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Catalan numbers and level 2 weight structures of $A^{(1)}_{p-1}$

By

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Abstract

Motivated by a connection between representation theory of the degenerate affine Hecke algebra of type A and Lie theory associated with $A^{(1)}_{p-1}$, we determine the complete set of representatives of the orbits for the Weyl group action on the set of weights of level 2 integrable highest weight representations of $\widehat{\mathfrak{sl}}_p = \mathfrak{g}(A^{(1)}_{p-1})$. Applying a crystal technique, we show that Catalan numbers appear in their weight multiplicities.

§ 1. Introduction

Let p be a prime number and let F be an algebraically closed field of characteristic p and let $(A^{(1)}_{p-1}, \Pi = \{\alpha_i\}_{0\le i<p}, \Pi^\vee, \mathcal{P}, \mathcal{P}^\vee)$ be the Cartan datum and let $W = W(A^{(1)}_{p-1})$ be the Weyl group. For each positive integral weight $\Lambda \in \mathcal{P}^+$ and $n \ge 0$, let us consider $\mathcal{H}^\Lambda_n$, the cyclotomic degenerate affine Hecke algebra of type A [Kle, Chapter 7.3]. The following gives a motivation in this paper.

Theorem 1.1 ([Kle, Theorem 9.5.1,Corollary 9.6.2]). *As $\widehat{\mathfrak{sl}}_p$-module, we have*

$$\bigoplus_{n\ge 0} \mathsf{K}_0(\mathcal{H}^\Lambda_n\text{-}mod) \otimes_{\mathbb{Z}} \mathbb{C} \cong L(\Lambda).$$

Further, under this isomorpshim, the weight space decomposition of $L(\Lambda)$ corresponds to the block decomposition of $\{\mathcal{H}^\Lambda_n\}_{n\ge 0}$.

Here $\mathsf{K}_0(\mathcal{C})$ stands for the Grothendieck group of an abelian category $\mathcal{C}$, and we omit the definition of the action of $\widehat{\mathfrak{sl}}_p$ on the LHS (for the detail, see [Kle] and the references therein). Thus, if two weights share a property coming from Lie theory,

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we expect that the corresponding blocks share some properties. A famous example is Chuang-Rouquier's $\mathfrak{sl}_2$-categorification asserts that if two weights μ_1, μ_2 of $L(\Lambda)$ are in the same W-orbit, then the corresponding blocks are derived equivalent [CR].

Motivated by this, we are interested in $P(\Lambda)/W$ where $P(\Lambda) = \{\mu \in \mathfrak{h}^* \mid L(\Lambda)_\mu \neq 0\}$ is the set of weights of $L(\Lambda)$. $P(\Lambda)$ is described as follows:

Proposition 1.2 ([Kac, Chapter 12.6]). *Let $\Lambda \in \mathcal{P}^+$ be positive level k over an affine algebra. We have*

$$P(\Lambda) = \bigsqcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \mid n \geq 0\}$$

where $\max(\Lambda)$ *is the set of all maximal weights of $L(\Lambda)$ defined as follows.*

$$\max(\lambda) = \{\lambda \in P(\Lambda) \mid \lambda + \delta \notin P(\Lambda)\}.$$

Because $\max(\Lambda)$ is clearly W-invariant (i.e., $\max(\Lambda) = W \cdot (\max(\Lambda) \cap \mathcal{P}^+)$), we are interested in $\max(\Lambda) \cap \mathcal{P}^+$ and it is described as follows:

Proposition 1.3 ([Kac, Proposition 12.6]). *Let $\Lambda \in \mathcal{P}^+$ be positive level k over an affine algebra. The map $\lambda \mapsto \overline{\lambda}$ defines a bijection from* $\max(\Lambda) \cap \mathcal{P}^+$ *onto $kC_{af} \cap (\overline{\Lambda} + \overline{\mathcal{Q}})$. In particular, the set of dominant maximal weights of $L(\Lambda)$ is finite (For the necessary notations, see [Kac]).*

It is well-known that $\max(\Lambda_0) \cap \mathcal{P}^+ = \{\Lambda_0\}$, hence, we deal with the next non-trivial case, i.e., level 2 case. The following is the main result of this paper.

