}}$ is a two-sided ideal of $\mathscr{S}^{\mathbf{p}}$. Thus we can define a quotient algebra $\overline{\mathscr{S}}^{\mathbf{p}} = \mathscr{S}^{\mathbf{p}}/\widehat{\mathscr{S}}^{\mathbf{p}}$. Clealy, $\overline{\mathscr{S}}^{\mathbf{p}}$ has a free R-basis

$$\overline{\mathcal{C}}(\Lambda) = \{\overline{\varphi}_{ST} \,|\, S,T \in \mathcal{T}_0^{\mathbf{p}}(\lambda) \text{ for some } \lambda \in \Lambda^+\},$$

where $\overline{\varphi}_{ST}$ is the image of $\varphi_{ST} \in \mathscr{S}^{\mathbf{p}}$ under the natural surjection $\mathscr{S}^{\mathbf{p}} \to \overline{\mathscr{S}}^{\mathbf{p}}$. Moreover, we have the following theorem.

Theorem 2.6 ([SW1, Theorem 2.13])**.** *$\overline{\mathscr{S}}^{\mathbf{p}}$ is a cellular algebra with the cellular basis $\overline{\mathcal{C}}(\Lambda)$.*

2.7. By a general theory of cellular algebras, for each $\lambda \in \Lambda^+$, we can define the right standard (cell) $\overline{\mathscr{S}}^{\mathbf{p}}$-module $\overline{Z}^{\lambda}$ with a free R-basis $\{\overline{\varphi}_T \,|\, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)\}$. There exists a canonical bilinear form $\langle\, ,\rangle_{\mathbf{p}} : \overline{Z}^{\lambda} \times \overline{Z}^{\lambda} \to R$. Set $\operatorname{rad} \overline{Z}^{\lambda} = \{\overline{x} \in \overline{Z}^{\lambda} \,|\, \langle \overline{x}, \overline{y} \rangle = 0 \text{ for any } \overline{y} \in \overline{Z}^{\lambda}\}$, then $\operatorname{rad} \overline{Z}^{\lambda}$ is a $\overline{\mathscr{S}}^{\mathbf{p}}$-submodule of $\overline{Z}^{\lambda}$, and $\overline{L}^{\lambda} = \overline{Z}^{\lambda}/\operatorname{rad} \overline{Z}^{\lambda}$ is an absolutely irreducible $\overline{\mathscr{S}}^{\mathbf{p}}$-module or zero. Note that $\overline{L}^{\lambda} \neq 0$ for any $\lambda \in \Lambda^+$ since $\overline{\varphi}_{T^{\lambda}} \in \overline{Z}^{\lambda}$ for any $\lambda \in \Lambda^+$. Thus, we have the following corollary.

Corollary 2.8. *Suppose that R is a field. Then $\{\overline{L}^{\lambda} \,|\, \lambda \in \Lambda^+\}$ is a complete set of pairwise non-isomorphic (right) simple $\overline{\mathscr{S}}^{\mathbf{p}}$-modules.*

2.9. In the next section, we shall discuss the relationship between $\mathscr{S}$ and $\overline{\mathscr{S}}^{\mathbf{p}}$ via $\mathscr{S}^{\mathbf{p}}$. One of the merits for considering $\overline{\mathscr{S}}^{\mathbf{p}}$ is a decomposition of $\overline{\mathscr{S}}^{\mathbf{p}}$ to a direct sum of tensor products of smaller rank cyclotomic q-Schur algebras than the original $\mathscr{S}$. Here, in order to describe this decomposition, we prepare some notations. Recall that $p_k = \sum_{i=1}^{k-1} r_i$ with $p_1 = 0$ for $k = 1, \cdots, g$. For $\mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \in \Lambda$, we set $\mu^{[k]} = (\mu^{(p_k+1)}, \cdots, \mu^{(p_k+r_k)})$. Then we can write $\mu = (\mu^{[1]}, \cdots, \mu^{[g]})$. Similarly, for a semistandard tableau $T = (T^{(1)}, \cdots, T^{(r)}) \in \mathcal{T}_0(\lambda, \mu)$, we write $T = (T^{[1]}, \cdots, T^{[g]})$ with $T^{[k]} = (T^{(p_k+1)}, \cdots, T^{(p_k+r_k)})$.

For $\alpha = (n_1, \cdots, n_g) \in \mathbb{Z}_{\geq 0}^g$ such that $n_1 + \cdots + n_g = n$, we set

$$\Lambda_{n_k} = \{(\mu^{(p_k+1)}, \cdots, \mu^{(p_k+r_k)}) \,|\, \mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \in \Lambda \text{ such that } \alpha_{\mathbf{p}}(\mu) = \alpha\}$$

for $k = 1, \cdots, r$. Set $\mathbf{m}^{[k]} = (m_{p_k+1}, \cdots, m_{p_k+r_k})$. Then $\Lambda_{n_k} = \Lambda_{n_k,r_k}(\mathbf{m}^{[k]})$ is the set of r_k-compositions with size n_k and a length determined by $\mathbf{m}^{[k]}$. Let $\Lambda_{n_k}^+ = \Lambda_{n_k,r_k}^+(\mathbf{m}^{[k]})$ be the set of r_k-partitions contained in Λ_{n_k}. The following lemma is easily varified.

Lemma 2.10. *Let $\alpha = (n_1, \cdots, n_g) \in \mathbb{Z}^g_{\geq 0}$ be such that $n_1 + \cdots + n_g = n$. Then*

(i). The map $\mu \mapsto (\mu^{[1]}, \cdots, \mu^{[g]})$ gives a bijection between $\{\mu \in \Lambda \mid \alpha_{\mathbf{p}}(\mu) = \alpha\}$ and $\Lambda_{n_1} \times \cdots \times \Lambda_{n_g}$.

(ii). The map $\lambda \mapsto (\lambda^{[1]}, \cdots, \lambda^{[g]})$ gives a bijection between $\{\lambda \in \Lambda^+ \mid \alpha_{\mathbf{p}}(\lambda) = \alpha\}$ and $\Lambda^+_{n_1} \times \cdots \times \Lambda^+_{n_g}$.

(iii). For each $\lambda \in \Lambda^+$ and $\mu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, the map $T \mapsto (T^{[1]}, \cdots, T^{[g]})$ gives a bijection between $\mathcal{T}_0(\lambda, \mu)$ and $\mathcal{T}_0(\lambda^{[1]}, \mu^{[1]}) \times \cdots \times \mathcal{T}_0(\lambda^{[g]}, \mu^{[g]})$.

