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Torsion of abelian schemes and rational points on moduli spaces

By

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Abstract

In [CT], we proved, in characteristic 0, certain 1-dimensional base versions of the uniform boundedness conjecture for p-primary torsion of abelian varieties and of Fried's modular tower conjecture related to the regular inverse Galois problem. In this paper, we prove these results in arbitrary characteristics.

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§0. Notation.

Let S be a scheme and p a prime number. Given a commutative group scheme A over S, we use the following notations:

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$$A[N] \stackrel{\text{def}}{=} \mathrm{Ker}([N] : A \to A),$$

where $[N]$ stands for the multiplication-by-N endomorphism ($N \geq 1$),

$$A_{\mathrm{tors}} \stackrel{\text{def}}{=} \varinjlim_{N} A[N] \text{ (i.e., } A_{\mathrm{tors}}(T) = \varinjlim_{N} A[N](T) \text{ for each } S\text{-scheme } T),$$

$$A[p^\infty] \stackrel{\text{def}}{=} \varinjlim_{n} A[p^n] \text{ (i.e., } A[p^\infty](T) = \varinjlim_{n} A[p^n](T) \text{ for each } S\text{-scheme } T),$$

$$A[p^n]^* \stackrel{\text{def}}{=} A[p^n] \smallsetminus A[p^{n-1}] \; (A[p^0]^* \stackrel{\text{def}}{=} A[p^0]).$$

If, moreover, $S = \mathrm{Spec}(k)$ for some field k of characteristic $\neq p$, we also use the following notations:

$$T_p(A) \stackrel{\text{def}}{=} \varprojlim_{n} A[p^n](\overline{k}) \text{ (the } p\text{-adic Tate module of } A),$$

$$T_p(A)^* \stackrel{\text{def}}{=} T_p(A) \smallsetminus pT_p(A) = \varprojlim_{n} A[p^n]^*(\overline{k}),$$

$$V_p(A) \stackrel{\text{def}}{=} T_p(A) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

and

$$\mathbf{Z}_p(1) \stackrel{\text{def}}{=} T_p(\mathbf{G}_m),$$

where $\mathbf{G}_m \stackrel{\text{def}}{=} \mathbf{P}^1 \smallsetminus \{0, \infty\}$ is the multiplicative group scheme.

§1. Introduction.

In this §, we shall explain two important problems in arithmetic geometry that have motivated our study — the uniform boundedness conjecture for torsion of abelian varieties and the regular inverse Galois problem (especially, the modular tower conjecture).

⟨*Uniform boundedness conjecture for torsion of abelian varieties*⟩

Let k be an algebraic number field (i.e., $d \stackrel{\text{def}}{=} [k : \mathbf{Q}] < \infty$) and g an integer ≥ 0.

Uniform Boundedness Conjecture (UB).
There exists a constant $N = N(k, g) \geq 0$, such that for any g-dimensional abelian variety A over k and any $v \in A_{\mathrm{tors}}(k)$, the order of v is $\leq N$.

(UB) is trivially valid for $g = 0$ and has been proved for $g = 1$ (by Mazur, Kamienny, Abramovich for smaller values of d and by Merel [Me] for d general) via intensive study of geometry of modular curves. However, it is widely open for $g > 1$.

We also have the following variant. Let p be a (fixed) prime number.

p-Uniform Boundedness Conjecture (pUB).
There exists a constant $N = N(k, g, p)$, such that for any g-dimensional abelian variety A over k and any $v \in A[p^\infty](k)$, the order of v is $\leq N$.

(pUB) is trivially valid for $g = 0$ and has been proved for $g = 1$ by Manin [Ma2]. Although (pUB) is clearly weaker than (UB), it is still widely open for $g > 1$.

$\langle$*Inverse Galois problem*$\rangle$

Let G be a finite group and k an algebraic number field.

Inverse Galois Problem (IGP).
Does there exist a Galois extension L/k, such that $\mathrm{Gal}(L/k) \simeq G$*?*

The following variant of (IGP) is more closely related to our study in this paper.

Regular Inverse Galois Problem (RIGP).
Does there exist a Galois extension $\mathcal{L}/k(T)$ (T: indeterminate), such that $\mathcal{L}$ is regular over k (i.e. $\mathcal{L} \cap \overline{k} = k$) and that $\mathrm{Gal}(\mathcal{L}/k(T)) \simeq G$*?*

It is well-known that (RIGP) implies (IGP) (by Hilbert's irreducibility theorem).

While (IGP) is purely number-theoretic, (RIGP) is arithmetico-geometric in nature, as follows. First, we have the following one-to-one correspondences:

$$\begin{aligned}
&\text{a Galois extension } \mathcal{L}/k(T) \text{ with } \mathrm{Gal}(\mathcal{L}/k(T)) \simeq G \\
\overset{1:1}{\longleftrightarrow}\ &\text{a (branched) connected Galois cover } Y \to \mathbf{P}^1_k \text{ with } \mathrm{Aut}(Y/\mathbf{P}^1_k) \simeq G \\
\overset{1:1}{\longleftrightarrow}\ &\text{a surjection } \pi_1(\mathbf{P}^1_k \smallsetminus S) \twoheadrightarrow G, \text{ considered modulo } \mathrm{Inn}(G) \\
&\text{(where } S \text{ runs over the finite sets of closed points of } \mathbf{P}^1_k).
\end{aligned}$$

Here, π_1 stands for the étale fundamental group of scheme (with a suitable base point).

Moreover, in these one-to-one correspondences, the condition for the first object $\mathcal{L}/k(T)$ that $\mathcal{L}$ is regular over k corresponds to the condition for the second object $f : Y \to \mathbf{P}^1_k$ that $Y_{\overline{k}}$ is connected (or, equivalently, f is "geometric"), and to the condition for the third object $\pi_1(\mathbf{P}^1_k \smallsetminus S) \twoheadrightarrow G$ that the induced map $\pi_1(\mathbf{P}^1_{\overline{k}} \smallsetminus S_{\overline{k}}) \to G$ is already surjective. Note that $\pi_1(\mathbf{P}^1_k \smallsetminus S)$ fits into the following exact sequence of profinite groups:

$$1 \to \pi_1(\mathbf{P}^1_{\overline{k}} \smallsetminus S_{\overline{k}}) \to \pi_1(\mathbf{P}^1_k \smallsetminus S) \to \mathrm{Gal}(\overline{k}/k) \to 1,$$

where $\pi_1(\mathbf{P}^1_k \smallsetminus S)$ and $\pi_1(\mathbf{P}^1_{\overline{k}} \smallsetminus S_{\overline{k}})$ are sometimes referred to as the arithmetic and the geometric fundamental groups, respectively.

It is well-known that (as k is of characteristic 0) the geometric fundamental group $\pi_1(\mathbf{P}^1_{\overline{k}} \smallsetminus S_{\overline{k}})$ is a free profinite group of rank $r - 1$, where r is the cardinality of the point set $S_{\overline{k}}$. Thus, it is easy to construct a surjection $\pi_1(\mathbf{P}^1_{\overline{k}} \smallsetminus S_{\overline{k}}) \twoheadrightarrow G$. (Indeed, this is possible if and only if G is generated by at most $(r - 1)$ elements.) However, it is a subtle descent problem to extend a surjection $\pi_1(\mathbf{P}^1_{\overline{k}} \smallsetminus S_{\overline{k}}) \twoheadrightarrow G$ to $\pi_1(\mathbf{P}^1_k \smallsetminus S)$.

The above geometric interpretation of (RIGP) further leads to a modular interpretation of (RIGP), as follows. For this, let us introduce moduli spaces of covers of curves, i.e., Hurwitz spaces. (For details, see, e.g., [BR].)

Definition. Let G be a finite group and r an integer ≥ 0.
(i) We denote by $\mathcal{H}_{G,\mathbf{P}^1,r}$ the moduli stack (over $\mathbf{Z}[1/|G|]$) of pairs $(f, \boldsymbol{t})$, where $f : Y \to \mathbf{P}^1$ is a geometric Galois cover equipped with $\mathrm{Aut}(Y/\mathbf{P}^1) \simeq G$, and $\boldsymbol{t} \subset \mathbf{P}^1$ is an étale divisor of degree r that contains the branch locus of f. It is known that $\mathcal{H}_{G,\mathbf{P}^1,r}$ is a Deligne-Mumford stack and that the associated coarse space $H_{G,\mathbf{P}^1,r}$ is a scheme.
(ii) Let g be an integer ≥ 0 with the hyperbolicity condition $2 - 2g - r < 0$. Then we denote by $\mathcal{H}_{G,g,r}$ the moduli stack (over $\mathbf{Z}[1/|G|]$) of pairs $(f, \boldsymbol{t})$, where $f : Y \to X$ is a geometric Galois cover over a proper, smooth, geometrically connected curve X of genus g equipped with $\mathrm{Aut}(Y/X) \simeq G$, and $\boldsymbol{t} \subset X$ is an étale divisor of degree r that contains the branch locus of f. It is known that $\mathcal{H}_{G,g,r}$ is a Deligne-Mumford stack and that the associated coarse space $H_{G,g,r}$ is a scheme.