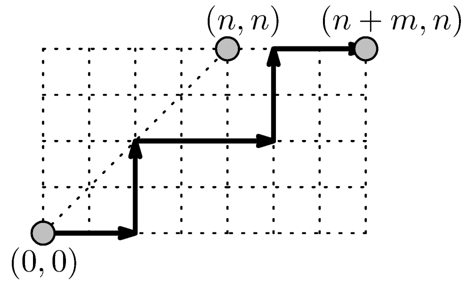
Theorem 1.4. *Let $p \geq 2$ be an integer and consider a level 2 weight $\Lambda = \Lambda_0 + \Lambda_s$ of $\widehat{\mathfrak{sl}}_p$ for some $0 \leq s < p$. The set of all dominant maximal weights* $\max(\Lambda) \cap \mathcal{P}^+$ *and their multiplicities are described as follows.*

(i) $\max(\Lambda) \cap \mathcal{P}^+ = \{\Lambda\} \sqcup \{\lambda^s_l \mid 1 \leq l \leq \lfloor \frac{p-s}{2} \rfloor\} \sqcup \{\mu^s_l \mid 1 \leq l \leq \lfloor \frac{s}{2} \rfloor\}$, *where*

$$\begin{cases} \lambda^s_l = \Lambda - l\alpha_0 - \begin{pmatrix} l\alpha_1 + \cdots l\alpha_s \\ +(l-1)\alpha_{s+1} + (l-2)\alpha_{s+2} + \cdots + \alpha_{l+s-1} \\ +\alpha_{p-l+1} + \cdots + (l-2)\alpha_{p-2} + (l-1)\alpha_{p-1} \end{pmatrix}, \\ \mu^s_l = \Lambda - l\alpha_0 - \begin{pmatrix} (l-1)\alpha_1 + (l-2)\alpha_2 + \cdots + \alpha_{l-1} \\ +\alpha_{s-l+1} + \cdots + (l-2)\alpha_{s-2} + (l-1)\alpha_{s-1} \\ +l\alpha_s + \cdots l\alpha_{p-1} \end{pmatrix}. \end{cases}$$

(ii) $\mathsf{mult}\,\lambda_l^s = \mathsf{D}_{l,s}$, $\mathsf{mult}\,\mu_l^s = \mathsf{D}_{l,p-s}$.

Here $\mathsf{D}_{n,m}$ is defined as the number of lattice paths from $(0,0)$ to $(n+m,n)$ with steps $(1,0)$ and $(0,1)$ that does not exceed the diagonal $y = x$.



Note that $\mathsf{D}_{n,0}$ is the usual Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, hence we have $\mathsf{mult}\,\lambda_l^0 = C_l$. Applying the reflection principle of André [Sta, Solutions 6.20.a], we have

$$\mathsf{D}_{n,m} = \binom{2n+m}{n} - \binom{2n+m}{n-1} = \frac{m+1}{n+m+1}\binom{2n+m}{n}.$$

We remark that our proof of Theorem 1.4 (i) is only a calculation along with Proposition 1.2 and Proposition 1.3, hence contains nothing new. However, our proof of Theorem 1.4 (ii) uses a recently proved result [AKT, Theorem 9.5] on $U_q(\widehat{sl}_p)$-crystals, which combinatorially characterize the connected component (usually called Kleshchev bipartition in the representation theoretic context) $B(\Lambda_0 + \Lambda_s) \subseteq B(\Lambda_0) \otimes B(\Lambda_s)$ in the tensor product.

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§ 2. Some auxiliary inequalities

Definition 2.1. For $l \geq 1$, we define T_l and U_l as follows.

$$\begin{cases} T_l = \{\boldsymbol{x} = {}^t(x_1, \cdots, x_l) \in \mathbb{Z}^l_{\geq 0} \mid A_l\boldsymbol{x} \geq \mathbf{0} \text{ and } x_1 = x_l = 1\} \\ U_l = \{{}^t(1, 2, \cdots, p-1, p^{\langle l+2-2p\rangle}, p-1, \cdots, 2, 1) \mid 1 \leq p \leq \lfloor (l+1)/2 \rfloor\}. \end{cases}$$

Note that $A_l = (2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j})_{1\leq i,j\leq l}$ is the $l \times l$ Cartan matrix of type A and $p^{\langle l+2-2p\rangle}$ is an abbreviation of $\underbrace{p, \cdots, p}_{l+2-2p}$.