For $\alpha = (n_1, \cdots, n_g) \in \mathbb{Z}^g_{\geq 0}$ such that $n_1 + \cdots + n_g = n$, let $\mathscr{S}(\Lambda_{n_k})$ be the cyclotomic q-Schur algebra with respect to Λ_{n_k} which is associated to the Ariki-Koike algebra $\mathscr{H}_{n_k, r_k}$ with parameters $q, Q_{p_k+1}, \cdots, Q_{p_k+r_k}$. Then $\mathscr{S}(\Lambda_{n_k})$ has a cellular basis $\{\varphi_{S^{[k]}T^{[k]}} \mid S^{[k]}, T^{[k]} \in \mathcal{T}_0(\lambda^{[k]})$ for some $\lambda^{[k]} \in \Lambda^+_{n_k}\}$. We have the following theorem for the structure of $\overline{\mathscr{S}}^{\mathbf{p}}$.

Theorem 2.11 ([SW1, Theorem 4.15])**.** *There exists an isomorphism of R-algebras*

$$\overline{\mathscr{S}}^{\mathbf{p}} \cong \bigoplus_{\substack{(n_1, \cdots, n_g) \in \mathbb{Z}^g_{\geq 0} \\ n_1 + \cdots + n_g = n}} \mathscr{S}(\Lambda_{n_1}) \otimes \cdots \otimes \mathscr{S}(\Lambda_{n_g}), \tag{2.1}$$

where $\overline{\varphi}_{ST}$ is mapped to $\varphi_{S^{[1]}T^{[1]}} \otimes \cdots \otimes \varphi_{S^{[g]}T^{[g]}}$.

For $\lambda^{[k]} \in \Lambda^+_{n_k}$, let $W^{\lambda^{[k]}}$ be the Weyl module, and $L^{\lambda^{[k]}} = W^{\lambda^{[k]}} / \operatorname{rad} W^{\lambda^{[k]}}$ with respect to $\mathscr{S}(\Lambda_{n_k})$. Then we have the following corollary.

Corollary 2.12. *Let $\lambda \in \Lambda^+$. Then under the isomorphism in (2.1), we have the following isomorphisms.*

(i). $\overline{Z}^{\lambda} \cong W^{\lambda^{[1]}} \otimes \cdots \otimes W^{\lambda^{[g]}}$.

(ii). $\overline{L}^{\lambda} \cong L^{\lambda^{[1]}} \otimes \cdots \otimes L^{\lambda^{[1]}}$.

§ 3. Relations among $\mathscr{S}$, $\mathscr{S}^{\mathbf{p}}$ and $\overline{\mathscr{S}}^{\mathbf{p}}$

In this section, we study the relationship among $\mathscr{S}$, $\mathscr{S}^{\mathbf{p}}$ and $\overline{\mathscr{S}}^{\mathbf{p}}$. We shall prove a product formula for the decomposition numbers of $\mathscr{S}$ by using those relations and the decomposition of $\overline{\mathscr{S}}^{\mathbf{p}}$ in Theorem 2.11.

3.1. Summarizing the previous results, we have the following diagram.

$$\begin{array}{l}\mathscr{S}^{\mathbf{p}} \hookrightarrow \mathscr{S}(\Lambda) \\ \downarrow\!\!\downarrow \\ \overline{\mathscr{S}}^{\mathbf{p}} \cong \bigoplus\limits_{\substack{(n_1,\cdots,n_g) \\ n_1+\cdots+n_g=n}} \mathscr{S}(\Lambda_{n_1}) \otimes \cdots \otimes \mathscr{S}(\Lambda_{n_g})\end{array}$$

We consider the decomposition numbers for $\mathscr{S}, \mathscr{S}^{\mathbf{p}}, \overline{\mathscr{S}}^{\mathbf{p}}$ and $\mathscr{S}(\Lambda_{n_k})$ respectively,

$$[W^\lambda : L^\mu]_{\mathscr{S}}, \quad [Z^{(\lambda,0)} : L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}}, \quad [\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathscr{S}}^{\mathbf{p}}}, \quad [W^{\lambda^{[k]}} : L^{\mu^{[k]}}]_{\mathscr{S}(\Lambda_{n_k})}.$$

First note, by Corollary 2.12, that

Lemma 3.2. *Suppose that R is a field. Then, for $\lambda, \mu \in \Lambda^+$, we have*

$$[\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathscr{S}}^{\mathbf{p}}} = \begin{cases} \prod_{k=1}^g [W^{\lambda^{[k]}} : L^{\mu^{[k]}}]_{\mathscr{S}(\Lambda_{n_k})} & \textit{if } \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu) = (n_1, \cdots, n_g), \\ 0 & \textit{otherwise.} \end{cases}$$

Next we compare $\mathscr{S}^{\mathbf{p}}$ and $\overline{\mathscr{S}}^{\mathbf{p}}$. Here $\mathscr{S}^{\mathbf{p}}$ and $\overline{\mathscr{S}}^{\mathbf{p}}$ are constructed by using the cellular basis of $\mathscr{S}$, and the cellular structure of $\mathscr{S}$ induces the standardly based algebra structure on $\mathscr{S}^{\mathbf{p}}$ and the cellular structure on $\overline{\mathscr{S}}^{\mathbf{p}}$. We regard $\overline{\mathscr{S}}^{\mathbf{p}}$-modules as $\mathscr{S}^{\mathbf{p}}$-modules through the natural surjection $\pi : \mathscr{S}^{\mathbf{p}} \to \overline{\mathscr{S}}^{\mathbf{p}}$. Then we have the following lemma.

Lemma 3.3. *For each $\lambda \in \Lambda^+$, we have following isomorphisms of $\overline{\mathscr{S}}^{\mathbf{p}}$-modules.*

(i). $Z^{(\lambda,0)} \xrightarrow{\sim} \overline{Z}^\lambda$, where $\varphi_T^{(\lambda,0)}$ is mapped to $\overline{\varphi}_T$ for $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$.

(ii). $L^{(\lambda,0)} \cong \overline{L}^\lambda$.

This lemma implies that a composition series of $\overline{Z}^\lambda$ as a $\overline{\mathscr{S}}^{\mathbf{p}}$-module coincides with a composition series of $Z^{(\lambda,0)}$ as a $\mathscr{S}^{\mathbf{p}}$-module through the map π when R is a field. Then we have the following corollary.

Corollary 3.4. *Suppose that R is a field. Then, for $\lambda, \mu \in \Lambda^+$, we have*

$$[Z^{(\lambda,0)}; L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}} = [\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathscr{S}}^{\mathbf{p}}}.$$

Finally, we compare $\mathscr{S}^{\mathbf{p}}$ with $\mathscr{S}$. Since $\mathscr{S}^{\mathbf{p}}$ is a subalgebra of $\mathscr{S}$, we can regard $\mathscr{S}$-modules as $\mathscr{S}^{\mathbf{p}}$-modules by the restriction. On the other hand, we can induce up an $\mathscr{S}^{\mathbf{p}}$-module M to the $\mathscr{S}$-module $M \otimes_{\mathscr{S}^{\mathbf{p}}} \mathscr{S}$. We have the following results ([SW1, Lemma 3.5, Proposition 3.6, Lemma 3.9, Lemma 3.10]).

