Unramified extensions and geometric \( \mathbb{Z}_p \)-extensions of global function fields

By

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Abstract

We study on finite unramified extensions of global function fields (that is, function fields of one variable over a finite field). We show two results. One is an extension of Perret’s result about the ideal class group problem. Another is a construction of a geometric \( \mathbb{Z}_p \)-extension which has a certain property.

\[ 1. \text{ Main theorems} \]

Throughout the present paper, we fix a prime number \( p \) and a finite field \( \mathbb{F} \) of characteristic \( p \). Let \( q \) be the number of elements of \( \mathbb{F} \). Recall that a global function field is a function field of one variable over a finite field. Let \( k \) be a global function field with full constant field \( \mathbb{F} \). We also recall that a finite algebraic extension \( K/k \) is geometric if and only if the constant field of \( K \) is also \( \mathbb{F} \).

It is known that there is a finite abelian group \( G \) which is not isomorphic to the divisor class group of degree 0 of any global function field (Stichtenoth [20]). On the other hand, Perret [16] showed the following:

\[ \text{Theorem 1.1 ([16]).} \quad \text{For any given finite abelian group} \ G, \ \text{there is a finite separable geometric extension} \ k/\mathbb{F}(T) \ \text{such that} \ Cl(\mathcal{O}) \cong G, \ \text{where} \ \mathcal{O} \ \text{is the integral closure of} \ \mathbb{F}[T] \ \text{in} \ k \ \text{and} \ Cl(\mathcal{O}) \ \text{is the ideal class group of} \ \mathcal{O}. \]

This theorem is shown by using the following:

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Theorem 1.2 ([16]). For any given finite abelian group $G$, there is a global function field $k$ with full constant field $\mathbb{F}$ and a non-empty finite set $S$ of places of $k$ such that $\text{Cl}_S(k) \cong G$, where $\text{Cl}_S(k)$ is the $S$-class group of $k$.

Let $S$ be a non-empty finite set of places of $k$, and $H_S(k)$ the $S$-Hilbert class field of $k$, that is, the maximal unramified abelian extension field of $k$ in which all places of $S$ split completely (see [17]). We note that $\text{Cl}_S(k) \cong \text{Gal}(H_S(k)/k)$ by class field theory. Hence Theorem 1.2 also implies the existence of $k$ and $S$ which satisfy $\text{Gal}(H_S(k)/k) \cong G$. (More precisely, we can take $k$ and $S$ such that $H_S(k)/k$ is a geometric extension. See [16].)

In the present paper, we extend the above result to non-abelian finite groups. We will show the following:

Theorem 1.3. For any given finite group $G$, there is a global function field $k$ with full constant field $\mathbb{F}$ and a non-empty finite set $S$ of places of $k$ such that $\text{Gal}(\tilde{H}_S(k)/k) \cong G$, where $\tilde{H}_S(k)$ denotes the maximal unramified Galois extension field of $k$ in which all places of $S$ split completely. Moreover, we can take $k$ and $S$ such that $\tilde{H}_S(k)/k$ is a geometric extension.

See Ozaki [15] for the number field case.

We will prove Theorem 1.3 in section 2. Our proof is due to Perret’s idea (see [16]). That is, we will construct an unramified $G$-extension, and take a sufficiently large set $S$ of places such that $\text{Gal}(\tilde{H}_S(k)/k) \cong G$. (We use the term “$G$-extension” as a Galois extension whose Galois group is isomorphic to $G$.) To construct an unramified $G$-extension, we shall show an analog (Theorem 2.2) of Fröhlich’s classical result [4] for number fields.

In section 3, we shall apply Perret’s idea to Iwasawa theory. Let $k$ be a global function field with full constant field $\mathbb{F}$, $S$ a non-empty finite set of places of $k$. We recall that a $\mathbb{Z}_p$-extension is an infinite Galois extension whose Galois group is topologically isomorphic to the additive group of the ring $\mathbb{Z}_p$ of $p$-adic integers. Let $k_\infty/k$ be a geometric $\mathbb{Z}_p$-extension, that is, $k_\infty/k$ is a $\mathbb{Z}_p$-extension which satisfies that every finite subextension over $k$ is a geometric extension (see, e.g., [7]). (Recall that $p$ is the characteristic of $\mathbb{F}$. ) We assume that

(A) only finitely many places of $k$ ramify in $k_\infty/k$, and

(B) all places of $S$ split completely in $k_\infty/k$.