Lemma 2.2. *If $\boldsymbol{x} = {}^t(x_1, \cdots, x_l) \in T_l$, then we have $x_k \geq 1$ for all $1 \leq k \leq l$.*

Proof. Suppose to the contrary, that there exist some $\boldsymbol{x} \in T_l$ and $1 \leq k \leq l$ such that $x_k = 0$. We denote by k_0 the minimum among such k. Note that $1 < k_0 < l$. Now we have the following contradiction.

$$(A_l\boldsymbol{x})_{k_0} = -x_{k_0-1} + 2x_{k_0} - x_{k_0+1} = -x_{k_0-1} - x_{k_0+1} \leq -x_{k_0-1} \leq -1.$$

$\square$

Proposition 2.3. *We have $T_l = U_l$ for all $l \geq 1$.*

Proof. By direct calculation, it is easily checked that $U_l \subseteq T_l$. Thus, it is enough to show by induction on l that $T_l \subseteq U_l$. The case $l = 1, 2$ follows from $T_1 = \{(1)\}$ and $T_2 = \{{}^t(1, 1)\}$. Let us assume $l \geq 3$. If $x_2 = 1$, then $(A_l\boldsymbol{x})_2 = -x_1 + 2x_2 - x_3 \geq 0$ and Lemma 2.2 implies $x_3 = 1$. By repeating this, we have $x_1 = \cdots = x_l = 1$. This is the case $p = 1$. Now we may assume that $x_2 = x_{l-1} = 2$ because

$$\begin{cases} (A_l\boldsymbol{x})_1 = 2x_1 - x_2 = 2 - x_2 \geq 0 \\ (A_l\boldsymbol{x})_{l-1} = -x_{l-1} + 2x_l = 2 - x_{l-1} \geq 0 \end{cases}$$

and $x_2 = 1 \Leftrightarrow x_{l-1} = 1$. Note that we have $A_l\mathbf{1}_l = {}^t(1, 0^{\langle l-2\rangle}, 1)$ for $\mathbf{1}_l \stackrel{\text{def}}{=} {}^t(1^{\langle l\rangle})$. This means that for $\boldsymbol{y} = \boldsymbol{x} - \mathbf{1}_l$, we have $(A_l\boldsymbol{y})_k \geq 0$ for all $2 \leq k \leq l-1$, i.e., we have $(A_l\boldsymbol{y})_k = (A_{l-2}\tilde{\boldsymbol{y}})_{k-1}$ for all $2 \leq k \leq l-1$ where $\tilde{\boldsymbol{y}} = {}^t(x_2 - 1, \cdots, x_{l-1} - 1)$. By Lemma 2.2 we have $\tilde{\boldsymbol{y}} \in \mathbb{Z}_{\geq 0}^{l-2}$, thus $\tilde{\boldsymbol{y}} \in T_{l-2}$. By the induction hypothesis, there exists some $1 \leq p \leq \lfloor (l-1)/2 \rfloor$ such that $\tilde{\boldsymbol{y}} = {}^t(1, 2, \cdots, p-1, p^{\langle l-2p\rangle}, p-1, \cdots, 2, 1)$. Therefore we have $\boldsymbol{x} = \boldsymbol{y} + \mathbf{1}_l = {}^t(1, 2, \cdots, p, p+1^{\langle l-2p\rangle}, p, \cdots, 2, 1)$. $\square$

Definition 2.4. We say that $\boldsymbol{y} = {}^t(y_1, \cdots, y_l) \in \mathbb{Z}^l$ is *almost non-negative* iff there exists $1 \leq i \leq l$ such that $y_i \geq -1$ and $y_j \geq 0$ for all $1 \leq j \neq i \leq l$.

Proposition 2.5. *Suppose $A_l\boldsymbol{x}$ is almost non-negative for $\boldsymbol{x} \in \mathbb{Z}^l$ and $l \geq 3$, then we have the following 2 logical implications for all $1 \leq k \leq l-2$.*

$$\begin{aligned} \big(P(\boldsymbol{x}, k) \text{ and } Q(\boldsymbol{x}, k)\big) &\Longrightarrow P(\boldsymbol{x}, k+1) \ (\text{and } Q(\boldsymbol{x}, k+1)) \\ R(\boldsymbol{x}, k) &\Longrightarrow R(\boldsymbol{x}, k+1) \text{ or } \big(P(\boldsymbol{x}, k+1) \text{ and } Q(\boldsymbol{x}, k+1)\big), \end{aligned}$$