Proposition 3.5. *For each $\lambda \in \Lambda^+$, we have the following.*

(i). There exists an injective $\mathscr{S}^{\mathbf{p}}$-homomorphism $Z^{(\lambda,0)} \to Z^{(\lambda,1)}$ such that $\varphi_T^{(\lambda,0)} \mapsto \varphi_T^{(\lambda,1)}$ $(T \in \mathcal{T}_0^{\mathbf{p}}(\lambda))$.

(ii). There exists an isomorphism of $\mathscr{S}^{\mathbf{p}}$-modules $Z^{(\lambda,1)} \xrightarrow{\sim} W^{\lambda}$ such that $\varphi_T^{(\lambda,1)} \mapsto \varphi_T$ $(T \in \mathcal{T}_0(\lambda))$.

(iii). There exists an isomorphism of $\mathscr{S}$-modules $Z^{(\lambda,0)} \otimes_{\mathscr{S}^{\mathbf{p}}} \mathscr{S} \xrightarrow{\sim} W^{\lambda}$.

(iv). Suppose that R is a field. Then L^{λ} contains $L^{(\lambda,0)}$ as a $\mathscr{S}^{\mathbf{p}}$-submodule.

(v). Suppose that R is a field. Then $L^{(\lambda,0)} \otimes_{\mathscr{S}^{\mathbf{p}}} \mathscr{S}$ has the unique maximal $\mathscr{S}$-submodule $N^{(\lambda,0)}$ such that $L^{(\lambda,0)} \otimes_{\mathscr{S}^{\mathbf{p}}} \mathscr{S}/N^{(\lambda,0)} \cong L^{\lambda}$ as $\mathscr{S}$-modules.

This proposition implies some relations for decomposition numbers between $\mathscr{S}$ and $\mathscr{S}^{\mathbf{p}}$ ([SW1, Prposition 3.11, 3.12]). As a consequence, we have the following proposition.

Proposition 3.6. *Suppose that R is a field. For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$[W^{\lambda} : L^{\mu}]_{\mathscr{S}} = [Z^{(\lambda,0)} : L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}}.$$

Combining the proposition with Lemma 3.2 and Corollary 3.4, we have the following product formula for decomposition numbers of $\mathscr{S}$.

Theorem 3.7 ([SW1, Theorem 4.17]). *Suppose that R is a field. For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu) = (n_1, \cdots, n_g)$, we have*

$$[W^{\lambda} : L^{\mu}]_{\mathscr{S}} = \prod_{k=1}^{g} [W^{\lambda^{[k]}} : L^{\mu^{[k]}}]_{\mathscr{S}(\Lambda_{n_k})}.$$

§ 4. Decomposition numbers with Jantzen filtration

In [W], we obtained a product formula for the v-decomposition numbers which are v-analogue of decomposition numbers with respect to a Jantzen filtration. In this section, we define v-decomposition numbers and explain the product formula. Note that Jantzen filtrations, thus v-decomposition numbers also, depend on a choice of discrete valuation ring R and parameters in R (see below). However, our results in this section hold for any choice of discrete valuation ring R and parameters in R.

4.1. Let R be a discrete valuation ring with the unique maximal ideal $\wp$ and $F = R/\wp$ be the residue field. Fix $\widehat{q}, \widehat{Q}_1, \cdots, \widehat{Q}_r \in R$ such that $\widehat{q}$ is invertible in R and let $q = \widehat{q}+\wp, Q_1 = \widehat{Q}_1+\wp, \cdots, Q_r = \widehat{Q}_r+\wp$ be their canonical image in F. Let $\mathscr{S}_R = \mathscr{S}_R(\Lambda)$ be the cyclotomic $\widehat{q}$-Schur algebra over R with parameters $\widehat{q}, \widehat{Q}_1, \cdots, \widehat{Q}_r$ and $\mathscr{S} = \mathscr{S}(\Lambda)$

be the cyclotomic q-Schur algebra over F with parameters $q, Q_1, \cdots, Q_r$. Then we have $\mathscr{S} \cong \mathscr{S}_R \otimes_R F \cong \mathscr{S}_R/\wp\mathscr{S}_R$.

We consider the subalgebra $\mathscr{S}_R^{\mathbf{p}}$ (resp. $\mathscr{S}^{\mathbf{p}}$) of $\mathscr{S}_R$ (resp. $\mathscr{S}$) and its quotient $\overline{\mathscr{S}}_R^{\mathbf{p}}$ (resp. $\overline{\mathscr{S}}^{\mathbf{p}}$) as in the previous section. We also use various notations in the previous section with subscripts R for objects over R if we need the distinction.

4.2. For $\lambda \in \Lambda^+$ and $i \in \mathbb{Z}_{\geq 0}$, we set

$$W_R^\lambda(i) = \{x \in W_R^\lambda \mid \langle x, y\rangle \in \wp^i \text{ for any } y \in W_R^\lambda\},$$

and define

$$W^\lambda(i) = (W_R^\lambda(i) + \wp W_R^\lambda)/\wp W_R^\lambda.$$

Then $W^\lambda(i)$ is an $\mathscr{S}$-submodule of W^λ and we have a filtration of W^λ

$$W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset W^\lambda(2) \cdots.$$

We call this filtration the **Jantzen filtration** associated with R and $(\widehat{q}, \widehat{Q}_1, \cdots, \widehat{Q}_r)$.

Similarly, for the $\mathscr{S}^{\mathbf{p}}$-module $Z^{(\lambda,0)}$ and the $\overline{\mathscr{S}}^{\mathbf{p}}$-module $\overline{Z}^\lambda$, by using the bilinear form $\beta_{(\lambda,0)} : {}^\diamond Z_R^{(\lambda,0)} \times Z_R^{(\lambda,0)} \to R$ and $\langle\, ,\, \rangle_{\mathbf{p}} : \overline{Z}_R^\lambda \times \overline{Z}_R^\lambda \to R$ respectively, and reduction modulo $\wp$, we can define Jantzen filtrations of $Z^{(\lambda,0)}$ and $\overline{Z}^\lambda$

$$Z^{(\lambda,0)} = Z^{(\lambda,0)}(0) \supset Z^{(\lambda,0)}(1) \supset Z^{(\lambda,0)}(2) \supset \cdots,$$

$$\overline{Z}^\lambda = \overline{Z}^\lambda(0) \supset \overline{Z}^\lambda(1) \supset \overline{Z}^\lambda(2) \supset \cdots.$$

Since W^λ (resp. $Z^{(\lambda,0)}, \overline{Z}^\lambda$) is a finite dimensional F-vector space, in the Jantzen filtration of W^λ (resp. $Z^{(\lambda,0)}, \overline{Z}^\lambda$), all but finitely many inclusions are equalities.