By definition, each geometric Galois cover $f : Y \to \mathbf{P}^1_k$ over k equipped with $\mathrm{Aut}(Y/\mathbf{P}^1_k) \simeq G$, whose branch locus in $\mathbf{P}^1_k$ is contained in an étale divisor $\boldsymbol{t}$ of degree r, defines a k-rational point in $H_{G,\mathbf{P}^1,r}(k)$ depending on the pair $(f, \boldsymbol{t})$. (When the branch locus coincides with $\boldsymbol{t}$, this k-rational point depends only on f, since $\boldsymbol{t}$ is then determined by f.) If, moreover, $\mathcal{H}_{G,\mathbf{P}^1,r}$ is representable (i.e., $\mathcal{H}_{G,\mathbf{P}^1,r} = H_{G,\mathbf{P}^1,r}$), then this defines a one-to-one correspondence between the set of isomorphism classes of such $(f, \boldsymbol{t})$ and $H_{G,\mathbf{P}^1,r}(k)$. Similarly, for a proper, smooth, geometrically connected curve X of genus g over k, each geometric Galois cover $f : Y \to X$ over k equipped with $\mathrm{Aut}(Y/X) \simeq G$, whose branch locus in X is contained in an étale divisor $\boldsymbol{t}$ of degree r, defines a k-rational point in $H_{G,g,r}(k)$ depending on the pair $(X, f, \boldsymbol{t})$. (When the branch locus coincides with $\boldsymbol{t}$, this k-rational point depends only on (X, f), since $\boldsymbol{t}$ is then determined by (X, f).) If, moreover, $\mathcal{H}_{G,g,r}$ is representable (i.e., $\mathcal{H}_{G,g,r} = H_{G,g,r}$), then this defines a one-to-one correspondence between the set of isomorphism classes of such $(X, f, \boldsymbol{t})$ and $H_{G,g,r}(k)$.

Facts. Here, dim stands for the relative dimension over the base.
(i) $\dim(H_{G,\mathbf{P}^1,r}) = r$, unless $H_{G,\mathbf{P}^1,r} = \emptyset$. $\mathcal{H}_{G,\mathbf{P}^1,r}$ is representable if and only if either $\mathcal{H}_{G,\mathbf{P}^1,r} = \emptyset$ or the center of G is trivial.
(ii) $\dim(H_{G,g,r}) = 3g - 3 + r$, unless $H_{G,g,r} = \emptyset$. $\mathcal{H}_{G,g,r}$ is representable if and only if, for any object $f : Y \to X$ classified by $\mathcal{H}_{G,g,r}$, the centralizer of G in $\mathrm{Aut}(Y)$ is trivial.
(iii) $H_{G,\mathbf{P}^1,r}/PGL_2 = H_{G,0,r}$ (for $r \geq 3$).

$\langle$*Fried's modular tower conjecture* $\rangle$

The modular tower conjecture is a conjecture arising from (RIGP) that was posed by M. Fried in the early 1990s. Here, we formulate some variants of this conjecture. For more details, see [F], [FK] and [D].

Let p be a prime number. Let $\mathbf{G} = \{G_{n+1} \twoheadrightarrow G_n\}_{n \geq 0}$ be a projective system of finite groups, such that $G \stackrel{\mathrm{def}}{=} \varprojlim G_n$ is p-obstructed in the following sense:

Definition. We say that a profinite group G is p-obstructed, if G contains an open subgroup that admits a quotient isomorphic to $\mathbf{Z}_p$.

Modular Tower Conjecture (MT).
Let k be an algebraic number field and r an integer ≥ 0.

(i) *There exists a constant* $N = N_1(p, \mathbf{G}, r, k)$*, such that, for any* $n \geq N$ *and for any geometric Galois cover* $f : Y \to \mathbf{P}^1_k$ *with group* G_n*, the degree of the* (*reduced*) *branch divisor of* f *in* $\mathbf{P}^1_k$ *is* $> r$*.*
(ii) *There exists a constant* $N = N_2(p, \mathbf{G}, r, k)$*, such that* $H_{G_n,\mathbf{P}^1,r}(k) = \emptyset$ *for any* $n \geq N$*.*
(iii) *Let* g *be an integer* ≥ 0 *with* $2 - 2g - r < 0$*. Then there exists a constant* $N = N_3(p, \mathbf{G}, g, r, k)$*, such that* $H_{G_n,g,r}(k) = \emptyset$ *for any* $n \geq N$*.*

Here, the projective systems $\{H_{G_n,\mathbf{P}^1,r}\}_{n\geq 0}$ and $\{H_{G_n,g,r}\}_{n\geq 0}$ of Hurwitz spaces are often referred to as "modular towers". Note that the following implications are immediate:

$$\text{(MT)(iii) for } g = 0 \implies \text{(MT)(ii)} \implies \text{(MT)(i)}.$$

Finally, the following (weaker) variant of (MT)(iii) has been already proved in arbitrary characteristics. Here, for a profinite group G, denote by Σ_G the set of prime numbers which divide the order of (some finite quotient of) G.

Theorem 1.1 ([C], Corollary 3.6. See also [BF], [K]). *Let* k *be a field finitely generated over the prime field of characteristic* $q \notin \Sigma_G$ (*hence, in particular,* $q \neq p$)*, and* g, r *integers* ≥ 0 *with* $2 - 2g - r < 0$*. Then we have*

$$\varprojlim H_{G_n,g,r}(k) = \emptyset.$$

§2. Main results.

The main results of this paper are solutions of certain 1-dimensional versions of (pUB) and (MT) over fields finitely generated over the prime field of arbitrary characteristic $\neq p$. To state the former in some more generality, we shall introduce the notion of non-Tate characters as in [CT].

Definition. Let p be a prime number and k a field of characteristic $q \neq p$. (Denote by Γ_k the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of k.) We say that a character $\chi : \Gamma_k \to \mathbf{Z}_p^*$ is non-Tate, if it does not appear as a subrepresentation of the p-adic representation associated with an abelian variety over k. Equivalently, χ is non-Tate if and only if, for any abelian variety A over k,

$$A[p^\infty](\chi) \stackrel{\text{def}}{=} \{T \in A[p^\infty](\overline{k}) \mid {}^\sigma T = \chi(\sigma)T \text{ for all } \sigma \in \Gamma_k\}$$

is finite.

When k is finitely generated over the prime field, the trivial character and the p-adic cyclotomic character are typical examples of non-Tate characters. (See [CT].)

Let d be an integer ≥ 0. Now, we can formulate the d-dimensional version of (pUB), as follows.

Conjecture (pUB$_d$). *Let p be a prime number. Let k be a field finitely generated over the prime field of characteristic $q \neq p$ and $\chi : \Gamma_k \to \mathbf{Z}_p^*$ a non-Tate character. Let S be a scheme of finite type over k with $\dim(S) \leq d$ and A an abelian scheme over S. Then there exists an integer $N = N(p, k, \chi, S, A)$, such that $A_s[p^\infty](\chi) \subset A_s[p^N](\overline{k})$ for any $s \in S(k)$.*

(pUB$_0$) follows immediately from the definition of non-Tate characters. Now, the first main result of this paper is:

Theorem A. (pUB$_1$) *holds.*

Next, we formulate the d-dimensional version of (MT) (in arbitrary characteristics).

Conjecture (MT$_d$). *Let p be a prime number and $\mathbf{G} = \{G_{n+1} \twoheadrightarrow G_n\}_{n \geq 0}$ a projective system of finite groups, such that $G \stackrel{\text{def}}{=} \varprojlim G_n$ is p-obstructed. Let g, r be integers ≥ 0 with $2 - 2g - r < 0$. Let k be a field finitely generated over the prime field of characteristic $q \notin \Sigma_G$ (hence, in particular, $q \neq p$), S a scheme of finite type over k with $\dim(S) \leq d$, and $\xi : S \to H_{G_0,g,r}$ a k-morphism. Then there exists an integer $N = N(p, \mathbf{G}, g, r, k, S, \xi)$, such that $S_n(k) = \emptyset$ for any $n \geq N$. Here, we set $S_n \stackrel{\text{def}}{=} S \times_{H_{G_0,g,r}} H_{G_n,g,r}$.*

Observe that (MT$_d$) implies (MT)(iii) for (g, r) with $3g - 3 + r \leq d$. (MT$_0$) (hence, (MT)(iii) for $(g, r) = (0, 3)$ and (MT)(i)(ii) for $r = 3$) follows immediately from Theorem 1.1. Now, the second main result of this paper is:

Theorem B. (MT$_1$) *holds.* (*In particular,* (MT)(iii) *for* $(g, r) = (0, 4), (1, 1)$ *and* (MT)(i)(ii) *for $r = 4$ hold.*)

Theorem A in characteristic 0 and the deduction Theorem A $\Longrightarrow$ Theorem B in arbitrary characteristics are main results of [CT]. In this paper, we shall give a proof of Theorem A in arbitrary characteristics, eventually in §4. Before that, we shall collect some preliminaries in the next §.

§3. Preliminaries.

In this §, we collect various arithmetico-geometric preliminaries for the proof of Theorem A in the next §. They were not needed in [CT], but are needed here to cover arbitrary characteristics. Some of them may be of some interest independent of the proof of Theorem A.

⟨*Non-Tate characters*⟩

Fix a prime p and let k be a field of characteristic $q \neq p$. The following was proved in [CT].

Lemma 3.1. *For any finitely generated extension K of k, $\chi : \Gamma_k \to \mathbf{Z}_p^*$ is non-Tate if and only if $\chi|_{\Gamma_K} : \Gamma_K \to \mathbf{Z}_p^*$ is non-Tate. Here, we set $\chi|_{\Gamma_K} \stackrel{\text{def}}{=} \chi \circ |_{k^{\text{sep}}}$, where $|_{k^{\text{sep}}} : \Gamma_K \to \Gamma_k$ stands for the restriction from K^{sep} to k^{sep} (with respect to a fixed embedding $k^{\text{sep}} \hookrightarrow K^{\text{sep}}$ over k).*

Here, we shall prove two more lemmas on non-Tate characters.