where $P(\boldsymbol{x}, k), Q(\boldsymbol{x}, k)$ and $R(\boldsymbol{x}, k)$ are statements defined by

$$\begin{aligned} P(\boldsymbol{x}, k) = \text{TRUE} &\stackrel{\text{def}}{\Longleftrightarrow} x_{k+1} \leq x_k \leq -1 \\ Q(\boldsymbol{x}, k) = \text{TRUE} &\stackrel{\text{def}}{\Longleftrightarrow} 1 \leq \exists p \leq k, (A_l\boldsymbol{x})_p = -1 \\ R(\boldsymbol{x}, k) = \text{TRUE} &\stackrel{\text{def}}{\Longleftrightarrow} x_{k+1} < x_k \leq 0. \end{aligned}$$

Proof. First let us assume $P(\boldsymbol{x}, k)$ and $Q(\boldsymbol{x}, k)$. Since $A_l\boldsymbol{x}$ is almost non-negative, we have $(A_l\boldsymbol{x})_{k+1} = -x_k + 2x_{k+1} - x_{k+2} \geq 0$. Hence we have

$$x_{k+2} \leq -x_k + 2x_{k+1} = x_{k+1} + (x_{k+1} - x_k) \leq -1.$$

This implies $P(\boldsymbol{x}, k+1)$. Now assume $R(\boldsymbol{x}, k)$. If $(A_l\boldsymbol{x})_{k+1} \geq 0$, then we have

$$x_{k+2} \leq -x_k + 2x_{k+1} = (x_{k+1} - x_k) + x_{k+1} < x_{k+1}.$$

Thus, we have the implication $R(\boldsymbol{x}, k) \Rightarrow R(\boldsymbol{x}, k+1)$. If $(A_l\boldsymbol{x})_{k+1} = -1$, then

$$x_{k+2} \leq 1 - x_k + 2x_{k+1} = 1 + (x_{k+1} - x_k) + x_{k+1} \leq x_{k+1} (< x_k \leq 0).$$

Thus, we have the implication $R(\boldsymbol{x}, k) \Rightarrow (P(\boldsymbol{x}, k+1)$ and $Q(\boldsymbol{x}, k+1))$. □

Corollary 2.6. *If $A_l\boldsymbol{x}$ is almost non-negative for $\boldsymbol{x} \in \mathbb{Z}^l$ and $l \geq 2$, then $x_1 \geq 0$.*

Proof. Suppose that $x_1 \leq -1$. We need to consider the following 2 cases.

case 1. $(A_l\boldsymbol{x})_1 \geq 0$**:** Since $x_2 \leq -2$, we have $R(\boldsymbol{x}, 1)$.

case 2. $(A_l\boldsymbol{x})_1 = -1$**:** Since $x_2 \leq -1$, we have $P(\boldsymbol{x}, 1)$ and $Q(\boldsymbol{x}, 1)$.

In either case, we have the following contradiction by Proposition 2.5.

case $R(\boldsymbol{x}, l-1)$**:** We have $(A_l\boldsymbol{x})_l = -x_{l-1} + 2x_l \leq -2$.

case $P(\boldsymbol{x}, l-1)$ **and** $Q(\boldsymbol{x}, l-1)$**:** We have $(A_l\boldsymbol{x})_l = -x_{l-1} + 2x_l \leq -1$ and $Q(\boldsymbol{x}, l-1)$.

□

Corollary 2.7. *Suppose that $A_l\boldsymbol{x}$ is almost non-negative for $\boldsymbol{x} \in \mathbb{Z}^l, x_1 = 0$ and $l \geq 2$ and further assume that there exists some $1 \leq k < l$ such that $x_{k+1} \neq 0$. We denote by k_0 the minimum among such k. Then we have $(A_l\boldsymbol{x})_{k_0} = -1$.*

Proof. Suppose to the contrary that we have

$$0 \leq (A_l\boldsymbol{x})_{k_0} = \begin{cases} 2x_1 - x_2 & (k_0 = 1) \\ -x_{k_0-1} + 2x_{k_0} - x_{k_0+1} & (1 < k_0 < l), \end{cases}$$

then we have $x_{k_0+1} < 0$ by the choice of k_0. This contradicts Corrollary 2.6. □

§ 3. Proof of Theorem 1.4 (i)

In the following, we denote by $\{\beta_k \mid 1 \leq k < p\}$ and $\{t_k \mid 1 \leq k < p\}$ the simple root system and the simple coroot system of the underlying Lie algebra $\overline{\mathfrak{g}}$ respectively where $\mathfrak{g} = \widehat{\mathfrak{sl}}_p$. We denote by θ the highest root of $\overline{\mathfrak{g}}$, i.e., $\theta = \beta_1 + \cdots + \beta_{p-1}$. We refer one more necessary fact from [Kac].