Let v be an indeterminate. For $\lambda, \mu \in \Lambda^+$, we define $d_{\lambda\mu}(v)$, $d_{\lambda\mu}^{(\lambda,0)}(v)$, $\overline{d}_{\lambda,\mu}(v) \in \mathbb{Z}[v]$ by

$$\begin{aligned}
d_{\lambda\mu}(v) &= \sum_{i\geq 0} [W^\lambda(i)/W^\lambda(i+1) : L^\mu]_{\mathscr{S}} \cdot v^i,\\
d_{\lambda\mu}^{(\lambda,0)}(v) &= \sum_{i\geq 0} [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}} \cdot v^i,\\
\overline{d}_{\lambda\mu}(v) &= \sum_{i\geq 0} [\overline{Z}^\lambda(i)/\overline{Z}^\lambda(i+1) : \overline{L}^\mu]_{\overline{\mathscr{S}}^{\mathbf{p}}} \cdot v^i.
\end{aligned}$$

We call them **v-decomposition numbers**. Note that when we specialize v to 1, we have $d_{\lambda\mu}(1) = [W^\lambda : L^\mu]_{\mathscr{S}}$, $d_{\lambda,\mu}^{(\lambda,0)}(1) = [Z^{(\lambda,0)} : L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}}$ and $\overline{d}_{\lambda\mu}(1) = [\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathscr{S}}^{\mathbf{p}}}$.

For W^λ, $Z^{(\lambda,0)}$ and $\overline{Z}^\lambda$, by comparing bilinear forms, we have the following lemma which is a refined version of Lemma 3.3 and Proposition 3.5.

Lemma 4.3 ([W, Proposition 2.3, Lemma 2.5, Lemma 2.6]).
For each $\lambda \in \Lambda^+$, we have the following.

(i). Under the isomorphism $Z^{(\lambda,0)} \cong \overline{Z}^{\lambda}$ as $\mathscr{S}^{\mathbf{p}}$-modules, we have $Z^{(\lambda,0)}(i) \cong \overline{Z}^{\lambda}(i)$ for any $i \in \mathbb{Z}_{\geq 0}$.

(ii). Let $f_\lambda : Z^{(\lambda,0)} \to W^{\lambda}$ be the injective $\mathscr{S}^{\mathbf{p}}$-homomorphism given by Proposition 3.5 (i) and (ii). Then we have $f_\lambda^{-1}(W^{\lambda}(i)) = Z^{(\lambda,0)}(i)$ for any $i \in \mathbb{Z}_{\geq 0}$.

(iii). For $i \in \mathbb{Z}_{\geq 0}$, let $\iota_i : Z^{(\lambda,0)}(i) \to Z^{(\lambda,0)}$ be the inclusion map. Then, under the isomorphism $Z^{(\lambda,0)} \otimes_{\mathscr{S}^{\mathbf{p}}} \mathscr{S} \cong W^{\lambda}$ as $\mathscr{S}$-modules, we have $(\iota_i \otimes \mathrm{id}_{\mathscr{S}})(Z^{(\lambda,0)}(i) \otimes_{\mathscr{S}^{\mathbf{p}}} \mathscr{S}) \subset W^{\lambda}(i)$.

This lemma implies analogues of Lemma 3.4 and Proposition 3.6, namely we have the following proposition.

Proposition 4.4. *For $\lambda, \mu \in \Lambda^+$ and $i \in \mathbb{Z}_{\geq 0}$, we have*

(i). $[Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}} = [\overline{Z}^{\lambda}(i)/\overline{Z}^{\lambda}(i+1) : \overline{L}^{\mu}]_{\overline{\mathscr{S}}^{\mathbf{p}}}$.

(ii). $[W^{\lambda}(i)/W^{\lambda}(i+1) : L^{\mu}]_{\mathscr{S}} = [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}]_{\mathscr{S}^{\mathbf{p}}}$ if $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$.

In particular, if $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ then we have

$$d_{\lambda\mu}(v) = d^{(\lambda,0)}_{\lambda\mu}(v) = \overline{d}_{\lambda\mu}(v).$$

In order to obtain the product formula for v-decomposition numbers, we need a v-analogue of Lemma 3.2.

By using a certain basis of $\overline{Z}^{\lambda}_R(i)$ and basis of $W_R^{\lambda^{[k]}}(i_k)$ $(i, i_k \in \mathbb{Z}_{\geq 0}, k = 1, \cdots, g)$ and reduction modulo $\wp$, we can prove the following lemma.

Lemma 4.5 ([W, Proposition 2.11]). *Let $\lambda \in \Lambda^+$ and $i \in \mathbb{Z}_{\geq 0}$. Under the isomorphism $\overline{Z}^{\lambda} \cong W^{\lambda^{[1]}} \otimes \cdots \otimes W^{\lambda^{[g]}}$ in Corollary 2.12, we have*

$$\overline{Z}^{\lambda}(i) = \sum_{\substack{(i_1,\cdots,i_g) \\ i_1+\cdots+i_g=i}} W^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W^{\lambda^{[g]}}(i_g).$$

This lemma implies the following proposition.

Proposition 4.6. *For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu) = (n_1, \cdots, n_g)$ and $i \in \mathbb{Z}_{\geq 0}$, we have*

$$[\overline{Z}^{\lambda}(i)/\overline{Z}^{\lambda}(i+1) : \overline{L}^{\lambda}]_{\overline{\mathscr{S}}^{\mathbf{p}}} = \sum_{\substack{(i_1,\cdots,i_g) \\ i_1+\cdots+i_g=i}} \prod_{k=1}^{g} [W^{\lambda^{[k]}}(i_k)/W^{\lambda^{[k]}}(i_k+1) : L^{\mu^{[k]}}]_{\mathscr{S}(\Lambda_{n_k})}.$$

Although this proposition is not a v-analogue of Lemma 3.2 directly, this plays a similar role of Lemma 3.2 in the v-analogue setting.

We denote by $d_{\lambda^{[k]}\mu^{[k]}}(v)$ v-decomposition numbers of $\mathscr{S}(\Lambda_{n_k})$ for $k = 1, \cdots, g$. Then, Proposition 4.4 and Proposition 4.6 imply the following theorem.

Theorem 4.7 ([W, Theorem 2.14]). *For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$d_{\lambda\mu}(v) = \overline{d}_{\lambda\mu}(v) = \prod_{k=1}^{g} d_{\lambda^{[k]}\mu^{[k]}}(v).$$

§ 5. Yvonne's conjecture

Let $\mathbf{F}_v[\boldsymbol{s}]$ be the v-deformed Fock space with a multi-charge $\boldsymbol{s}$. Uglov constructed in [U] the canonical bases of $\mathbf{F}_v[\boldsymbol{s}]$. Then Yvonne proposed a conjecture that v-decomposition numbers for $\mathscr{S}$ are described by Uglov's canonical bases of $\mathbf{F}_v[\boldsymbol{s}]$. Here we review some of them.