Lemma 3.2. *The following are all equivalent:*
(i) *The p-adic cyclotomic character $\chi_{\mathrm{cyc}} : \Gamma_k \to \mathbf{Z}_p^*$ has open image and is non-Tate.*
(ii) *For any finitely generated extension K of k and any character $\chi : \Gamma_K \to \mathbf{Z}_p^*$, $\chi(\Gamma_{K\overline{k}})$ is finite.*
(ii′) *For any finitely generated extension K of k and any (additive) character $\psi : \Gamma_K \to \mathbf{Z}_p$, $\psi(\Gamma_{K\overline{k}})$ is trivial.*

Proof. First, as $\mathbf{Z}_p^* \simeq \mathbf{Z}_p \times M$ with M finite, the equivalence (ii) $\iff$ (ii′) is clear.

Next, assume that (i) holds. We shall prove the assertion of (ii′) by induction on the transcendence degree d of K over k. If $d = 0$, the assertion is trivial. If $d > 0$, take a subextension k'/k of K/k with transcendence degree $d-1$. Replacing k' by its algebraic closure in K, we may assume that k' is algebraically closed in K, hence the natural map $\Gamma_K \to \Gamma_{k'}$ is surjective. Now, observe that the conditions of (i) are also satisfied when k is replaced by k'. Indeed, as the natural map $\Gamma_{k'} \to \Gamma_k$ has open image, the first condition is satisfied, and, by Lemma 3.1, the second condition is satisfied. Thus, if we assume the implication (i) $\Longrightarrow$ (ii′) for $d = 1$ (for k'), $\psi : \Gamma_K \to \mathbf{Z}_p$ factors through $\Gamma_K \twoheadrightarrow \Gamma_{k'}$, or, equivalently, induces a character $\overline{\psi} : \Gamma_{k'} \to \mathbf{Z}_p$. Applying the assumption of induction to the finitely generated extension k'/k and the character $\overline{\psi} : \Gamma_{k'} \to \mathbf{Z}_p$, we are done.

Thus, up to replacing K/k by K/k', it suffices to settle the case where $d = 1$ and k is algebraically closed in K. Let k^{perf} denote the perfect closure of k. Then, Kk^{perf} is regarded as the function field of a proper, smooth, geometrically connected curve $C_{k^{\mathrm{perf}}}$ over k^{perf}. Further, $C_{k^{\mathrm{perf}}} \to \mathrm{Spec}(k^{\mathrm{perf}})$ descends to a proper, smooth, geometrically connected curve $C_{\tilde{k}}$ over some finite subextension $\tilde{k}$ of k in k^{perf}, so that $K\tilde{k}$ is regarded as the function field of $C_{\tilde{k}}$. Now, by Lemma 3.1, we may replace K/k by $K\tilde{k}/\tilde{k}$ and assume that K is the function field of a proper, smooth, geometrically connected curve C over k.

For each closed point x of C, denote the residue field at x by $k(x)$ and let $\Gamma_K \supset D_x \supset I_x$ be the decomposition and the inertia subgroups (defined up to conjugacy). It is well-known that the maximal pro-p quotient I_x^p of I_x is isomorphic to $\mathbf{Z}_p$ (as $p \neq q$) and that the natural action of $\Gamma_{k(x)}$ on $I_x^p \simeq \mathbf{Z}_p$ (induced by the conjugate action of D_x on I_x) is via $\chi_{\mathrm{cyc}}|_{\Gamma_{k(x)}}$. Since the p-adic cyclotomic character $\chi_{\mathrm{cyc}} : \Gamma_k \to \mathbf{Z}_p^*$ has open image by assumption, so is $\chi_{\mathrm{cyc}}|_{\Gamma_{k(x)}}$, hence $(I_x^p)_{\Gamma_{k(x)}}$ is finite. (Here, for a (topological) group G and a (topological) G-module M, M_G denotes the coinvariant module of M, i.e., the maximal quotient of M on which G acts trivially.) Now, first, since $\mathbf{Z}_p$ is pro-p, $\psi|_{I_x} : I_x \to \mathbf{Z}_p$ factors through I_x^p. Next, since $\mathbf{Z}_p$ is abelian, $\psi|_{D_x}$ factors through D_x^{ab}, hence $\psi|_{I_x}$ factors through $(I_x^p)_{\Gamma_{k(x)}}$. Finally, since $\mathbf{Z}_p$ is torsion-free, $\psi|_{I_x}$ is trivial.

In summary, the image under ψ of the inertia subgroup at any closed point of C is trivial. Thus, $\psi : \Gamma_K \to \mathbf{Z}_p$ (resp. $\psi|_{\Gamma_{K\overline{k}}} : \Gamma_{K\overline{k}} \to \mathbf{Z}_p$) factors through $\Gamma_K \twoheadrightarrow \pi_1(C)$ (resp. $\Gamma_{K\overline{k}} \twoheadrightarrow \pi_1(C \times_k \overline{k})$). In particular, ψ induces a Γ_k-equivariant homomorphism $\psi_J : T_p(J) \to \mathbf{Z}_p$, where J is the Jacobian variety of C. Suppose that ψ_J is nontrivial. Then, by duality, we get a nontrivial Γ_k-equivariant homomorphism $\mathbf{Z}_p(1) \to T_p(J)$. This is absurd, since χ_{cyc} is assumed to be non-Tate. Therefore, ψ_J is trivial, or, equivalently, $\psi(\Gamma_{K\overline{k}})$ is trivial.

Finally, assume that (i) does not hold, or, equivalently, that either $\chi_{\mathrm{cyc}}(\Gamma_k) \subset \mathbf{Z}_p^*$ is not open or $\chi_{\rm cyc}$ is not non-Tate. In the first case, $\chi_{\rm cyc}(\Gamma_k)$ is finite, hence there exists a finite extension k' of k such that $\chi_{\rm cyc}(\Gamma_{k'})$ is trivial. This means that k' contains all p-power roots of unity of $\overline{k}$. Now, set $K \stackrel{\rm def}{=} k'(t)$, where t stands for an indeterminate. By Kummer theory, $K(t^{1/p^\infty}) \stackrel{\rm def}{=} \cup_{n\geq 0} K(t^{1/p^n})$ defines a $\mathbf{Z}_p$-extension of K, and the corresponding additive character

$$\psi : \Gamma_K \twoheadrightarrow \mathrm{Gal}(K(t^{1/p^\infty})/K) \simeq \mathbf{Z}_p$$

satisfies $\psi(\Gamma_{K\overline{k}}) = \mathbf{Z}_p$, which gives a counterexample for (ii$'$). In the second case, there exists an abelian variety A over k that admits an injective, Γ_k-equivariant homomorphism $\mathbf{Z}_p(1) \to T_p(A)$. By duality, we get a nontrivial, Γ_k-equivariant homomorphism $T_p(A^\vee) \to \mathbf{Z}_p$, where $A^\vee$ stands for the dual abelian variety of A. The latter Γ_k-equivariant homomorphism yields a homomorphism $\pi_1(A^\vee) \to \mathbf{Z}_p$ such that the image of $\pi_1(A^\vee_{\overline{k}})$ is nontrivial. So, $K \stackrel{\rm def}{=} k(A^\vee)$ gives a counterexample for (ii$'$). This completes the proof. $\square$

We say that k is p-arithmetic, if one (hence all) of the conditions in Lemma 3.2 is satisfied. Note that, if k is finitely generated over the prime field (of characteristic $q \neq p$), then k is p-arithmetic. Moreover, if k is finitely generated over a p-arithmetic field, then k is p-arithmetic.

Lemma 3.3. *Assume that k is p-arithmetic. Let T be a normal, integral scheme of finite type over k and $\chi_T : \pi_1(T) \to \mathbf{Z}_p^*$ a character. For each (not necessarily closed) point $t \in T$, denote by $\chi_t : \Gamma_{k(t)} \to \mathbf{Z}_p^*$ the character obtained by taking the composite of χ_T and the map $\Gamma_{k(t)} \to \pi_1(T)$ associated with the natural morphism* $\mathrm{Spec}(k(t)) \to T$ *with image t. Then, χ_{t_0} is non-Tate for some $t_0 \in T$ if and only if χ_t is non-Tate for all $t \in T$.*

Proof. Set $K \stackrel{\rm def}{=} k(T)$, and denote by $\tilde{k}$ the algebraic closure of k in K. Then, we see that $\tilde{k}$ is again p-arithmetic (as $\tilde{k}$ is a finite extension of k), and that the structure morphism $T \to \mathrm{Spec}(k)$ factors as the composite of a morphism $T \to \mathrm{Spec}(\tilde{k})$ and the natural morphism $\mathrm{Spec}(\tilde{k}) \to \mathrm{Spec}(k)$ (as T is normal). Moreover, T is geometrically connected over $\tilde{k}$ by definition. So, replacing k by $\tilde{k}$, we may assume that T is geometrically connected over k.

Then the natural map $\Gamma_{K\overline{k}} \to \pi_1(T \times_k \overline{k})$ is surjective. Indeed, by the functoriality property of π_1, we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma_{K\overline{k}} & \to & \pi_1(T \times_k \overline{k}) \\ \downarrow & & \downarrow \\ \Gamma_{Kk^{\rm sep}} & \to & \pi_1(T \times_k k^{\rm sep}), \end{array}$$

in which both vertical arrows are isomorphisms (cf. [GR], Exposé IX, Théorème 6.1). As T is normal and $k^{\rm sep}/k$ is separated, $T \times_k k^{\rm sep}$ is normal. Thus, the bottom horizontal arrow is surjective ([GR], Exposé V, Proposition 8.2), hence so is the top horizontal arrow, as desired.