Proposition 3.1 ([Kac, Proposition 12.5.(a)]). *Let $L(\Lambda)$ be an integrable module of positive level k over an affine algebra. Then*

$$\mathcal{P}(\Lambda) = W \cdot \{\lambda \in \mathcal{P}^+ \mid \lambda \leq \Lambda\}.$$

§ 3.1. Proof of Theorem 1.4 (i) : case $s = 0$

By the Proposition 1.3, $\max(\Lambda) \cap \mathcal{P}^+$ is bijective to $2C_{\mathrm{af}} \cap \overline{\mathcal{Q}}$. Note that

$$2C_{\mathrm{af}} \cap \overline{\mathcal{Q}} \cong \{\lambda = \sum_{k=1}^{p-1} x_k\beta_k \mid \lambda(t_k) \geq 0 \text{ for all } 1 \leq k < p \text{ and } (\lambda|\theta) \leq 2\}.$$

It is easy to see that for $\lambda = \sum_{k=1}^{p-1} x_k\beta_k$, the condition of RHS is equivalent to

$$\begin{cases} \lambda(t_1) = 2x_1 - x_2 & \geq 0 \\ \lambda(t_2) = -x_1 + 2x_2 - x_3 & \geq 0 \\ \qquad \vdots & \vdots \\ \lambda(t_{p-2}) = -x_{p-3} + 2x_{p-2} - x_{p-1} & \geq 0 \\ \lambda(t_{p-1}) = -x_{p-2} + 2x_{p-1} & \geq 0 \\ (\lambda|\theta) = x_1 + x_{p-1} & \leq 2. \end{cases}$$

$\lambda(t_k) \geq 0$ for all $1 \leq k < p$ implies $x_k \geq 0$ for all $1 \leq k < p$ because A_{p-1} is finite type. Therefore $(\lambda|\theta) \leq 2$ implies $(x_1, x_{p-1}) = (0,0), (0,1), (1,0), (1,1)$. We easily have $x_1 = 0 \Leftrightarrow x_{p-1} = 0$ and in this case $x_k = 0$ for all $1 \leq k < p$. Therefore, we have to consider the remaing case $(x_1, x_{p-1}) = (1,1)$. By definition, we have ${}^t(x_1, \cdots, x_{p-1}) \in T_{p-1}$.

If $\lambda \overset{\text{def}}{=} \Lambda + \sum_{k=0}^{p-1} q_k\alpha_k \in \max(\Lambda) \cap \mathcal{P}^+$ corresponds to $\overline{\lambda} = \sum_{k=1}^{p-1} x_k\beta_k \in 2C_{\mathrm{af}} \cap \overline{\mathcal{Q}}$ by the map in Proposition 1.3 where $q_k \in \mathbb{Z}_{\leq 0}$, then we have $x_k = q_k - q_0$ for all $1 \leq k < p$ since we have $\overline{\alpha_0} = -(\beta_1 + \cdots + \beta_{p-1})$ and for all $0 < m < p$ we have $\overline{\alpha_m} = \beta_m$. Here we need to consider the following 2 cases.

case 1. $x_k = 0$ **for all** $1 \leq k < p$**:** It is equivalently saying that we have $q_k = q_0$ for all $0 \leq k < p$. Since $\Lambda \in \max(\Lambda)$ and the basic null root of $\mathfrak{g}$ is $\delta = \alpha_0 + \cdots + \alpha_{p-1}$, we have $q_0 = 0$ by Proposition 1.2, i.e., $\lambda = \Lambda$.