5.1. Let $\Pi^r = \{\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)})\}$ be the set of r-partitions. We fix $\boldsymbol{s} = (s_1, \cdots, s_r) \in \mathbb{Z}^r$ and call it a multi-charge. Let $\mathbf{F}_v[\boldsymbol{s}]$ be the v-deformed Fock space of level r with multi-charge $\boldsymbol{s}$, namely $\mathbf{F}_v[\boldsymbol{s}]$ is a vector space over $\mathbb{C}(v)$ with a basis $\{|\lambda, \boldsymbol{s}\rangle \mid \lambda \in \Pi^r\}$ and the quantum group $U_v(\widehat{\mathfrak{sl}}_e)$ of type $A^{(1)}_{e-1}$ acts on $\mathbf{F}_v[\boldsymbol{s}]$. For the definition of this action, see [U].

For $\lambda \in \Pi^r$, $\boldsymbol{s} = (s_1, \cdots, s_r)$, and $M \in \mathbb{Z}$, we say that $|\lambda, \boldsymbol{s}\rangle$ is **M-dominant** if $s_i - s_{i+1} > M + |\lambda|$ for $i = 1, \cdots, r$.

In [U], Uglov defined a bar-involution $\overline{} : \mathbf{F}_v[\boldsymbol{s}] \to \mathbf{F}_v[\boldsymbol{s}]$, $x \mapsto \overline{x}$, which is semi-linear with respect to the involution on $\mathbb{C}(v)$ given by $v \mapsto v^{-1}$, and commutes with the action of $U_v(\widehat{\mathfrak{sl}}_e)$, i.e. $\overline{u \cdot x} = \overline{u} \cdot \overline{x}$ for $u \in U_v(\widehat{\mathfrak{sl}}_e)$, $x \in \mathbf{F}_v[\boldsymbol{s}]$ (here $\overline{u}$ is the usual bar-involution on $U_v(\widehat{\mathfrak{sl}}_e)$). Moreover, Uglov constructed the following basis of $\mathbf{F}_v[\boldsymbol{s}]$ which is now called the Uglov canonical basis.

Proposition 5.2 ([U, Proposition 4.11]). *There exists a unique basis $\{\mathcal{G}^+(\lambda, \boldsymbol{s}) \mid \lambda \in \Pi^r\}$ of $\mathbf{F}_v[\boldsymbol{s}]$ satisfying the following properties;*

(i). $\overline{\mathcal{G}^+(\lambda, \boldsymbol{s})} = \mathcal{G}^+(\lambda, \boldsymbol{s})$,

(ii). $\mathcal{G}^+(\lambda, \boldsymbol{s}) \equiv |\lambda, \boldsymbol{s}\rangle \mod v\mathcal{L}^+$,

where $\mathcal{L}^+$ is the $\mathbb{C}[v]$-lattice of $\mathbf{F}_v[\boldsymbol{s}]$ generated by $\{|\lambda, \boldsymbol{s}\rangle \mid \lambda \in \Pi^r\}$.

5.3. We define $\Delta^+_{\lambda\mu}(v) \in \mathbb{C}[v]$, for $\lambda, \mu \in \Pi^r$, by

$$\mathcal{G}^+(\lambda, \boldsymbol{s}) = \sum_{\mu \in \Pi^r} \Delta^+_{\lambda\mu}(v)\, |\mu, \boldsymbol{s}\rangle. \tag{5.1}$$

By [U], it is known that $\Delta^+_{\lambda\mu}(v) = 0$ unless $|\lambda| = |\mu|$. Moreover, $\Delta^+_{\lambda\mu}(v)$ can be interpreted by parabolic Kazhdan-Lusztig polynomials of an affine Weyl group.

5.4. In order to describe the Yvonne's conjecture, we consider the cyclotomic q-Schur algebra $\mathscr{S}(\Lambda)$ over $\mathbb{C}$ with parameters $(q; Q_1, \cdots, Q_r) = (\xi; \xi^{s_1}, \cdots, \xi^{s_r})$, where $\xi = \exp(2\pi i/e) \in \mathbb{C}$ and $\boldsymbol{s} = (s_1, \cdots, s_r)$ is a multi-charge. For $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)}) \in \Pi^r$, we define an r-partition $\lambda^\dagger$ by

$$\lambda^\dagger = ((\lambda^{(r)})', (\lambda^{(r-1)})', \cdots, (\lambda^{(1)})'),$$

where $(\lambda^{(i)})'$ denotes the dual partition of the partition $\lambda^{(i)}$. Let $d_{\lambda\mu}(v) \in \mathbb{Z}[v]$ be the v-decomposition number defined in 4.2 associated with a suitable valuation ring R and parameters in R (see [Y, 2.2] for details). In [Y], Yvonne gave the following conjecture.

Conjecture Suppose that $|\lambda, \boldsymbol{s}\rangle$ is 0-dominant. Then we have

$$d_{\lambda\mu}(v) = \Delta^+_{\mu^\dagger\lambda^\dagger}(v). \tag{5.2}$$

In view of this conjecture, one can expect that the counter-part of Theorem 4.7 will also hold for the Fock space. In the next section, we shall discuss the product formula for $\Delta^+_{\lambda\mu}(v)$.

Remark 1. In the case where $r = 1$ (namely $\mathscr{S}$ is a q-Schur algebra associated to the Iwahori-Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$), the formula (5.2) at $v = 1$ is proved by Valagnolo-Vasserot [VV], namely the following formula holds.

$$d_{\lambda\mu}(1) = \Delta^+_{\mu'\lambda'}(1). \tag{5.3}$$

This formula had been conjectured by Leclerc-Thibon in [LT] as a generalization of the LLT conjecture [LLT] for Iwahori-Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$. For v-decomposition numbers, the conjecture is still open even in the case where $r = 1$. For $r \geq 2$, even the formula (5.2) at $v = 1$ is not yet proved.

§ 6. Tensor products of the Fock spaces

In [SW2], we have proved the product formula for $\Delta^+_{\lambda\mu}(v)$ by decomposing the Fock space $\mathbf{F}_v[\boldsymbol{s}]$ into a tensor product $\mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \cdots \otimes \mathbf{F}_v[\boldsymbol{s}^{[g]}]$, where each $\mathbf{F}_v[\boldsymbol{s}^{[k]}]$ is the Fock space with smaller level than the original Fock space. The idea of the proof is the following. By careful calculations of the image of the standard basis under the bar-involution, we compare two bases, one is the original canonical basis of $\mathbf{F}_v[\boldsymbol{s}]$ and another is a tensor product of canonical basis of $\mathbf{F}_v[\boldsymbol{s}^{[k]}]$ $(k = 1, \cdots, g)$. This induces the product formula for $\Delta^+_{\lambda\mu}(v)$. Here we review some of them. In the last part, we also give a remark on the relation with the quasi R-matrix. Actually, our first approach for the product formula of $\Delta^+_{\lambda\mu}(v)$ was to make use of the quasi R-matrix. But it did not work well.