Now, since k is p-arithmetic and K is finitely generated over k, we see that $\chi_T(\pi_1(T \times_k \overline{k}))$ is finite. Thus, there exists a connected finite étale cover $T' \to T$ such that $\chi_T(\pi_1(T' \times_{k'} \overline{k}))$ is trivial (where k' denotes the algebraic closure of k in $k(T')$), or, equivalently, that $\chi_{T'} \stackrel{\text{def}}{=} \chi_T|_{\pi_1(T')}$ factors through $\pi_1(T') \twoheadrightarrow \Gamma_{k'}$. Let $\chi_{k'}$ denote the character of $\Gamma_{k'}$ induced by $\chi_{T'}$.

Suppose that χ_{t_0} is non-Tate for some $t_0 \in T$, and take a point $t'_0 \in T'$ above t_0. Then $k'(t'_0)$ is a finite extension of $k(t_0)$, hence $\chi_{t'_0}$ is non-Tate. Here, $\chi_{t'_0}$ is defined to be the composite of the natural map $\Gamma_{k'(t'_0)} \to \pi_1(T')$ and $\chi_{T'} : \pi_1(T') \to \mathbf{Z}_p^*$, hence coincides with the composite of the natural map $\Gamma_{k'(t'_0)} \to \Gamma_{k'}$ and $\chi_{k'} : \Gamma_{k'} \to \mathbf{Z}_p^*$. Now, since $k'(t'_0)$ is finitely generated over k' and $\chi_{t'_0}$ is non-Tate, we conclude that $\chi_{k'}$ is non-Tate by Lemma 3.1.

Finally, let $t \in T$ be any point and take a point $t' \in T'$ above t. Since $\chi_{k'}$ is non-Tate, $\chi_{t'}$ is non-Tate, hence χ_t is non-Tate. This completes the proof. □

⟨*A variant of the Serre-Tate criterion*⟩

Let p be a prime. Let R be a discrete valuation ring and K the field of fractions of R. Assume that the characteristic of the residue field of R is not p. Thus, in particular, the characteristic of K is not p. Let I be the inertia subgroup (determined up to conjugacy) for R in Γ_K.

Let A be an abelian variety over K. Then Γ_K acts naturally on the p-adic Tate module $T_p(A)$ of A. The Serre-Tate criterion for good reduction of abelian varieties ([ST]) tells that A has good reduction over R if and only if I acts trivially on $T_p(A)$.

Now, we shall prove the following variant of this criterion under the assumption that K is finitely generated over the prime field. We need this extra assumption to resort to the semisimplicity and the Tate conjecture, which were proved by Tate, Zarhin and Mori in positive characteristic (cf. [MB], Chapitre XII) and by Faltings in characteristic 0 (cf. [FW], Chapter VI).

Proposition 3.4. *Assume moreover that A is nontrivial and K-simple and that K is finitely generated over the prime field. Then A has good reduction over R if and only if there exists a nontrivial Γ_K-submodule T of $T_p(A)$ on which I acts trivially.*

Proof. The 'only if' part immediately follows from the 'only if' part of the original Serre-Tate criterion. (Take, say, $T = T_p(A)$.) To see the 'if' part, we resort to the semisimplicity and the Tate conjecture for the Γ_K-module $V_p(A)$. As $V_p(A)$ is a semisimple Γ_K-module, there exist a finite number of simple Γ_K-submodules $W_1, \dots, W_r$ of $V_p(A)$ which are mutually non-isomorphic as Γ_K-modules and positive integers $n_1, \dots, n_r$, such that

$$V_p(A) = V_1 \oplus \cdots \oplus V_r, \quad V_i \simeq W_i^{\oplus n_i}$$

as Γ_K-modules. Then we have

$$\mathrm{End}_{\Gamma_K}(V_p(A)) \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $D_i \stackrel{\text{def}}{=} \mathrm{End}_{\Gamma_K}(W_i)$ is a division algebra, as W_i is a simple Γ_K-module. On the other hand, as a consequence of the Tate conjecture, we have

$$\mathrm{End}_{\Gamma_K}(V_p(A)) \simeq D \times_{\mathbf{Q}} \mathbf{Q}_p,$$

where $D \stackrel{\text{def}}{=} \mathrm{End}_K(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a division algebra, as A is a K-simple abelian variety. Let F_i (resp. F) denote the center of D_i (resp. D), which is a finite extension field of $\mathbf{Q}_p$ (resp. $\mathbf{Q}$). As

$$D \times_{\mathbf{Q}} \mathbf{Q}_p \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

we have

$$F \times_{\mathbf{Q}} \mathbf{Q}_p \simeq F_1 \times \cdots \times F_r.$$

Thus, in particular, giving an $F \times_{\mathbf{Q}} \mathbf{Q}_p$-module M with $\dim_{\mathbf{Q}_p}(M) < \infty$ is equivalent to giving an F_i-vector space M_i with $d_i \stackrel{\text{def}}{=} \dim_{F_i}(M_i) < \infty$ for each $i \in \{1, \ldots, r\}$: $M \simeq M_1 \oplus \cdots \oplus M_r$. Here, we shall refer to M_i as the F_i-component of M. (For example, V_i is the F_i-component of $V_p(A)$.) Moreover, observe that M is free as an $F \times_{\mathbf{Q}} \mathbf{Q}_p$-module (i.e., $M_1 \oplus \cdots \oplus M_r$ is free as an $(F_1 \times \cdots \times F_r)$-module) if and only if $d_1 = \cdots = d_r$.

We note that both $V_p(A)$ and $V_p(A)^I$ are known to be free $F \times_{\mathbf{Q}} \mathbf{Q}_p$-modules (see, e.g., [T1], Lemmas (2.1) and (2.2)). Thus, the quotient $V_p(A)/V_p(A)^I$ is also a free $F \times_{\mathbf{Q}} \mathbf{Q}_p$-module.

Now, suppose that I acts trivially on a nontrivial Γ_K-submodule T of $T_p(A)$, and set $W = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \subset V_p(A)$. Take any simple Γ_K-submodule of W, then it is necessarily isomorphic to W_{i_0} for some $i_0 \in \{1, \ldots, r\}$. Thus, I acts trivially on W_{i_0}, hence on $V_{i_0} \simeq W_{i_0}^{\oplus n_{i_0}}$. Namely, we have $V_{i_0} \subset V_p(A)^I$, or, equivalently, the dimension of the F_{i_0}-component of $V_p(A)/V_p(A)^I$ is 0. Since $V_p(A)/V_p(A)^I$ is free as an $F \times_{\mathbf{Q}} \mathbf{Q}_p$-module, this implies that the dimension of the F_i-component of $V_p(A)/V_p(A)^I$ is 0 for all $i \in \{1, \ldots, r\}$. Namely, $V_p(A)/V_p(A)^I$ is trivial, or, equivalently, I acts trivially on $V_p(A)$. Now, by the original Serre-Tate criterion, A has good reduction, as desired. □

⟨*A consequence of Mordell's conjecture over function fields*⟩

We mean by a curve a separated, normal, geometrically integral, 1-dimensional scheme over a field, and by a proper curve a curve which is proper over the base field. (Observe that a curve in this sense is generically smooth over the base field in general, and smooth if the base field is perfect.) Mordell's conjecture over function fields, proved by Manin and Grauert in characteristic 0 and by Samuel in positive characteristic, is summarized as follows. Here, for an extension k/F of fields and a k-scheme S, we say that S is F-trivial, if there exists an F-scheme S_F such that S is k-isomorphic to $S_F \times_F k$, and say that S is F-isotrivial, if the $\overline{k}$-scheme $S \times_k \overline{k}$ is $\overline{F}$-trivial.

Theorem 3.5. *Let F be an algebraically closed field of characteristic $q \geq 0$, $\mathbf{F}$ the prime field of F, and k a field finitely generated over F. Let C be a proper curve over k, and assume that the normalization of $C \times_k \overline{k}$ is of genus ≥ 2. Then at least one of the following holds:*
(i) *$C(k)$ is finite;*
(ii) *there exists a curve C_F over F, such that C is k-isomorphic to $C_F \times_F k$ (i.e., C is F-trivial) and that, under the identification $C = C_F \times_F k$, $C(k) \smallsetminus C_F(F)$ is finite;*
(iii) *$q > 0$ and there exist a finite extension k' of k, a finite subfield $\mathbf{F}'$ of F, and a curve $C_{\mathbf{F}'}$ over $\mathbf{F}'$, such that $C \times_k k'$ is k'-isomorphic to $C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$. Moreover, given*

such $(k', \mathbf{F}', C_{\mathbf{F}'})$ *and an identification* $C \times_k k' = C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$*, there exists a finite subset* $\Xi \subset C(k')$*, such that*

$$C(k) \subset \{\phi_{\mathbf{F}'}^n(x) \mid x \in \Xi,\ n \geq 0\} \cup C_{\mathbf{F}'}(F) \subset C_{\mathbf{F}'}(k') = C(k'),$$

where $\phi_{\mathbf{F}'} : C_{\mathbf{F}'} \to C_{\mathbf{F}'}$ *denotes the* $|\mathbf{F}'|$*-th power Frobenius endomorphism.*

Proof. For $q = 0$, see [Ma1], [Gra]. (See also [Sa], Théorème in the introduction.) For $q > 0$, this follows from [Sa]. More specifically, we may assume that $C(k)$ is infinite. Then, by Théorème 6, loc. cit., C is smooth over k, and, by Théorème 5, loc. cit., we see that either (ii) or (iii) holds. (More precisely, for case (iii), Théorème 5, b), loc. cit., ensures that the assertion holds if $(k', \mathbf{F}', C_{\mathbf{F}'})$ is replaced by $(k'', \mathbf{F}'', C_{\mathbf{F}'} \times_{\mathbf{F}'} \mathbf{F}'')$, where k''/k is a certain (Galois) subextension of k'/k and $\mathbf{F}''/\mathbf{F}'$ is a (sufficiently large) finite subextension of $F/\mathbf{F}'$. Now, observe that the assertion for $(k'', \mathbf{F}'', C_{\mathbf{F}'} \times_{\mathbf{F}'} \mathbf{F}'')$ is stronger than that for $(k', \mathbf{F}', C_{\mathbf{F}'})$.) $\square$