case 2. ${}^t(x_1, \cdots, x_{p-1}) \in T_{p-1}$**:** Then there exists $1 \leq l \leq \lfloor p/2 \rfloor$ such that

$${}^t(q_1, \cdots, q_{p-1}) = {}^t(1 + q_0, \cdots, l - 1 + q_0, (l + q_0)^{\langle p+1-2l \rangle}, l - 1 + q_0, \cdots, 1 + q_0),$$

by Proposition 2.3. Because $q_k \leq 0$ for all $1 \leq k < p$, we have $q_0 = -l - r$ for some $r \in \mathbb{Z}_{\geq 0}$. Hence, we have $\lambda = \widetilde{\lambda} - r\delta$ where

$$\widetilde{\lambda} = \Lambda - l\alpha_0 - \begin{pmatrix} (l-1)\alpha_1 + (l-2)\alpha_2 + \cdots + \alpha_{l-1} \\ + \\ \alpha_{p+1-l} + \cdots + (l-2)\alpha_{p-2} + (l-1)\alpha_{p-1} \end{pmatrix}.$$

It is enough to show that in this case we have $r = 0$. Suppose to the contrary, we assume $r \geq 1$. Note that $\lambda + \delta \leq \Lambda$ and $\lambda + \delta \in \mathcal{P}^+$. Therefore, by Proposition 3.1, we have $\lambda + \delta \in \mathcal{P}(\Lambda)$, which is a contradiction to $\lambda \in \max(\Lambda)$.

§ 3.2. Proof of Theorem 1.4 (i) : case $s \neq 0$

By Proposition 1.3, $\max(\Lambda) \cap \mathcal{P}^+$ is bijective to $2C_{\mathrm{af}} \cap (\overline{\Lambda_s} + \overline{\mathcal{Q}})$. Note that

$$2C_{\mathrm{af}} \cap (\overline{\Lambda_s} + \overline{\mathcal{Q}}) \cong \{\lambda = \overline{\Lambda_s} + \sum_{k=1}^{p-1} x_k\beta_k \mid \lambda(t_k) \geq 0 \text{ for all } 1 \leq k < p \text{ and } (\lambda|\theta) \leq 2\}.$$

It is easy to see that for $\lambda = \overline{\Lambda_s} + \sum_{k=1}^{p-1} x_k\beta_k$, the condition of RHS is equivalent to

$$\begin{cases} \lambda(t_1) = 2x_1 - x_2 & \geq 0 \\ \lambda(t_2) = -x_1 + 2x_2 - x_3 & \geq 0 \\ \qquad \vdots & \vdots \\ \lambda(t_{s-1}) = -x_{s-2} + 2x_{s-1} - x_s & \geq 0 \\ \lambda(t_s) = 1 - x_{s-1} + 2x_s - x_{s+1} & \geq 0 \\ \lambda(t_{s+1}) = -x_s + 2x_{s+1} - x_{s+2} & \geq 0 \\ \qquad \vdots & \vdots \\ \lambda(t_{p-2}) = -x_{p-3} + 2x_{p-2} - x_{p-1} & \geq 0 \\ \lambda(t_{p-1}) = -x_{p-2} + 2x_{p-1} & \geq 0 \\ (\lambda|\theta) = 1 + x_1 + x_{p-1} & \leq 2. \end{cases}$$

If $p = 2$, then the above is

$$\begin{cases} \lambda(t_1) = 1 + 2x_1 \geq 0 \\ (\lambda|\theta) = 1 + 2x_1 \leq 2. \end{cases}$$

Thus we have $x_1 = 0$, which implies Theorem 1.4.

Therefore we may assume $p \geq 3$. Note that $A_{p-1}\boldsymbol{x}$ is almost non-negative where $\boldsymbol{x} = {}^t(x_1, \cdots, x_{p-1}) \in \mathbb{Z}^{p-1}$, hence $x_1 \geq 0$ by Corollary 2.6, and $x_{p-1} \geq 0$ by symmetry. Therefore, there are 3 possible pairs $(x_1, x_{p-1}) = (0,0), (1,0), (0,1)$

It is easy to see that, if $(x_1, x_{p-1}) = (0,0)$, then we have $x_i = 0$ for all $1 \leq i < p$. Now let us assume that $(x_1, x_{p-1}) = (0,1)$. In this case, we have

$$x_1 = \cdots = x_s = 0, x_{s+1} \neq 0, -x_{s-1} + 2x_s - x_{s+1} = -1$$

by Corollary 2.7. Thus we have $x_{s+1} = x_{p-1} = 1$, i.e., ${}^t(x_{s+1}, \cdots, x_{p-1}) \in T_{p-s-1}$. This contributes to $\{\lambda^s_l \mid 1 \leq l \leq \lfloor \frac{p-s}{2} \rfloor\}$ as in the proof of $s = 0$. Apply the same argument for $(x_1, x_l) = (1, 0)$, we see that this contributes to $\{\mu^s_l \mid 1 \leq l \leq \lfloor \frac{s}{2} \rfloor\}$.