6.1. We fix $\mathbf{p} = (r_1, \cdots, r_g)$ as in Section 2. Set $\boldsymbol{s}^{[k]} = (s_{p_k+1}, \cdots, s_{p_k+r_k})$ for $k = 1, \cdots, g$. Thus we can write $\boldsymbol{s} = (\boldsymbol{s}^{[1]}, \cdots, \boldsymbol{s}^{[g]})$. For $\lambda \in \Pi^r$, we express it as $(\lambda^{[1]}, \cdots, \lambda^{[g]})$ as in 2.9. For $\lambda, \mu \in \Pi^r$, we denote $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ if $|\lambda^{[k]}| = |\mu^{[k]}|$ for $k = 1, \cdots, g$ (this implies $|\lambda| = |\mu|$).

6.2. Let $\mathbf{F}_v[\boldsymbol{s}^{[k]}]$ be the v-deformed Fock space with multi-charge $\boldsymbol{s}^{[k]}$ and with a basis $\{|\lambda^{[k]}, \boldsymbol{s}^{[k]}\rangle \mid \lambda^{[k]} \in \Pi^{r_k}\}$. By proposition 5.2, $\mathbf{F}_v[\boldsymbol{s}^{[k]}]$ has the canonical basis $\{\mathcal{G}^+(\lambda^{[k]}, \boldsymbol{s}^{[k]}) \mid \lambda^{[k]} \in \Pi^{r_k}\}$. Put

$$\mathcal{G}^+(\lambda^{[k]}, \boldsymbol{s}^{[k]}) = \sum_{\mu^{[k]} \in \Pi^{r_k}} \Delta^+_{\lambda^{[k]}\mu^{[k]}}(v)\, |\mu^{[k]}, \boldsymbol{s}^{[k]}\rangle \tag{6.1}$$

with $\Delta^+_{\lambda^{[k]}\mu^{[k]}}(v) \in \mathbb{C}[v]$.

We have an isomorphism as vector spaces

$$\Phi : \mathbf{F}_v[\boldsymbol{s}] \xrightarrow{\sim} \mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \cdots \otimes \mathbf{F}_v[\boldsymbol{s}^{[g]}]$$

such that $|\lambda, \boldsymbol{s}\rangle \mapsto |\lambda^{[1]}, \boldsymbol{s}^{[1]}\rangle \otimes \cdots \otimes |\lambda^{[g]}, \boldsymbol{s}^{[g]}\rangle$. Under this isomorphism,

$$\left\{G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) = \mathcal{G}^+(\lambda^{[1]}, \boldsymbol{s}^{[1]}) \otimes \cdots \otimes \mathcal{G}^+(\lambda^{[g]}, \boldsymbol{s}^{[g]}) \mid \lambda \in \Pi^r\right\}$$

is also a basis of $\mathbf{F}_v[\boldsymbol{s}]$.

Note that $\Delta^+_{\lambda^{[k]}\mu^{[k]}}(v) = 0$ unless $|\lambda^{[k]}| = |\mu^{[k]}|$, then we have

$$G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) = \sum_{\mu \in \Pi^r} \left(\prod_{k=1}^{g} \Delta^+_{\lambda^{[k]}\mu^{[k]}}(v) \right) |\mu, \boldsymbol{s}\rangle. \tag{6.2}$$

Note that if $|\lambda, \boldsymbol{s}\rangle$ is M-dominant then $|\lambda^{[k]}, \boldsymbol{s}^{[k]}\rangle$ is also M-dominant for $k = 1, \cdots, g$, and that

$$\prod_{k=1}^{g} \Delta^+_{\lambda^{[k]}\mu^{[k]}}(v) = 0 \text{ unless } \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu). \tag{6.3}$$

Remark 2. Let $\overline{} \otimes \cdots \otimes \overline{}$ be the bar-involution on $\mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \cdots \otimes \mathbf{F}_v[\boldsymbol{s}^{[g]}]$ with $x^{[1]} \otimes \cdots \otimes x^{[g]} \mapsto \overline{x^{[1]}} \otimes \cdots \otimes \overline{x^{[g]}}$, where $\overline{x^{[k]}}$ is the bar-involution on $\mathbf{F}_v[\boldsymbol{s}^{[k]}]$ defined in [U]. This bar-involution $\overline{} \otimes \cdots \otimes \overline{}$ commute with the action of $U_v(\widehat{\mathfrak{sl}}_e) \otimes \cdots \otimes U_v(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \cdots \otimes \mathbf{F}_v[\boldsymbol{s}^{[g]}]$, but it does not commute with the action of $U_v(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_v[\boldsymbol{s}]$ under the isomorphism Φ.

For $\lambda \in \Lambda^+$, $G^+_{\mathbf{p}}(\lambda, \boldsymbol{s})$ is not invariant under the bar-involution $\overline{}$ on $\mathbf{F}_v[\boldsymbol{s}]$ though it is invariant under the bar-involution $\overline{} \otimes \cdots \otimes \overline{}$ on $\mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \cdots \otimes \mathbf{F}_v[\boldsymbol{s}^{[g]}]$. Thus $G^+_{\mathbf{p}}(\lambda, \boldsymbol{s})$ does not coincide with $\mathcal{G}^+(\lambda, \boldsymbol{s})$.

In [SW2], by comparing $\{\mathcal{G}^+(\lambda, \boldsymbol{s}) \mid \lambda \in \Pi^r\}$ with $\{G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) \mid \lambda \in \Pi^r\}$, we proved the following product formula for $\Delta^+_{\lambda\mu}(v)$.

Theorem 6.3 ([SW2, Theorem 2.9]). *For $\lambda, \mu \in \Pi^r$ such that $|\lambda, \boldsymbol{s}\rangle$ is M-dominant for $M > 2|\lambda|$, and that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$\Delta^+_{\lambda\mu}(v) = \prod_{k=1}^{g} \Delta^+_{\lambda^{[k]}\mu^{[k]}}(v). \tag{6.4}$$

We consider the special case $\mathbf{p} = (1, \cdots, 1)$. In this case, the right-hand side of the formula in Theorem 3.7 is a product of decomposition numbers of the q-Schur algebra of type A. On the other hand, the right-hand side of the formula in Theorem 6.3 is a product of coefficients for the Fock space with level 1. Thus by applying the result of Varagnolo-Vasserot (5.3) to those products, we have the following corollary.