In the case of fields finitely generated over prime fields, we have the following stronger result (due to Faltings for $q = 0$):

Theorem 3.6. *Let* $\mathbf{F}$ *be the prime field of characteristic* $q \geq 0$*, and* k *a field finitely generated over* $\mathbf{F}$*. Let* C *be a proper curve over* k*, and assume that the normalization of* $C \times_k \overline{k}$ *is of genus* ≥ 2*. Then at least one of the following holds:*
(i) $C(k)$ *is finite;*
(ii) $q > 0$ *and there exist a finite extension* k' *of* k*, a finite subfield* $\mathbf{F}'$ *of* k'*, and a curve* $C_{\mathbf{F}'}$ *over* $\mathbf{F}'$*, such that* $C \times_k k'$ *is* k'*-isomorphic to* $C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$*. Moreover, given such* $(k', \mathbf{F}', C_{\mathbf{F}'})$ *and an identification* $C \times_k k' = C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$*, there exists a finite subset* $\Xi \subset C(k')$*, such that*

$$C(k) \subset \{\phi_{\mathbf{F}'}^n(x) \mid x \in \Xi,\ n \geq 0\} \subset C_{\mathbf{F}'}(k') = C(k'),$$

where $\phi_{\mathbf{F}'} : C_{\mathbf{F}'} \to C_{\mathbf{F}'}$ *denotes the* $|\mathbf{F}'|$*-th power Frobenius endomorphism.*

Proof. For $q = 0$, this is a theorem of Faltings ([FW], Chapter VI, Theorem 3). For $q > 0$, suppose that $C(k)$ is infinite. Then $C(k\overline{\mathbf{F}})$ is infinite, a fortiori. By applying Theorem 3.5 to the curve $C \times_k k\overline{\mathbf{F}}$ over the field $k\overline{\mathbf{F}}$ finitely generated over the algebraically closed field $\overline{\mathbf{F}}$, we conclude that we are in the situation of either (ii) or (iii) of Theorem 3.5. In fact, case (ii) cannot occur. Indeed, if case (ii) occurs, then there exists a curve $C_{\overline{\mathbf{F}}}$ over $\overline{\mathbf{F}}$, such that $C \times_k k\overline{\mathbf{F}}$ is $k\overline{\mathbf{F}}$-isomorphic to $C_{\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} k\overline{\mathbf{F}}$ and that (under the identification $C \times_k k\overline{\mathbf{F}} = C_{\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} k\overline{\mathbf{F}}$) $C(k\overline{\mathbf{F}}) \smallsetminus C_{\overline{\mathbf{F}}}(\overline{\mathbf{F}})$ is finite. Further, the $k\overline{\mathbf{F}}$-isomorphism $C \times_k k\overline{\mathbf{F}} \simeq C_{\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} k\overline{\mathbf{F}}$ descends to $k\mathbf{F}'$ for some finite extension $\mathbf{F}'$ of $\mathbf{F}$. More precisely, there exists a finite extension $\mathbf{F}'$ of $\mathbf{F}$ and a curve $C_{\mathbf{F}'}$ over $\mathbf{F}'$ such that $C_{\overline{\mathbf{F}}}$ is $\overline{\mathbf{F}}$-isomorphic to $C_{\mathbf{F}'} \times_{\mathbf{F}'} \overline{\mathbf{F}}$ and that $C \times_k k\mathbf{F}'$ is $k\mathbf{F}'$-isomorphic to $C_{\mathbf{F}'} \times_{\mathbf{F}'} k\mathbf{F}'$. Now, (under a suitable identification) we have

$$C(k) \cap C_{\overline{\mathbf{F}}}(\overline{\mathbf{F}}) \subset C(k\mathbf{F}') \cap C_{\mathbf{F}'}(\overline{\mathbf{F}}) = C_{\mathbf{F}'}(k\mathbf{F}' \cap \overline{\mathbf{F}}) = C_{\mathbf{F}'}(\mathbf{F}'_1),$$

where $\mathbf{F}'_1$ is the algebraic closure of $\mathbf{F}$ in $k\mathbf{F}'$, which is a finite extension of $\mathbf{F}'$. Thus,

$$C(k) = (C(k) \smallsetminus C_{\overline{\mathbf{F}}}(\overline{\mathbf{F}})) \cup (C(k) \cap C_{\overline{\mathbf{F}}}(\overline{\mathbf{F}})) \subset (C(k\overline{\mathbf{F}}) \smallsetminus C_{\overline{\mathbf{F}}}(\overline{\mathbf{F}})) \cup C_{\mathbf{F}'}(\mathbf{F}'_1)$$

is finite, which contradicts the assumption.

Finally, assume that case (iii) of Theorem 3.5 occurs. Then, first, there exist a finite extension k'_1 of $k\overline{\mathbf{F}}$, a finite subfield $\mathbf{F}'$ of $\overline{\mathbf{F}}$, and a curve $C_{\mathbf{F}'}$ over $\mathbf{F}'$, such that $C \times_k k'_1$ is k'_1-isomorphic to $C_{\mathbf{F}'} \times_{\mathbf{F}'} k'_1$. This isomorphism descends to one over a finite extension k' of $k\mathbf{F}'$ included in k'_1, as desired. Second, suppose that we are given k', $\mathbf{F}'$, $C_{\mathbf{F}'}$ and $C \times_k k' = C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$ as in the second assertion of (ii). Then Theorem 3.5 ensures that there exists a finite subset $\Xi_1 \subset C(k'\overline{\mathbf{F}})$, such that

$$C(k\overline{\mathbf{F}}) \subset \{\phi^n_{\mathbf{F}'}(x) \mid x \in \Xi_1,\ n \geq 0\} \cup C_{\mathbf{F}'}(\overline{\mathbf{F}}) \subset C_{\mathbf{F}'}(k'\overline{\mathbf{F}}) = C(k'\overline{\mathbf{F}}).$$

Let $\mathbf{F}''$ denote the algebraic closure of $\mathbf{F}'$ in k'. Considering the action of the Galois group $\mathrm{Gal}(k'\overline{\mathbf{F}}/k') \simeq \mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F}'')$ and taking the Galois-invariant parts, we obtain

$$C(k) \subset C(k\mathbf{F}'') \subset \{\phi^n_{\mathbf{F}'}(x) \mid x \in \Xi_0,\ n \geq 0\} \cup C_{\mathbf{F}'}(\mathbf{F}'') \subset C_{\mathbf{F}'}(k') = C(k'),$$

where $\Xi_0 \stackrel{\text{def}}{=} \Xi_1 \cap C(k')$. (Observe that $x \in \Xi_1$ is $\mathrm{Gal}(k'\overline{\mathbf{F}}/k')$-invariant if and only if so is $\phi^n_{\mathbf{F}'}(x)$.) Thus, the finite subset $\Xi \stackrel{\text{def}}{=} \Xi_0 \cup C_{\mathbf{F}'}(\mathbf{F}'')$ of $C_{\mathbf{F}'}(k') = C(k')$ has the desired properties. $\square$

The following consequence of Theorem 3.6 will be used in next §.

Proposition 3.7. *Let* $\mathbf{F}$ *be the prime field of characteristic* $q \geq 0$*, and* k *a field finitely generated over* $\mathbf{F}$*. Let* C *be a proper curve over* k*, and assume that the normalization of* $C \times_k \overline{k}$ *is of genus* ≥ 2*. Let* S *be a nonempty open subscheme of* C (*which is a curve over* k)*. When* $S(k)$ *is infinite, put the extra assumption that* S *is* $\mathbf{F}$*-isotrivial.* (*Note that* C *is automatically* $\mathbf{F}$*-isotrivial by* Theorem 3.6.) *Then there exists an* $\mathbf{F}$*-morphism* $f : \mathcal{S} \to T$ *between separated, normal, integral schemes of finite type over* $\mathbf{F}$*, such that the following hold:* (a) *the function field* $\mathbf{F}(T)$ *of* T *is* $\mathbf{F}$*-isomorphic to* k*;* (b) *under the identification* $\mathbf{F}(T) = k$*,* S *is* k*-isomorphic to the generic fiber* $\mathcal{S}_k$ *of* f*; and* (c) *under the identification* $S = \mathcal{S}_k$*, we have* $S(k) = \mathcal{S}(T)$*, i.e., each element of* $S(k) = \mathcal{S}_k(k)$ *uniquely extends to an element of* $\mathcal{S}(T)$*.*

Proof. Since k is a finitely generated extension of $\mathbf{F}$, there exists a separated (or even affine), normal, integral scheme T of finite type over $\mathbf{F}$, such that $k = \mathbf{F}(T)$. On the other hand, fix a finite k-morphism $C \to \mathbf{P}^1_k$, and define $\mathcal{C}$ to be the normalization of $\mathbf{P}^1_T$ in (the function field of) C. Then, as C is normal and finite over $\mathbf{P}^1_k$, we have $\mathcal{C}_k = C$. Set $D \stackrel{\text{def}}{=} C \setminus S$, and define $\mathcal{D}$ to be the topological closure of D in $\mathcal{C}$. Set $\mathcal{S} \stackrel{\text{def}}{=} \mathcal{C} \setminus \mathcal{D}$, which is separated, normal, integral and of finite type over $\mathbf{F}$. Then it is easy to see that the $\mathbf{F}$-morphism $\mathcal{S} \to T$ thus constructed satisfies (a) and (b). Further, if we consider the base change from T to a nonempty open subscheme U, all the conditions are preserved. Thus, it suffices to find U such that $f_U : \mathcal{S}_U \stackrel{\text{def}}{=} \mathcal{S} \times_T U \to U$ satisfies (c).