§ 4. Proof of Theorem 1.4 (ii)

We apply crystal theory to prove Theorem 1.4 (ii), i.e., we show the following.

$$\begin{cases} \mathsf{D}_{l,s} = \#\{\lambda \otimes \mu \in B(\Lambda_0) \otimes B(\Lambda_s) \mid \lambda \otimes \mu \in B(\Lambda_0 + \Lambda_s), \mathrm{wt}(\lambda \otimes \mu) = \lambda^s_l\}, \\ \mathsf{D}_{l,p-s} = \#\{\lambda \otimes \mu \in B(\Lambda_0) \otimes B(\Lambda_s) \mid \lambda \otimes \mu \in B(\Lambda_0 + \Lambda_s), \mathrm{wt}(\lambda \otimes \mu) = \mu^s_l\}. \end{cases}$$

Here $B(\Lambda_0 + \Lambda_s)$ stands for the naturally embedded one in $B(\Lambda_0) \otimes B(\Lambda_s)$.

We adapt Misra-Miwa realization [MM] for $U_q(\widehat{sl}_p)$-crystal $B(\Lambda_m)$ for $0 \leq m < p$. We need not know the details of this realization such as the definition of Kashiwara operator. All we need to know is the following basic things and a recently proved result [AKT, Theorem 9.5].

- The underlying set of $B(\Lambda_m)$ is the set of all p-restricted partitions.
- For each $\lambda \in B(\Lambda_m)$ and each box $x = (i,j) \in \lambda$ (this means x is the box inside λ located at i-th row and j-th column), x has the quantity $\mathsf{Res}(x) = m - i + j \pmod{p\mathbb{Z}} (\in \mathbb{Z}/p\mathbb{Z})$, called the residue of x.
- For each $\lambda \in B(\Lambda_m)$,

$$\mathrm{wt}(x) = \Lambda_m - \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \#\{x \in \lambda \mid \mathsf{Res}(x) = i\} \cdot \alpha_i.$$

Theorem 4.1 ([AKT, Theorem 9.5]). *Let $\lambda \in B(\Lambda_0), \mu \in B(\Lambda_m)$. Then $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$ if and only if $\tau_m(\mathsf{base}(\lambda)) \supseteq \mathsf{roof}(\mu)$.*

Here base, τ_m [AKT] and roof [KLMW] are explicit combinatorially defined maps

$$\begin{cases} \mathsf{base}, \mathsf{roof} : \{p\text{-restricted partition}\} \longrightarrow \{p\text{-core partition}\} \\ \tau_m : \{p\text{-core partition}\} \longrightarrow \{p\text{-core partition}\} \end{cases}$$

and $\lambda' \supseteq \mu'$ means that λ' contains μ' as Young diagrams. We need not know the precise definitions of maps $\mathsf{base}, \mathsf{roof}$ and τ_m, however we need the following minimum.

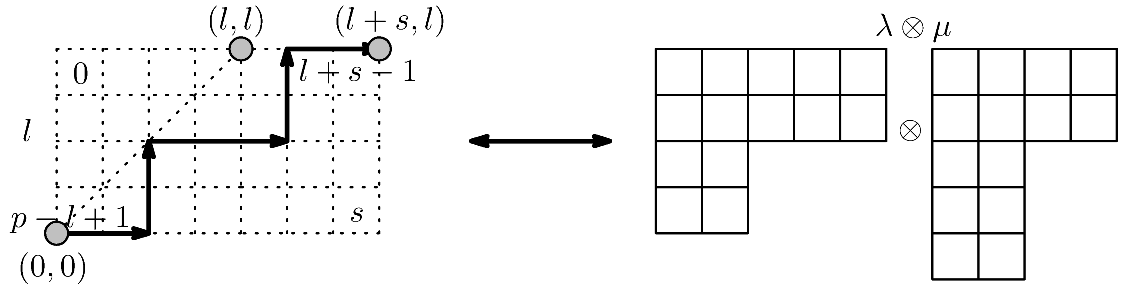
- For a p-core partition λ, we have $\lambda = \mathsf{base}(\lambda) = \mathsf{ceil}(\lambda)$ [AKT, Definition 2.5,2.8].
- For a p-core partition $\lambda = (\lambda_1, \cdots, \lambda_k)$, we have $\tau_m(\lambda) = (\nu_1, \cdots, \nu_{k+m})$ [AKT, Proposition 9.4] where

$$\nu_i = \begin{cases} \lambda_i + (p-m) & (1 \le i \le m) \\ \min\{\lambda_i + (p-m), \lambda_{i-m}\} & (m < i \le k) \\ \min\{p-m, \lambda_{i-m}\} & (k < i \le k+m). \end{cases}$$