Under these assumptions, we can treat Iwasawa theory for the $S$-class group (see [17]). For a non-negative integer $n$, let $k_n$ be the $n$th layer of $k_\infty/k$. That is, $k_n$ is the unique subfield of $k_\infty$ which is a cyclic extension over $k$ of degree $p^n$. Moreover, let $A_n$ be the
Sylow $p$-subgroup of the $S$-class group of $k_n$. (Here we use the same symbol $S$ as the set of places of $k_n$ lying above $S$.) We put $X_S = \lim \leftarrow A_n$, where the projective limit is taken with respect to the norm maps. We call $X_S$ the Iwasawa module of $k_\infty/k$ for the $S$-class group. We put $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$. Note that $\Lambda \cong \mathbb{Z}_p[[T]]$. It is known that $X_S$ is a finitely generated torsion $\Lambda$-module, and the “Iwasawa type formula” holds for $A_n$ (see [17]). That is, there are non-negative integers $\lambda, \mu, \text{ and an integer } \nu$ such that $|A_n| = p^{\lambda n+\mu p^n+\nu}$ for all sufficiently large $n$. Aiba [1] studied these invariants $\lambda, \mu, \text{ and } \nu$ for certain geometric $\mathbb{Z}_p$-extensions.

There is a natural problem: characterize the $\Lambda$-modules which appear as $X_S$. (For the number field case, the same problem is dealt in, e.g., [14], [5].) Concerning this problem, we shall give the following result including “non-abelian” cases.

**Theorem 1.4.** For any given finite $p$-group $G$, there exist a global function field $k$ with full constant field $F$, a non-empty finite set $S$ of places of $k$, and a geometric $\mathbb{Z}_p$-extension $k_\infty/k$ satisfying the above assumptions (A) and (B) such that $\mathrm{Gal}(\tilde{L}_S(k_n)/k_n) \cong G$ (as groups) for all $n \geq 0$, where $\tilde{L}_S(k_n)$ is the maximal unramified Galois pro-$p$-extension field of $k_n$ in which all places lying above $S$ split completely.

For the number field case, Ozaki [14] showed that every “finite $\Lambda$-module” appears as $X_S$. Theorem 1.4 for $G$ abelian gives a weak analog of Ozaki’s result. That is, every finite $\Lambda$-module on which $\mathrm{Gal}(k_\infty/k)$ acts trivially appears as $X_S$. We will prove Theorem 1.4 in section 3.

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§ 2. Proof of Theorem 1.3

§ 2.1. Function field analog of Fröhlich’s result

At first, we shall show that for any finite group $G$, there is an unramified geometric extension $K/k$ of global function fields such that $\mathrm{Gal}(K/k) \cong G$. Recall that any finite group can be embedded into a finite symmetric group. Hence it is sufficient to consider the case that $G$ is a finite symmetric group. For the number field case, Fröhlich already showed the following result.

**Theorem 2.1 ([4]).** For every positive integer $n$, there is an unramified Galois extension $K/k$ of algebraic number fields such that $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group of degree $n$.

We will show the following:
Theorem 2.2. For every positive integer $n$, there is a global function field $k$ with full constant field $\mathbb{F}$ and an unramified geometric Galois extension $K/k$ such that $\text{Gal}(K/k) \cong \mathfrak{S}_n$. More precisely, there exist a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that $K/k$ is unramified and that $\text{Gal}(K/k) \cong \mathfrak{S}_n$.

To prove this, we follow Fröhlich’s original argument (see also Malinin [10]). That is, we construct a certain (ramified) $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$ and then we take a certain base change of this extension. Let $\infty$ be the infinite place of $\mathbb{F}(T)$.

Lemma 2.3. There is a Galois extension $k'$ over $\mathbb{F}(T)$ which satisfies all of the following properties.

- $k'/\mathbb{F}(T)$ is a geometric extension,
- $\text{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n$, and
- $\infty$ is unramified in $k'/\mathbb{F}(T)$.