Assume first that $S(k)$ is a finite set $\{x_1, \ldots, x_r\}$. Each $x_i : \mathrm{Spec}(k) \to S = \mathcal{S}_k$ defines a rational map $T \dashrightarrow \mathcal{S}$, or, more precisely, there exists a nonempty open subscheme U_i of T, such that x_i (uniquely) extends to a morphism $U_i \to \mathcal{S}$ over T. Now, it is easy to see that $U \stackrel{\text{def}}{=} U_1 \cap \cdots \cap U_r$ has the desired property.

Next, assume that $S(k)$ is infinite. Then $C(k)$ is infinite, a fortiori, hence, by Theorem 3.6, $q > 0$ and there exist a finite extension k' of k, a finite subfield $\mathbf{F}'$ of k', a curve $C_{\mathbf{F}'}$ over $\mathbf{F}'$, such that $C \times_k k'$ is k'-isomorphic to $C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$. On the other hand, by our extra assumption, S is also $\mathbf{F}$-isotrivial, hence, by replacing k' and $\mathbf{F}'$ by suitable finite extensions, we may also assume that there exists a curve $S_{\mathbf{F}'}$ over $\mathbf{F}'$, such that $S \times_k k'$ is k'-isomorphic to $S_{\mathbf{F}'} \times_{\mathbf{F}'} k'$. Further, define $C_{1,\mathbf{F}'}$ to be the smooth compactification of $S_{\mathbf{F}'}$ (note that $\mathbf{F}'$ is perfect). Since both $C_{\mathbf{F}'} \times_{\mathbf{F}'} \overline{k} = C \times_k \overline{k}$ and $C_{1,\mathbf{F}'} \times_{\mathbf{F}'} \overline{k} \supset S_{\mathbf{F}'} \times_{\mathbf{F}'} \overline{k} = S \times_k \overline{k}$ are smooth compactifications of $S \times_k \overline{k}$, they are canonically $\overline{k}$-isomorphic to each other. This $\overline{k}$-isomorphism descends uniquely to an $\overline{\mathbf{F}}$-isomorphism between $C_{\mathbf{F}'} \times_{\mathbf{F}'} \overline{\mathbf{F}}$ and $C_{1,\mathbf{F}'} \times_{\mathbf{F}'} \overline{\mathbf{F}}$. Thus, up to replacing $\mathbf{F}'$ by a finite extension, we may assume that this $\overline{\mathbf{F}}$-isomorphism descends to an $\mathbf{F}'$-isomorphism between $C_{\mathbf{F}'}$ and $C_{1,\mathbf{F}'}$. In particular, we obtain an open immersion $S_{\mathbf{F}'} \hookrightarrow C_{\mathbf{F}'}$ over $\mathbf{F}'$ which is compatible (over k') with the original open immersion $S \hookrightarrow C$ over k.

Let T' be the normalization of T in k' and denote by π the natural finite morphism $T' \to T$. Let $\mathcal{S}'$ be the normalization of $\mathcal{S}$ in the function field of $S \times_k k'$. On the other hand, set $\mathcal{S}'_1 \stackrel{\text{def}}{=} S_{\mathbf{F}'} \times_{\mathbf{F}'} T'$. The generic fibers of the morphisms of finite type $\mathcal{S}' \to T'$ and $\mathcal{S}'_1 \to T'$ are $S \times_k k'$ and $S_{\mathbf{F}'} \times_{\mathbf{F}'} k'$, respectively, which are identified with each other. Accordingly, there exists a nonempty open subscheme V' of T' over which these two families coincide with each other.

Now, by Theorem 3.6, there exists a finite subset $\Xi \subset C(k')$, such that, under the identification $C \times_k k' = C_{\mathbf{F}'} \times_{\mathbf{F}'} k'$, we have

$$C(k) \subset \{\phi_{\mathbf{F}'}^n(x) \mid x \in \Xi,\ n \geq 0\} \subset C_{\mathbf{F}'}(k') = C(k').$$

From this, we obtain

$$S(k) \subset \{\phi_{\mathbf{F}'}^n(x) \mid x \in \Xi_S,\ n \geq 0\} \subset S_{\mathbf{F}'}(k') = S(k'),$$

where $\Xi_S \stackrel{\text{def}}{=} \Xi \cap S(k')$. (Observe $(\phi_{\mathbf{F}'})^{-1}(S_{\mathbf{F}'}) = S_{\mathbf{F}'}$.)

Write $\Xi_S = \{x_1, \ldots, x_r\}$. Each $x_i : \operatorname{Spec}(k') \to S \times_k k' = \mathcal{S}'_{V'} \times_{V'} k'$ (where $\mathcal{S}'_{V'} \stackrel{\text{def}}{=} \mathcal{S}' \times_{T'} V'$) defines a rational map $V' \dashrightarrow \mathcal{S}'_{V'}$, or, more precisely, there exists a nonempty open subscheme V'_i of V', such that x_i (uniquely) extends to a morphism $V'_i \to \mathcal{S}'_{V'}$ over V'. Set $U'_1 \stackrel{\text{def}}{=} V'_1 \cap \cdots \cap V'_r$, then x_i extends to a morphism $U'_1 \to \mathcal{S}'_{V'}$ over V' for all $i = 1, \ldots, r$. Moreover, as $U'_1 \subset V'$, we have $\mathcal{S}'_{V'} = \mathcal{S}'_{1,V'} \stackrel{\text{def}}{=} S_{\mathbf{F}'} \times_{\mathbf{F}'} V'$, from which we conclude that $\phi_{\mathbf{F}'}^n(x_i)$ extends to a morphism $U'_1 \to \mathcal{S}'_{V'}$ over V' for all $i = 1, \ldots, r$ and all $n \geq 1$. Thus, any element of $S(k) \subset S(k')$ extends (uniquely) to a morphism $U'_1 \to \mathcal{S}'_{V'}$ over V', or, equivalently, a morphism $U'_1 \to \mathcal{S}'_{U'_1}$ over U'_1.

Finally, set $U \stackrel{\text{def}}{=} T \smallsetminus \pi(T' \smallsetminus U'_1)$ and $U' \stackrel{\text{def}}{=} \pi^{-1}(U)$. Now, any element $x \in S(k) \subset S(k')$ extends to a morphism $U' \to \mathcal{S}'_{U'} \to \mathcal{S}_U$. By the following Lemma 3.8, this implies that x extends to a morphism $U \to \mathcal{S}_U$, as desired.

Lemma 3.8. *Let U be a separated, normal, integral scheme, k the function field of U, k' an algebraic extension of k, and U' the normalization of U in k'. Then, in the* *category of separated schemes, the following diagram is co-cartesian:*

$$\begin{array}{ccc} \mathrm{Spec}(k') & \to & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ U' & \to & U. \end{array}$$

Namely, for any separated scheme Y, the natural map $Y(U) \to Y(k) \times_{Y(k')} Y(U')$ is a bijection.

Proof. Let k'' be the normal closure of k' over k and U'' the normalization of U' in k''. In the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(k'') & \to & \mathrm{Spec}(k') & \to & \mathrm{Spec}(k) \\ \downarrow & & \downarrow & & \downarrow \\ U'' & \to & U' & \to & U, \end{array}$$

if the left square and the big rectangle are co-cartesian, then the right square is also co-cartesian. Thus, replacing k' by k'', we may assume that k'/k is a normal extension. Set $G \stackrel{\text{def}}{=} \mathrm{Aut}(k'/k)$. Then G induces an action on U' over U, and the underlying topological space of U can be regarded as the quotient space of U' by this G-action. Indeed, this is well-known set-theoretically. As for the topology, note that the natural morphism $\pi : U' \to U$ is closed.

Let $f_k : \mathrm{Spec}(k) \to Y$ and $f_{U'} : U' \to Y$ be morphisms whose restrictions to $\mathrm{Spec}(k')$ coincide with each other. For each $\sigma \in G$, the restriction of $f_{U'} \circ \sigma$ to $\mathrm{Spec}(k')$ coincides with that of $f_{U'}$. Since Y is separated, this implies that $f_{U'} \circ \sigma = f_{U'}$. Thus, by the preceding argument, there exists a continuous map $\phi : U \to Y$, such that $f_{U'} = \phi \circ \pi$. Now, considering suitable affine open neighborhoods of $x \in U$ and $\phi(x) \in Y$ for each $x \in U$, we may reduce the problem to the case that U (hence also U') and Y are affine.