In the following we show

$$\mathsf{D}_{l,s} = \#\{\lambda \otimes \mu \in B(\Lambda_0) \otimes B(\Lambda_s) \mid \lambda \otimes \mu \in B(\Lambda_0 + \Lambda_s), \mathrm{wt}(\lambda \otimes \mu) = \lambda_l^s\}.$$

Let $\lambda \in B(\Lambda_0), \mu \in B(\Lambda_s)$ and further assume that we have $\mathrm{wt}(\lambda \otimes \mu) = \mathrm{wt}(\lambda) + \mathrm{wt}(\mu) = \lambda_l^s$. Comparing $\mathrm{wt}(\lambda \otimes \mu)$ with λ_l^s, it is easily seen that λ and μ exactly divide $l \times (l+s)$ rectangle as follows.



Note that $2l + s - 1 < p$ and $p - l + 1 > l + s - 1$ since $1 \le l \le \lfloor \frac{p-s}{2} \rfloor$. Especially, λ and μ are both p-core partitions. It is enough to show the following claim.

Claim 4.2. *$\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_s)$ if and only if the path which divides λ and μ is a lattice path from $(0,0)$ to $(l+s,l)$ with steps $(0,1)$ and $(1,0)$ that does not exceed the diagonal $y = x$ (we say such a lattice path a good path).*

Proof. By Theorem 4.1 and the above remarks, $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_s)$ if and only if $\tau_s(\lambda) \supseteq \mu$. First, let us assume the path is not a good path. It is equivalent to assume that there exists some $1 \le i_0 \le l$ such that $\lambda_{i_0} = l - i_0$ and automatically $\mu_{s+i_0} = l - i_0 + 1$. Thus, we have

$$\min\{\lambda_{s+i_0} + (p-s), \lambda_{i_0}\} \le \lambda_{i_0} < \mu_{s+i_0}$$

where we put $\lambda_{s+i_0} = 0$ if $s + i_0 > l(\lambda)$. Hence, we have $\tau_s(\lambda) \not\supseteq \mu$.

Conversely, let us assume that the path is a good path. In this case, we have $\tau_s(\lambda) \supseteq \mu$ as follows.

- For $1 \leq i \leq s$, we have $\lambda_i + (p-s) \geq \mu_i$ since $p-s > l \geq \mu_i$.
- For $s+1 \leq i < l$, we have $\lambda_i + (p-s) > l \geq \mu_i$ since $p-s > l \geq \mu_i$. Because the path is a good path, we have $\lambda_{i-s} > \mu_i$. Thus, we have $\min\{\lambda_{i-s}, \lambda_i+(p-s)\} \geq \mu_i$.
- For $l+1 \leq i \leq l(\mu)$. Because the path is a good path, we have $\lambda_{i-s} > \mu_i$. Thus, we have $\min\{\lambda_{i-s}, p-s\} \geq \mu_i$ since we have $p-s > l \geq \mu_i$.

$\square$

The proof of $\mathsf{mult}\,\mu_l^s = \mathsf{D}_{l,p-s}$ is similar.

§ 5. A remark

Let X be an affine Dynkin diagram belongs to an infinite series except the series $C_n^{(1)}$ (i.e., $X = A_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$) and consider the corresponding affine Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(X)$ and its level 2 weight $\Lambda \in \mathcal{P}^+$. By our computer calculation, it seems that for each $b \in \max(\Lambda) \cap \mathcal{P}^+$, $\mathsf{mult}(b) = \mathsf{D}_{x,y}$ or $\mathsf{mult}(b) = \binom{x}{y}$ for some $x = x(b)$ and $y = y(b)$. But our motivation comes from a connection between Lie theory and representation theory of some algebras, we shall stop here.

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