Proof. At first, we must see that there is an $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$. This follows from the fact that $\mathbb{F}(T)$ is a Hilbertian field (see, e.g, [3, Corollary 16.2.7]). We put $A = \mathbb{F}[T]$. For an element $r$ of $A$, let $\deg(r)$ be the degree of $r$ as a polynomial of $T$.

Fix a monic separable polynomial $F(X) \in A[X]$ of degree $n$ such that the splitting field of $F(X)$ over $\mathbb{F}(T)$ is an $\mathfrak{S}_n$-extension.

We claim that there is an element $N_F \in A$ which satisfies the following property: if a monic polynomial $G(X) \in A[X]$ of degree $n$ satisfies $G(X) \equiv F(X) \pmod{N_F}$, then the splitting field of $G(X)$ over $\mathbb{F}(T)$ is also an $\mathfrak{S}_n$-extension. We shall show this claim. By using the Chebotarev density theorem, we can take an irreducible monic polynomial $p_1$ such that if $G(X) \equiv F(X) \pmod{p_1}$ then $G(X)$ is irreducible and separable. Similarly, we can take distinct irreducible monic polynomials $p_2, p_3$ of $A = \mathbb{F}(T)$ which are distinct from $p_1$ and satisfy the following properties: (i) if $G(X) \equiv F(X) \pmod{p_2}$ then the Galois group of $G(X)$ contains a cycle of length $n - 1$ (as a subgroup of $\mathfrak{S}_n$), and (ii) if $G(X) \equiv F(X) \pmod{p_3}$ then the Galois group of $G(X)$ contains a transposition. We put $N_F = p_1p_2p_3$. This $N_F$ satisfies the above claim. Moreover, we can take $N_F$ which is prime to $T$ by the Chebotarev density theorem. We also fix such $N_F$.

To construct a geometric $\mathfrak{S}_n$-extension which is unramified at the infinite place, we take $G(X)$ as follows:

\[
\begin{align*}
G(X) & \equiv F(X) \pmod{N_F}, \\
G(X) & \equiv (\text{a product of distinct monic polynomials of degree 1}) \pmod{r}, \text{ and} \\
G(X) & \equiv (\text{a separable polynomial}) \pmod{T},
\end{align*}
\]

where $r$ is a monic irreducible polynomial of $A = \mathbb{F}[T]$ such that $n < q^{\deg(r)}$, $\deg(r)$ is odd, and $r$ is prime to $TN_F$. By the first congruence, we see that the splitting field $k'$
of \(G(X)\) is an \(\mathcal{S}_n\)-extension. We shall show that the constant field of \(k'\) is \(F\). Let \(\overline{F}\) be the algebraic closure of \(F\). We note that \(M := k' \cap \overline{F}(T)\) is a finite cyclic extension over \(\overline{F}(T)\). Since \(\text{Gal}(k'/\overline{F}(T)) \cong \mathcal{S}_n\), \(M\) must be \(\overline{F}(T)\) or the unique quadratic subfield in \(k'/\overline{F}(T)\). If \(M \neq \overline{F}(T)\), then no odd degree place of \(\overline{F}(T)\) splits in \(M\). However, we see that the place of \(\overline{F}(T)\) corresponding to \(r\) splits completely in \(k'\) by the second congruence. It is a contradiction.

By the third congruence, we see that the place of \(\overline{F}(T)\) corresponding to \(T\) is unramified in \(k'\). We replace the indeterminate \(T\) by \(U = 1/T\), then the infinite place of \(\overline{F}(U)\) is unramified in \(k'\) (and the former two conditions are also satisfied).

We shall prove Theorem 2.2. We may assume that \(n \geq 2\). Fix a geometric \(\mathcal{S}_n\)-extension \(k'/\overline{F}(T)\) satisfying the properties of Lemma 2.3. We put \(m = n!\). We can take a separable monic polynomial \(F(X) \in A[X]\) of degree \(m\) (as a polynomial of \(X\)) whose splitting field over \(\overline{F}(T)\) is \(k'\). Let \(M'\) be the unique quadratic subextension field of \(\overline{F}(T)\) contained in \(k'\).

We define the following notation.