So, write $U = \mathrm{Spec}(A)$, $U' = \mathrm{Spec}(A')$ and $Y = \mathrm{Spec}(R)$. In this case, the assertion is equivalent to saying that the natural map

$$\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, k) \times_{\mathrm{Hom}(R,k')} \mathrm{Hom}(R, A')$$

is a bijection. But this is a consequence of the equality $A = k \cap A'$ in k', which follows from the fact that A' is integral over A and that A is integrally closed. □

Thus, the proof of Proposition 3.7 is completed. □

Remark 3.9. In Proposition 3.7, the extra assumption that S is $\mathbf{F}$-isotrivial, when $S(k)$ is infinite, cannot be removed. Indeed, take a proper curve T of genus ≥ 2 over a finite extension $\mathbf{F}'$ of $\mathbf{F}$, set $\mathcal{C} \stackrel{\text{def}}{=} T \times_{\mathbf{F}'} T$ and $\mathcal{S} \stackrel{\text{def}}{=} \mathcal{C} \setminus \Delta$, where Δ stands for the diagonal. Moreover, set $k \stackrel{\text{def}}{=} \mathbf{F}(T) = \mathbf{F}'(T)$, and define C and S to be the generic fibers of the natural morphisms $\mathcal{C} \to T$ and $\mathcal{S} \to T$ obtained by the second projection pr. Then we may identify $C(k) = \mathcal{C}(T) = \mathrm{Hom}_{\mathbf{F}'}(T,T)$ and $S(k) = \mathrm{Hom}_{\mathbf{F}'}(T,T) \setminus \{\mathrm{id}_T\}$. (Here, the natural identification $C(k) = \mathcal{C}(T)$ is ensured essentially by the valuative criterion for properness.) Under these identifications, consider the subset $\{\phi^n \mid n > 0\} \subset S(k)$, where $\phi = \phi_{\mathbf{F}'} : T \to T$ is the $|\mathbf{F}'|$-th power Frobenius endomorphism. Let U be an open subscheme of T. Then $\phi^n \in S(k)$ extends to an element of $\mathcal{S}(U)$ if and only if $U \subset T \setminus \mathrm{pr}(\Gamma_{\phi^n} \cap \Delta)$, where Γ_{ϕ^n} stands for the graph of ϕ^n. However, $\mathrm{pr}(\Gamma_{\phi^n} \cap \Delta)$ coincides with the set of closed points of T that are $\mathbf{F}'_n$-rational, where $\mathbf{F}'_n$ denotes the unique degree n extension of $\mathbf{F}'$. Thus, there does not exist a nonempty open subscheme U of T such that $\phi^n \in S(k)$ extends to an element of $\mathcal{S}(U)$ for any $n \geq 0$. (Such a U must be contained in the one-point set consisting of the generic point!)

§4. Proof of Theorem A.

Before starting the proof of Theorem A, we shall recall the main geometric result of [CT]. Let k be a field of characteristic $q \neq p$, S a smooth, separated, geometrically connected curve (necessarily of finite type) over k, η the generic point of S, and $K = k(\eta)$ the function field of S. Let A be an abelian scheme over S such that the generic fiber A_η is of dimension d. Since $A \to S$ is an abelian scheme and $q \neq p$, $A[p^n] = \mathrm{Ker}([p^n] : A \to A)$ is finite étale over S. Thus, the natural action of the absolute Galois group Γ_K of K on $A_\eta[p^n](\overline{K})$ ($n \geq 0$) factors through $\pi_1(S)$. This, in turn, defines actions of $\pi_1(S)$ on $A_\eta[p^\infty](\overline{K})$ and on the Tate module $T_p(A_\eta)$. We denote by $\rho = \rho_{A,p}$ the corresponding representation $\pi_1(S) \to \mathrm{Aut}_{\mathbf{Z}_p}(T_p(A_\eta))$.

For each $v \in A_\eta[p^\infty](\overline{K})$, write $\pi_1(S)_v \subset \pi_1(S)$ for the stabilizer of v. This is an open subgroup of $\pi_1(S)$, and, by Galois theory, corresponds to a connected finite étale cover $S_v \to S$ (defined over a finite extension k_v/k). We denote by g_v the genus of the smooth compactification of $S_v \times_{k_v} \overline{k}$. Finally, denote by $(A_\eta)_0$ the largest abelian subvariety of A_η which is isogenous to a k-isotrivial abelian variety.

Theorem 4.1. *Assume that k is algebraically closed. Then, for any $c \geq 0$, there exists an integer $N = N(p,k,S,A,c) \geq 0$ such that, for all $v \in A_\eta[p^\infty](\overline{K})$, either $g_v \geq c$ or $p^N v \in (A_\eta)_0$.*

Theorem 4.1 was proved in [CT] in arbitrary characteristics. There, roughly speaking, we estimate the genus by observing the Galois representation $\pi_1(S) \to GL(T_p(A_\eta))$ and using the Riemann-Hurwitz genus formula. Here, the main ingredient to estimate the ramification terms in the genus formula is the Serre-Oesterlé theorem ([Se][O]) on the asymptotic behavior of the number of points on reduction modulo p^n of p-adic analytic subsets of $\mathbf{Z}_p^m$. For more details, see [CT].

In [CT], we deduced Theorem A in characteristic 0 from Theorem 4.1. More specifically, we introduce a sequence of (disconnected) finite étale covers $\{S_{n,\chi}\}_{n\geq 0}$ of S with the property that $s_n \in S_{n,\chi}(k)$ lying above $s \in S(k)$ corresponds to an element of order exactly p^n in $A_s[p^\infty](\chi)$. It is thus enough to prove that $S_{n,\chi}(k) = \emptyset$ for $n \gg 0$. According to Theorem 4.1, we may reduce this problem, roughly speaking, to either the case where A_η is isotrivial or the case where the genus of each component of $S_{n,\chi}$ is ≥ 2 for $n \gg 0$. The proof of [CT] for the first case works well in arbitrary characteristics, while that for the second case, which resorts to Mordell's conjecture (Faltings' finiteness theorem) and a compactness argument to produce a projective system of rational points, fails in positive characteristic as it is, since a curve of genus ≥ 2 over a field finitely generated over the (finite) prime field may admit infinitely many rational points. To remedy this, we take a model T of the finitely generated base field k, consider models of the base scheme S and the abelian scheme A over T, and produce a projective system of rational points on the fiber at a suitable closed point of T (whose residue field is finite). Now, to make this argument work, we need various extra arguments resorting to the results of §3. See below for more details.

Proof of Theorem A. We divide the proof into several steps.

Step 1. First reductions. First, (pUB$_0$) follows from the definition of non-Tate characters. (Indeed, if S is of finite type over k and $\dim(S) = 0$, $S(k)$ is a finite set. So, we may treat only finitely many abelian varieties A_s ($s \in S(k)$).)

Next, to prove (pUB$_1$), assume that S is of finite type over k and $\dim(S) = 1$. By replacing S by S^{red}, we may assume that S is reduced. By (pUB$_0$), we may replace S by an open dense subscheme freely. So, we may assume that S is regular and separated. Further, treating S componentwise, we may assume that S is connected. We may also assume that $S(k)$ is nonempty, since otherwise there is nothing to do. Since S is regular and 1-dimensional, any point of $S(k)$ is a smooth point, hence the smooth locus of S is nonempty (and open). Thus, again by replacing S by an open dense subscheme, we may assume that S is smooth and separated. Finally, since S is smooth, connected with $S(k) \neq \emptyset$, S is geometrically connected. Thus, in summary, we may assume that S is a smooth, separated, geometrically connected curve over k.

For each $n \geq 0$, set $\chi_n \stackrel{\text{def}}{=} \chi \bmod p^n : \Gamma_k \to (\mathbf{Z}/p^n)^*$. Set $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then, up to replacing k by the fixed field of $\mathrm{Ker}(\chi_{i(p)})$ in k^{sep}, one may assume that $\chi_{i(p)} : \Gamma_k \to (\mathbf{Z}/p^{i(p)})^*$ is trivial. (Here, we have used the fact that the restriction of a non-Tate character to an open subgroup is non-Tate. See Lemma 3.1.) This technical reduction ensures that $\mathrm{Im}(\chi_n) \subset (\mathbf{Z}/p^n)^*$ is contained in the order $p^{n-i(p)}$ cyclic subgroup $1 + p^{i(p)}\mathbf{Z}/p^n\mathbf{Z}$ of $(\mathbf{Z}/p^n)^*$, when $n \geq i(p)$.

Step 2. Relation with rational points on various covers. For each $v_n \in A_\eta[p^n]^*(\overline{K})$ ($n \geq 0$), we shall define a connected finite étale cover $S_{v_n,\chi}$ of S. To do this, write $\pi_1(S)_{\langle v_n \rangle}$ for the stabilizer of $\langle v_n \rangle = (\mathbf{Z}/p^n) \cdot v_n$ under $\pi_1(S)$ and $S_{\langle v_n \rangle} \to S$ for the resulting connected finite étale cover (defined over a finite extension $k_{\langle v_n \rangle}/k$). Consider the projection morphism $\mathrm{pr}_{\langle v_n \rangle} : \pi_1(S_{\langle v_n \rangle}) \to \Gamma_k$ (whose image coincides with $\Gamma_{k_{\langle v_n \rangle}}$) and the natural representation $\rho_{\langle v_n \rangle} : \pi_1(S_{\langle v_n \rangle}) \to \mathrm{Aut}_{\mathbf{Z}/p^n}(\langle v_n \rangle)$. These, together with $\chi_n = \chi \bmod p^n$, define a representation

$$\rho_{v_n,\chi} : \pi_1(S_{\langle v_n \rangle}) \to \mathrm{Aut}_{\mathbf{Z}/p^n}(\langle v_n \rangle),\ \gamma \mapsto \chi_n(\mathrm{pr}_{\langle v_n \rangle}(\gamma))^{-1}\rho_{\langle v_n \rangle}(\gamma).$$

Now, define $S_{v_n,\chi} \to S_{\langle v_n \rangle}$ to be the connected finite étale Galois cover (defined over a finite extension $k_{v_n,\chi}/k_{\langle v_n \rangle}$) corresponding to the open normal subgroup $\mathrm{Ker}(\rho_{v_n,\chi}) \subset \pi_1(S_{\langle v_n \rangle})$, and denote by $g_{v_n,\chi}$ the genus of the smooth compactification of $S_{v_n,\chi} \times_{k_{v_n,\chi}} \overline{k}$.