Corollary 6.4 ([SW2, Corollary 2.10]). *Let $\lambda, \mu \in \Lambda^+$ be such that $|\lambda^{(i)}| = |\mu^{(i)}|$ for $i = 1, \cdots, r$, and suppose that $|\lambda, \boldsymbol{s}\rangle$ is M-dominant for $M > 2|\lambda|$. Then we have*

$$d_{\lambda\mu}(1) = [W^\lambda : L^\mu]_{\mathscr{S}} = \Delta^+_{\mu^\dagger\lambda^\dagger}(1).$$

Remark 3. We remark that the product formulas, Theorem 4.7 and Theorem 6.3, are compatible with Yvonne's conjecture (5.2). In fact, by applying (5.2) on both sides of Theorem 4.7, we have

$$\Delta^+_{\mu^\dagger\lambda^\dagger}(v) = \prod_{k=1}^{g} \Delta^+_{(\mu^{[k]})^\dagger(\lambda^{[k]})^\dagger}(v) \quad \text{if } \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}(\mu)}. \tag{6.5}$$

As the following argument shows, this is equivalent to (6.4). (Note that this is not straightforward since $(\lambda^{[k]})^\dagger \neq (\lambda^\dagger)^{[k]}$ in general.) Let $\widetilde{\mathbf{p}} = (\widetilde{r}_1, \cdots, \widetilde{r}_g) = (r_g, \cdots, r_2, r_1)$ and write λ as $\lambda = (\widetilde{\lambda}^{[1]}, \cdots, \widetilde{\lambda}^{[g]})$ with respect to $\widetilde{\mathbf{p}}$ in a similar way as in the case of $\mathbf{p}$. Then by applying Theorem 6.3 for $\widetilde{\mathbf{p}}$, we have

$$\Delta^+_{\mu^\dagger\lambda^\dagger}(v) = \prod_{k=1}^{g} \Delta^+_{(\widetilde{\mu^\dagger})^{[k]}(\widetilde{\lambda^\dagger})^{[k]}}(v) \quad \text{if } \alpha_{\widetilde{\mathbf{p}}}(\lambda^\dagger) = \alpha_{\widetilde{\mathbf{p}}}(\mu^\dagger), \tag{6.6}$$

But $\alpha_{\widetilde{\mathbf{p}}}(\lambda^\dagger) = \alpha_{\widetilde{\mathbf{p}}}(\mu^\dagger)$ if and only if $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, and we have

$$(\lambda^{[k]})^\dagger = (\widetilde{\lambda^\dagger})^{[g-k+1]} \quad (k = 1, \cdots, g),$$

and similar for μ. Hence (6.5) follows from (6.6).

Example For $r=3$, $\mathbf{p}=(2,1)$, $\lambda=((2,1),(1,1),(2,2))\in\Lambda^+$, we have $\lambda^\dagger=((2,2),(2),(2,1))$. Thus $\lambda^{[1]}=((2,1),(1,1))$, $\lambda^{[2]}=(2,2)$, $(\lambda^{[1]})^\dagger=((2),(2,1))$, $(\lambda^{[2]})^\dagger=(2,2)$, $(\lambda^\dagger)^{[1]}=((2,2),(2))$ and $(\lambda^\dagger)^{[2]}=(2,1)$. On the other hand, we have $\widetilde{\mathbf{p}}=(1,2)$ $\widetilde{(\lambda^\dagger)}^{[1]}=(2,2)$ and $\widetilde{(\lambda^\dagger)}^{[2]}=((2),(2,1))$.

6.5. In the remainder of this paper, we give a rough sketch of the proof of Theorem 6.3. In order to show the formula (6.4), we have only to see the case where $\mathbf{p}=(r_1,r_2)$, since we can obtain the formula (6.4) for the general case by inductive arguments from the case of $\mathbf{p}=(r_1,r_2)$. Thus, from now on, we assume that $\mathbf{p}=(r_1,r_2)$.

In [U], the Fock space $\mathbf{F}_v[\boldsymbol{s}]$ is realized as the subspace of a semi-infinite wedge product $\Lambda^{s+\frac{\infty}{2}}$ such that $s=s_1+\cdots+s_r$ (for definition, see [U, 4.1]), and the bar-involution $\overline{}$ on $\mathbf{F}_v[\boldsymbol{s}]$ is defined as the restriction of the bar-involution on $\Lambda^{s+\frac{\infty}{2}}$. Let $\mathbf{P}(s)$ be the set of semi-infinite sequence $\mathbf{k}=(k_1,k_2,\cdots)\in\mathbb{Z}^\infty$ such that $k_i=s-i+1$ for all sufficient large i. Then, $\Lambda^{s+\frac{\infty}{2}}$ is spanned by a set of semi-infinite wedges $\{u_{\mathbf{k}}=u_{k_1}\wedge u_{k_2}\wedge\cdots\mid\mathbf{k}\in\mathbf{P}(s)\}$ (see [U, Section 3 and 4] for the definition of semi-infinite wedges). For $\mathbf{k}=(k_1,k_2,\cdots)\in\mathbf{P}(s)$, we call $u_{\mathbf{k}}$ an ordered wedge if $k_1>k_2>\cdots$, and call $u_{\mathbf{k}}$ an unordered wedge otherwise. It is known that $\Lambda^{s+\frac{\infty}{2}}$ has a basis consisting of ordered semi-infinite wedges which correspond to r-partitions with a various charge $\boldsymbol{s}$ such that $s_1+\cdots+s_r=s$. The bar-involution on $\Lambda^{s+\frac{\infty}{2}}$ maps an ordered wedge to an unordered wedge. On the other hand, one can represent an unordered wedge as a linear combination of ordered wedges by using the straightening law given in [U, Proposition 3.16]. Thus, the bar-involution on $\Lambda^{s+\frac{\infty}{2}}$ can be described by the straightening law for wedges. This straightening law is compatible with the decomposition of the Fock space $\mathbf{F}_v[\boldsymbol{s}]$. Then careful calculations, by making use of the straightening law, leads to the following proposition.

Proposition 6.6 ([SW2, Proposition 2.13]). *Suppose that $|\lambda,\boldsymbol{s}\rangle$ is M-dominant for $M>2|\lambda|$. Under the isomorphism $\Phi:\mathbf{F}_v[\boldsymbol{s}]\xrightarrow{\sim}\mathbf{F}_v[\boldsymbol{s}^{[1]}]\otimes\mathbf{F}_v[\boldsymbol{s}^{[2]}]$, we have*

$$\overline{|\lambda,\boldsymbol{s}\rangle}=\overline{|\lambda^{[1]},\boldsymbol{s}^{[1]}\rangle}\otimes\overline{|\lambda^{[2]},\boldsymbol{s}^{[2]}\rangle}+\sum_{\substack{\mu\in\Pi^r\\ \mathbf{a}_{\mathbf{p}}(\lambda)>\mathbf{a}_{\mathbf{p}}(\mu)}}\alpha_{\lambda\mu}\,|\mu,\boldsymbol{s}\rangle$$

with $\alpha_{\lambda\mu}\in\mathbb{C}[v,v^{-1}]$.