- \(\{p_1, \ldots, p_t\}\) : the set of distinct places of \(\overline{F}(T)\) which ramify in \(k'\) (hence are distinct from \(\infty\)).
- \(p_{t+1}\) : a place \(\not= \infty, p_1, \ldots, p_t\) of \(\overline{F}(T)\) which is inert in \(M'\) and has degree \(> \frac{\log(m)}{\log(q)}\).
- \(p_{t+2}\) : a place \(\not= \infty\) of \(\overline{F}(T)\) which splits completely in \(k'\) and has odd degree \(> \frac{\log(m)}{\log(q)}\) (hence is distinct from \(p_1, \ldots, p_t, p_{t+1}\)).
- \(p_1, \ldots, p_{t+2}\) : irreducible monic polynomials of \(A = \overline{F}[T]\) corresponding to \(p_1, \ldots, p_{t+2}\), respectively.

Note that we can take \(p_{t+1}\) (resp. \(p_{t+2}\)) by using Theorem 9.13B of [18], which is an effective version of the Chebotarev density theorem for global function fields. (See also [12], etc.) Indeed, by this theorem, there is a place of \(\overline{F}(T)\) of arbitrary sufficiently large degree which is inert in \(M'\) (resp. splits completely in \(k'\)), as \(M'/\overline{F}(T)\) is a geometric cyclic extension (resp. \(k'/\overline{F}(T)\) is a geometric Galois extension).

By using Lemma 2.3, we can also construct an \(\mathcal{S}_m\)-extension over \(\overline{F}(T)\). Let \(H(X)\) be a monic polynomial in \(A[X]\) of degree \(m\) which gives an \(\mathcal{S}_m\)-extension. Then there is an element \(N_H\) of \(A\) having the following property: if a monic polynomial \(G(X) \in A[X]\) of degree \(m\) satisfies \(G(X) \equiv H(X) \pmod{N_H}\), then the splitting field of \(G(X)\) over \(\overline{F}(T)\) is also an \(\mathcal{S}_m\)-extension (see the proof of Lemma 2.3). We can also take \(N_H\) such that it is prime to \(p_1, \ldots, p_{t+2}\).

We take a monic polynomial \(G(X)\) of \(A[X]\) (having degree \(m\)) which satisfies the following conditions (2.1)–(2.4).

\[(2.1)\quad G(X) \equiv H(X) \pmod{N_H}.\]
If \( G(X) \) satisfies (2.1), then \( G(X) \) gives an \( \mathfrak{S}_m \)-extension. Let \( L \) be the splitting field of \( G(X) \) over \( \mathbb{F}(T) \).

\[
(2.2) \quad G(X) \equiv \text{(a product of distinct monic polynomials of degree 1)} \pmod{p_{t+1}}.
\]

If \( G(X) \) satisfies (2.1) and (2.2), then we see that \( p_{t+1} \) splits in the unique quadratic subextension, say \( M_L \), over \( \mathbb{F}(T) \) contained in \( L \). On the other hand, \( p_{t+1} \) is inert in the unique quadratic subextension \( M' \) over \( \mathbb{F}(T) \) contained in \( k' \). We claim that \( k' \cap L = \mathbb{F}(T) \). Indeed, suppose that \( k' \cap L \neq \mathbb{F}(T) \). Then \( k' \cap L \) is a quadratic extension over \( \mathbb{F}(T) \). If \( n = 2 \), this is clear. For \( n \geq 3 \), we have \( \text{Gal}(L/\mathbb{F}(T)) \cong \mathfrak{S}_m \), where \( m = n! \geq 5 \). Observe also that \( k' \cap L \neq L \), as \( m > n \). Now, since the alternating group \( \mathfrak{A}_m \) is the unique nontrivial proper normal subgroup of \( \mathfrak{S}_m \) when \( m \geq 5 \) (see, e.g., [19]), \( k' \cap L \) is a quadratic extension over \( \mathbb{F}(T) \). Since this quadratic extension is contained in both \( k' \) and \( L \), it must coincide with both \( M' \) and \( M_L \) at a time. This contradicts the above observation on the behavior of \( p_{t+1} \) in \( M' \) and \( M_L \). Thus, we have proved the claim. Then we see \( \text{Gal}(Lk'/L) \cong \mathfrak{S}_n \).

\[
(2.3) \quad G(X) \equiv \text{(a product of distinct monic polynomials of degree 1)} \pmod{p_{t+2}}.
\]

If \( G(X) \) satisfies (2.1)–(2.3), then the odd degree place \( p_{t+2} \) splits completely in \( Lk'/\mathbb{F}(T) \). We claim that \( Lk'/\mathbb{F}(T) \) is a geometric extension. Note that the degree of a place of \( k' \) lying above \( p_{t+2} \) is also odd because \( p_{t+2} \) splits completely in \( k' \). Since \( \text{Gal}(Lk'/k') \cong \mathfrak{S}_m \) and an odd degree place splits completely in \( Lk'/k' \), we see that \( Lk'/k' \) is also a geometric extension. Hence the claim follows. By using Krasner’s lemma, we can see that there is a positive integer \( s_i \) for each \( i = 1, \ldots, t \) depending only on \( F(X) \) such that \( G(X) \equiv F(X) \pmod{p_i^{s_i}} \) when \( G(X) \) satisfies (2.1)–(2.3) and

\[
(2.4) \quad G(X) \equiv F(X) \pmod{p_i^{s_i}} \text{ for } i = 1, \ldots, t,
\]

then we can see that \( Lk'/L \) is unramified at all places.

We can take \( G(X) \) satisfying (2.1)–(2.4). By the above arguments, the extension \( Lk'/L \) satisfies the assertion of Theorem 2.2.

\[ \square \]

Remark. When \( G \) is abelian, an unramified geometric \( G \)-extension was constructed by Angles [2]. Moret-Bailly [11] also gives a result which is very close to ours. Probably, it seems that one can prove our main theorems by using the result given in [11] instead of Theorem 2.2.

\[ \text{§ 2.2. Proof of Theorem 1.3} \]

Since \( G \) is embedded into \( \mathfrak{S}_n \) for some \( n > 0 \), Theorem 2.2 implies that there exists a global function field \( k \) with full constant field \( \mathbb{F} \) and an unramified geometric Galois extension \( K/k \) such that \( \text{Gal}(K/k) \cong G \).
Proposition 2.4. There is a non-empty finite set $S$ of places of $k$ such that (i) all places of $S$ split completely in $K$, and (ii) $\tilde{H}_S(k)/k$ is a finite extension.

Proof. The crucial point of this proposition is choosing a set $S$ to satisfy (ii). For a positive integer $N$, we put

$$B_N = \{p \mid p \text{ is a place of } k \text{ having degree } N, \ p \text{ splits completely in } K/k\}.$$  

Since $K/k$ is a geometric extension, Theorem 9.13B of [18] implies that

$$|B_N| = \frac{q^N}{|G|N} + O\left(\frac{q^{N/2}}{N}\right),$$

(recall that $q$ is the number of elements of $\mathbb{F}$). In particular, if $N$ is sufficiently large, then we obtain the inequality

$$|B_N| > \frac{q^{N/2} - 1}{N} \text{Max}(g - 1, 0),$$

where $g$ is the genus of $k$. We fix an integer $N$ which satisfies the above inequality. According to Ihara’s theorem [8, Theorem 1(FF)], if $S \supset B_N$, then $\tilde{H}_S(k)/k$ is a finite extension. Hence we can take $S$ to satisfy the conditions (i) and (ii).

The rest of the proof of Theorem 1.3 is quite similar to Perret’s argument given in [16]. We remark that $K$ is contained in $\tilde{H}_S(k)$. For a nontrivial element $\sigma$ of $\text{Gal}(\tilde{H}_S(k)/K)$, we can take a place $\mathfrak{p}$ of $\tilde{H}_S(k)$ corresponding to $\sigma$ by the Chebotarev density theorem. We can take $\mathfrak{p}$ which is unramified in $\tilde{H}_S(k)/K$. Let $p$ be the place of $k$ which is lying below $\mathfrak{p}$. Since the decomposition field of $\mathfrak{p}$ in $\tilde{H}_S(k)/k$ contains $K$ and $K/k$ is a Galois extension, we see that $p$ splits completely in $K/k$. Then we see $\tilde{H}_S(k) \supset \tilde{H}_{S \cup \{p\}}(k) \supset K$. Replacing $S \cup \{p\}$ by $S$ and repeating the above operation, we can see that $\tilde{H}_S(k) = K$ for some finite set $S$. This implies $\text{Gal}(\tilde{H}_S(k)/K) \cong G$.