This proposition combined with (6.2) implies,

$$\overline{G^+_{\mathbf{p}}(\lambda,\boldsymbol{s})}=G^+_{\mathbf{p}}(\lambda,\boldsymbol{s})+\sum_{\substack{\mu\in\Pi^r\\ \mathbf{a}_{\mathbf{p}}(\lambda)>\mathbf{a}_{\mathbf{p}}(\mu)}}b'_{\lambda\mu}\,G^+_{\mathbf{p}}(\mu,\boldsymbol{s}) \tag{6.7}$$

with $b'_{\lambda\mu} \in \mathbb{C}[v, v^{-1}]$. Let

$$\overline{G^+_{\mathbf{p}}(\lambda, \boldsymbol{s})} = \sum_{\substack{\mu \in \Pi^r \\ |\mu| = |\lambda|}} R_{\lambda\mu} G^+_{\mathbf{p}}(\mu, \boldsymbol{s}) \tag{6.8}$$

with $R_{\lambda\mu} \in \mathbb{C}[v, v^{-1}]$. Then, by (6.7), the matrix $(R_{\lambda\mu})_{|\lambda|=|\mu|}$ is unitriangular with respect to the order which is compatible with $\mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)$. Thus, by a standard argument for constructing the canonical bases, we have the following theorem.

Theorem 6.7 ([SW2, Theorem 2.15])**.** *Suppose that $|\lambda, \boldsymbol{s}\rangle$ is M-dominant for $M > 2|\lambda|$. Then we have*

$$\mathcal{G}^+(\lambda, \boldsymbol{s}) = G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) + \sum_{\substack{\mu \in \Pi^r \\ \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)}} b_{\lambda\mu} G^+_{\mathbf{p}}(\mu, \boldsymbol{s})$$

with $b_{\lambda\mu} \in v\mathbb{C}[v]$.

We now expand both sides of the formula in Theorem 6.7 to linear combinations of the standard basis $\{|\mu, \boldsymbol{s}\rangle \mid \mu \in \Pi^r\}$ as in (5.1) and (6.2) respectively. Then by comparing coefficients of $|\mu, \boldsymbol{s}\rangle$ of both sides, we have (6.4) for the case $\mathbf{p} = (r_1, r_2)$. Thus, Theorem 6.3 is proved.

Remark 4. Recall that $\mathbf{F}_v[\boldsymbol{s}] \cong \mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \mathbf{F}_v[\boldsymbol{s}^{[2]}]$ and $G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) = \mathcal{G}^+(\lambda^{[1]}\boldsymbol{s}^{[1]}) \otimes \mathcal{G}^+(\lambda^{[2]}\boldsymbol{s}^{[2]})$ in the case of $\mathbf{p} = (r_1, r_2)$. Here we discuss the connection of the quasi R-matrix with Theorem 6.3.

One can define the action of $U_v(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_v[\boldsymbol{s}]$ by the composite of the action of $U_v(\widehat{\mathfrak{sl}}_e) \otimes U_v(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_v[\boldsymbol{s}^{[1]}] \otimes \mathbf{F}_v[\boldsymbol{s}^{[2]}]$ and the coproduct Δ of $U_v(\widehat{\mathfrak{sl}}_e)$ defined in [U, 3.5]. In fact, if $|\lambda, \boldsymbol{s}\rangle$ is 0-dominant then this action on $\mathbf{F}_v[\boldsymbol{s}]$ coincides with the Uglov's action. Let Θ be the quasi R-matrix. (For definition, see [Lu, Ch.4]. Note that the coproduct in [Lu] is different from ours.) Then $\psi = (\overline{} \otimes \overline{}) \circ \Theta$ is a bar-involution on $\mathbf{F}_v[\boldsymbol{s}]$ such that ψ commutes with the action of $U_v(\widehat{\mathfrak{sl}}_e)$ on $\mathbf{F}_v[\boldsymbol{s}]$ defined by using the coproduct Δ. Thus, from the definition of Θ and the actions of $U_v(\widehat{\mathfrak{sl}}_e)$, one can see, for $\lambda \in \Pi^r$, that

$$\psi\big(G^+_{\mathbf{p}}(\lambda, \boldsymbol{s})\big) = G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) + \sum_{\substack{\mu \in \Pi^r \\ \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)}} \widetilde{b}'_{\lambda\mu} G^+_{\mathbf{p}}(\mu, \boldsymbol{s}) \tag{6.9}$$

with $\widetilde{b}'_{\lambda\mu} \in \mathbb{C}[v, v^{-1}]$. Then, by a standard argument for constructing the canonical bases, one can see that there exists a unique basis $\{\widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s}) \mid \lambda \in \Pi^r\}$ of $\mathbf{F}_v[\boldsymbol{s}]$ satisfying the following properties;

(i). $\psi\big(\widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s})\big) = \widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s})$, (ii). $\widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s}) \equiv |\lambda, \boldsymbol{s}\rangle \mod v\mathcal{L}^+$.

Moreover, we have

$$\widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s}) = G^+_{\mathbf{p}}(\lambda, \boldsymbol{s}) + \sum_{\substack{\mu\in\Pi^r \\ \mathbf{a_p}(\lambda)>\mathbf{a_p}(\mu)}} \widetilde{b}_{\lambda\mu}\, G^+_{\mathbf{p}}(\mu, \boldsymbol{s}) \tag{6.10}$$

with $\widetilde{b}_{\lambda\mu} \in v\mathbb{C}[v]$. In fact, by using the standard argument, for constructing canonical bases, we obtain (i) and (6.10). The property (ii) is obtained from (6.10).

Set

$$\widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s}) = \sum_{\mu\in\Pi^r} \widetilde{\varDelta}^+_{\lambda\mu}(v)|\mu, \boldsymbol{s}\rangle$$

with $\widetilde{\varDelta}^+_{\lambda\mu}(v) \in \mathbb{C}[v]$. Then, by a similar argument as in the case of $\varDelta^+_{\lambda\mu}(v)$, (6.10) implies that, for $\lambda, \mu \in \Pi^r$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$,

$$\widetilde{\varDelta}^+_{\lambda\mu}(v) = \varDelta^+_{\lambda^{[1]}\mu^{[1]}}(v) \cdot \varDelta^+_{\lambda^{[2]}\mu^{[2]}}(v). \tag{6.11}$$

But we note that $\widetilde{\mathcal{G}}^+(\lambda, \boldsymbol{s}) \neq \mathcal{G}^+(\lambda, \boldsymbol{s})$ in general since there exists $\mu \in \Pi^r$ such that $\psi(|\mu, \boldsymbol{s}\rangle) \neq \overline{|\mu, \boldsymbol{s}\rangle}$. So we can not show that $\widetilde{\varDelta}^+_{\lambda\mu}(v) = \varDelta^+_{\lambda\mu}(v)$, and so Theorem 6.3 is not obtained from the argument by using the quasi R-matrix.

But, as a consequence of Theorem 6.3, we see that $\varDelta^+_{\lambda\mu}(v) = \widetilde{\varDelta}^+_{\lambda\mu}(v)$ for $\lambda, \mu \in \Pi^r$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ and that $|\lambda, \boldsymbol{s}\rangle$ is M-dominant for $M > 2|\lambda|$. In fact, the right-hand side of the formula (6.11) coincides with the right-hand side of the formula (6.4) in the case of $\mathbf{p} = (r_1, r_2)$.

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