We recall that $K/k$ is a geometric extension. Hence the final part of the theorem follows. $\square$

§ 3. Proof of Theorem 1.4

Firstly, we shall show the following:

Theorem 3.1. Let $k$ be a finite Galois extension over $\mathbb{F}(T)$. Then, there exist a non-empty finite set $S$ of places of $\mathbb{F}(T)$ and a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ which satisfy the following properties.

• $F_{\infty} \cap k = \mathbb{F}(T)$,

• all places of $S$ split completely in $k$,

• both of $F_{\infty}/\mathbb{F}(T)$ and $F_{\infty}k/k$ satisfy the assumptions (A) and (B) in section 1, and

• the Sylow $p$-subgroup of $\text{Cl}_S(F_nk)$ is trivial for all $n \geq 0$,

where $F_n$ is the $n$th layer of $F_{\infty}/\mathbb{F}(T)$. (We use the same symbol $S$ as the set of places lying above $S$.)

Proof. We take a place $p_0$ of $\mathbb{F}(T)$ which splits completely in $k$. We also take a place $r$ of $\mathbb{F}(T)$ which is distinct from $p_0$ and unramified in $k$. We claim that there is a geometric $\mathbb{Z}_p$-extension $F_{\infty}/\mathbb{F}(T)$ unramified outside $r$ which satisfies that

• $r$ is totally ramified, and

• $p_0$ splits completely.

We shall show this claim. Let $M$ be the maximal pro-$p$ abelian extension over $\mathbb{F}(T)$ which is unramified outside $r$. We know that $\text{Gal}(M/\mathbb{F}(T))$ is isomorphic to a countable infinite product of the additive group of $\mathbb{Z}_p$ (see [21], [9]). Hence there are infinitely many geometric $\mathbb{Z}_p$-extensions which satisfy the above conditions.

By the above choice of $F_{\infty}$, we see $F_1 \cap k = \mathbb{F}(T)$. We put $k_1 = F_1k$. Then $k_1/\mathbb{F}(T)$ is a Galois extension, and $p_0$ splits completely in $k_1$. We set $S_0 = \{ p_0 \}$, and we use the same symbol to denote the set of places lying above $p_0$. We can see that $H_{S_0}(k_1)$ is a finite Galois extension over $\mathbb{F}(T)$. We take a nontrivial element $\sigma_1$ of $\text{Gal}(H_{S_0}(k_1)/k_1)$.

By using the above argument, we can take a geometric $\mathbb{Z}_p$-extension $F'_{\infty}/\mathbb{F}(T)$ unramified outside $r$ which satisfies

• $F'_{\infty} \cap F_{\infty} = \mathbb{F}(T)$,

• $r$ is totally ramified in $F'_{\infty}F_{\infty}$, and

• $p_0$ splits completely in $F'_{\infty}$.

Let $F'_1$ be the initial layer of $F'_{\infty}/\mathbb{F}(T)$. Then we see that $F'_1 \cap k_1 = \mathbb{F}(T)$ and $k_1F'_1 \cap H_{S_0}(k_1) = k_1$. We note that

$$\text{Gal}(F'_1H_{S_0}(k_1)/k_1) \cong \text{Gal}(F'_1k_1/k_1) \times \text{Gal}(H_{S_0}(k_1)/k_1), \quad \text{Gal}(F'_1k_1/k_1) \cong \text{Gal}(F'_1/\mathbb{F}(T)).$$

Hence there is an isomorphism

$$\text{Gal}(F'_1/\mathbb{F}(T)) \times \text{Gal}(H_{S_0}(k_1)/k_1) \sim \text{Gal}(F'_1H_{S_0}(k_1)/k_1).$$
Let $\tau$ be a generator of the cyclic group $\text{Gal}(F'_1/\mathbb{F}(T))$, and $\tau_1$ an element of $\text{Gal}(F'_1^{\infty}H_{S_0}(k_1)/k_1)$ which is the image of $(\tau, \sigma_1)$ under the above isomorphism. We can regard $\tau$ as an element of $\text{Gal}(F'_1H_{S_0}(k_1)/\mathbb{F}(T))$. By the Chebotarev density theorem, there is a place $\mathfrak{p}_1$ of $F'_1H_{S_0}(k_1)$ which corresponds to $\tau_1$. Let $p_1$ be the place of $\mathbb{F}(T)$ lying below $\mathfrak{p}_1$. We can take $\mathfrak{p}_1$ such that $p_1$ is not ramified in $F'_1H_{S_0}(k_1)$. Then we see that $p_1$ splits completely in $k_1$ and is inert in $F'_1$. We put $S_1 = S_0 \cup \{p_1\}$.

In general, $p_1$ may not split completely in $F_\infty$. This is a problem because we need the assumption (B). We remark that $F_\infty F'_\infty/\mathbb{F}(T)$ is a $\mathbb{Z}_p^2$-extension unramified outside $\mathfrak{p}$. We recall that $p_1$ does not split in $F'_1$. Hence the decomposition field of $F_\infty F'_\infty/\mathbb{F}(T)$ for $p_1$ is a $\mathbb{Z}_p$-extension over $\mathbb{F}(T)$. We denote it by $F''_\infty$. We also note that $F''_\infty/\mathbb{F}(T)$ is the unique $\mathbb{Z}_p$-extension contained in $F_\infty F'_\infty$ such that $p_1$ splits completely. Then the initial layer of $F''_\infty/\mathbb{F}(T)$ must coincide with $F_1$. We replace $F_\infty$ by $F''_\infty$.

We note that $H_{S_1}(k_1) \supseteq H_{S_1}(k_1)$ by the definition of $p_1$. Similarly, we can choose a place $p_2$, put $S_2 = S_1 \cup \{p_2\}$, and modify the $\mathbb{Z}_p$-extension such that all places of $S_2$ splits completely. Repeating this operation, we see that $H_{S_1}(k_1) = k_1$ for some finite set $S_1$. From the above construction, we see that $F_\infty \cap k = \mathbb{F}(T)$ and that $F_\infty k/k$ satisfies the assumptions (A) and (B).

Finally, we shall give an Iwasawa-theoretic argument. In $F_\infty k/k$, all ramified places (which are lying above $\mathfrak{p}$) are totally ramified. From this, we also see $H_{S_1}(k) = k$. Let $A_n$ be the Sylow $p$-subgroup of $\text{Cl}_{S_1}(kF_n)$. By the above results, we see that both of $A_0$ and $A_1$ are trivial. In this situation, we can use the method given in Fukuda [6]. Namely, if all places which ramify in $F_\infty k/k$ are totally ramified and both of $A_0$ and $A_1$ are trivial, then $A_n$ is trivial for all $n \geq 0$. (See [6, Theorem 1]. We note that the same method is also applicable for our situation.) Hence we see that $A_n$ is trivial for all $n \geq 0$.

We shall show Theorem 1.4. We fix a finite $p$-group $G$. By using Theorem 2.2, we can take a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that $K/k$ is unramified and $\text{Gal}(K/k) \cong G$. By Theorem 3.1, we can take a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ and a set $S$ of places of $\mathbb{F}(T)$ such that $F_\infty \cap K = \mathbb{F}(T)$, all places of $S$ split completely in $K$, both of $F_\infty/\mathbb{F}(T)$ and $F_\infty K/K$ satisfy the assumptions (A) and (B), and $A_n$ is trivial for all $n \geq 0$ (where $A_n$ is the Sylow $p$-subgroup of $\text{Cl}_{S_1}(F_n K)$, and $F_n$ is the $n$th layer of $F_\infty/\mathbb{F}(T)$). We note that $F_\infty k/k$ also satisfies the assumptions (A) and (B). We claim that $\hat{L}_S(F_n K) = F_n K$ for all $n \geq 0$. Indeed, if $\hat{L}_S(F_n K)/F_n K$ is nontrivial, then there is a nontrivial finite Galois $p$-subextension over $F_n K$. Moreover, there is a nontrivial finite abelian $p$-subextension over $F_n K$ because every $p$-group is solvable. Since $A_n$ is trivial, it is a contradiction. We have shown the above claim. This implies that $\hat{L}_S(F_n k) = F_n K$ because $F_n K/F_n k$ is unramified and all places of $F_n k$ lying above $S$ split completely in $F_n K$. Hence
Gal(\(L_S(F_n k)/F_n k\)) \(\cong G\) for all \(n \geq 0\). Then the theorem follows.

References