Lemma 4.2. (i) $S_{v_n,\chi} \times_{k_{v_n,\chi}} \overline{k} = S_{v_n} \times_{k_{v_n}} \overline{k}$ *as covers of* $S \times_k \overline{k}$. *In particular,* $g_{v_n,\chi} = g_{v_n}$ *is independent of* χ.
(ii) *For any* k*-rational point* $s : \mathrm{Spec}(k) \to S$*, consider the specialization isomorphism*

$$\mathrm{sp}_s : A_\eta[p^\infty](\overline{K})\ (\xleftarrow{\sim} A[p^\infty](\overline{K})) \xrightarrow{\sim} A_s[p^\infty](\overline{k}).$$

Then $\mathrm{sp}_s(v_n) \in A_s[p^\infty](\chi)$ *if and only if* $s : \mathrm{Spec}(k) \to S$ *lifts to a* k*-rational point* $s_{v_n,\chi} : \mathrm{Spec}(k) \to S_{v_n,\chi}$.

Proof. For (i), just observe that

$$\begin{aligned}\pi_1(S_{v_n,\chi} \times_{k_{v_n,\chi}} \overline{k}) &= \mathrm{Ker}(\rho_{v_n,\chi}) \cap \mathrm{Ker}(\mathrm{pr}_{\langle v_n\rangle}) \\ &= \mathrm{Ker}(\rho_{\langle v_n\rangle}) \cap \mathrm{Ker}(\mathrm{pr}_{\langle v_n\rangle}) = \pi_1(S_{v_n} \times_{k_{v_n}} \overline{k}).\end{aligned}$$

For (ii), denote again by s the section $\Gamma_k \hookrightarrow \pi_1(S)$ of $\pi_1(S) \twoheadrightarrow \Gamma_k$ induced (up to conjugacy) by $s : \mathrm{Spec}(k) \to S$, which identifies Γ_k with the decomposition group at s. Then the existence of the lift $s_{v_n,\chi} : \mathrm{Spec}(k) \to S_{v_n,\chi}$ of $s : \mathrm{Spec}(k) \to S$ is equivalent to the inclusion $s(\Gamma_k) \subset \pi_1(S_{v_n,\chi}) (= \mathrm{Ker}(\rho_{v_n,\chi}))$, which can be rewritten as $s(\sigma)\cdot v_n = \chi(\sigma) v_n$ ($\sigma \in \Gamma_k$) or, applying the specialization isomorphism, as $\sigma \cdot \mathrm{sp}_s(v_n) = \chi(\sigma)\mathrm{sp}_s(v_n)$. $\square$

Now, we shall introduce a projective system $(S_{n,\chi})_{n\geq 0}$ of (disconnected) finite étale covers of S. For each $n \geq 0$, define

$$S_{n,\chi} \stackrel{\mathrm{def}}{=} \coprod_{v_n \in A_\eta[p^n]^*(\overline{K})} S_{v_n,\chi}.$$

Observe that $(S_{n,\chi})_{n\geq 0}$ forms a projective system with transition maps induced by the canonical morphisms $S_{v_n,\chi} \to S_{pv_n,\chi}$ over k. At the level of k-rational points, we have:

Claim 4.3. (i) $\varprojlim S_{n,\chi}(k) = \emptyset$.
(ii) *The assertion of Theorem A is equivalent to saying that* $S_{n,\chi}(k) = \emptyset$ *for any* $n \gg 0$.
(iii) *Suppose that* $S_{n,\chi}(k) \neq \emptyset$ *for any* $n \geq 0$. *Then there exists an element* $(v_n)_{n\geq 0} \in \varprojlim A_\eta[p^n]^*(\overline{K})$, *such that* $S_{v_n,\chi}(k) \neq \emptyset$ *for any* $n \geq 0$.

Indeed, for (i), suppose that $\varprojlim S_{n,\chi}(k) \neq \emptyset$ and take $(s_n)_{n\geq 0} \in \varprojlim S_{n,\chi}(k)$. Then, by the definition of $S_{n,\chi}$, there exists an element $(v_n)_{n\geq 0} \in \varprojlim A_\eta[p^n]^*(\overline{K})$, such that $(s_n)_{n\geq 0} \in \varprojlim S_{v_n,\chi}(k)$. Set $s \stackrel{\mathrm{def}}{=} s_0$. Then, by Lemma 4.2(ii), $\mathrm{sp}_s(v_n) \in A_s[p^n](\chi)$ for all $n \geq 0$. Thus, Γ_k acts on $(\mathrm{sp}_s(v_n))_{n\geq 0} \in T_p(A_s)^*$ via χ, which contradicts the assumption that χ is non-Tate. For (ii), again by Lemma 4.2(ii), the assertion of Theorem A is equivalent to saying that there exists an $N \geq 0$ such that $S_{v_n,\chi}(k) \neq \emptyset \implies n \leq N$, hence also to saying that $S_{n,\chi}(k) = \emptyset$ for any $n \gg 0$. (iii) follows from the fact that $A_\eta[p^n]^*(\overline{K})$ is finite for each $n \geq 0$.

Step 3. Second reductions. Now, suppose that the assertion of Theorem A fails for our abelian scheme $A \to S$. Then, by Claim 4.3(ii)(iii), there exists an element $v = (v_n)_{n\geq 0} \in \varprojlim A_\eta[p^n]^*(\overline{K})$, such that $S_{v_n,\chi}(k) \neq \emptyset$ for any $n \geq 0$. We fix such a

$v = (v_n)_{n\geq 0}$. Let K_n be the function field of $S_{v_n,\chi}$ and $\tilde{K} = \tilde{K}_{A,v}$ the union of K_n: $\tilde{K} \stackrel{\text{def}}{=} \varinjlim K_n$.

To execute the proof of Theorem A in arbitrary characteristics, we need some more reductions. First, we may replace $A \to S$ by $A \times_S S_{v_m,\chi} \to S_{v_m,\chi}$ for any $m \geq 0$. Indeed, v_n can be regarded as an element of $(A \times_S S_{v_m,\chi})[p^n]^*(\overline{K_m})$, and we have

$$(S_{v_m,\chi})_{v_n,\chi} = \begin{cases} S_{v_m,\chi}, & n < m, \\ S_{v_n,\chi}, & n \geq m. \end{cases}$$

Thus, $(S_{v_m,\chi})_{v_n,\chi}(k) \neq \emptyset$ for any $n \geq 0$. In particular, we may assume that $\mathrm{End}_{\tilde{K}}(A_\eta \times_K \tilde{K}) = \mathrm{End}_K A_\eta$, by replacing S by $S_{v_m,\chi}$ for $m \gg 0$. (Indeed, since $\mathrm{End}_{\tilde{K}}(A_\eta \times_K \tilde{K}) \subset \mathrm{End}_{\overline{K}}(A_\eta \times_K \overline{K})$ is a finitely generated (abelian) group, all elements of $\mathrm{End}_{\tilde{K}}(A_\eta \times_K \tilde{K})$ are already defined over K_m for some $m \geq 0$.)

Second, the generic fiber A_η of $A \to S$ is decomposed up to isogeny into a direct product of K-simple abelian varieties. Namely, we have a K-isogeny $A_\eta \to A_\eta^{(1)} \times \cdots \times A_\eta^{(r)}$, where $A_\eta^{(i)}$ is a K-simple abelian variety for $i = 1, \ldots, r$. We have more: as a consequence of the above first reduction step, $A_\eta^{(i)} \times_K \tilde{K}$ is $\tilde{K}$-simple. Let $v^{(i)} = (v_n^{(i)})_{n\geq 0}$ be the image of v in $T_p(A_\eta^{(i)}) = \varprojlim A_\eta^{(i)}[p^n](\overline{K})$. As the natural map $T_p(A_\eta) \to T_p(A_\eta^{(1)}) \times \cdots \times T_p(A_\eta^{(r)})$ is injective, there exists an $i_0 = 1, \ldots, r$ such that $v^{(i_0)} \neq 0$. Since the natural map $A_\eta[p^\infty](\overline{K}) \to A_\eta^{(i_0)}[p^\infty](\overline{K})$ is surjective, the action of Γ_K on $A_\eta^{(i_0)}[p^\infty](\overline{K})$ factors through $\pi_1(S)$, hence $A_\eta^{(i_0)}$ has good reduction everywhere on S by the (original) Serre-Tate criterion (cf. §3), or, equivalently, can be regarded as the generic fiber of a (unique) abelian scheme $A^{(i_0)}$ over S. Now, since the natural map $A_\eta[p^\infty](\overline{K}) \to A_\eta^{(i_0)}[p^\infty](\overline{K})$ is $\pi_1(S)$-equivariant, we obtain a k-morphism $S_{v_n,\chi} \to S_{v_n^{(i_0)},\chi}$ naturally. This implies that $S_{v_n^{(i_0)},\chi}(k) \neq \emptyset$ for any $n \geq 0$ and that $\tilde{K}_{A^{(i_0)},v^{(i_0)}} \subset \tilde{K}_{A,v}$. Now, replacing (A, v) by $(A^{(i_0)}, p^{-a}v^{(i_0)})$, where $a \geq 0$ is defined to satisfy $v^{(i_0)} \in p^a(T_p(A_\eta^{(i_0)})^*)$, we may assume that A_η is $\tilde{K}$-simple, and, a fortiori, K-simple. Then, in particular, either $(A_\eta)_0 = A_\eta$ (Case 1) or $(A_\eta)_0 = 0$ (Case 2).

Step 4. Case 1: $(A_\eta)_0 = A_\eta$. For each $n \geq 0$, there exists a k-rational point $s_n \in S_{v_n,\chi}(k)$, which yields a splitting $s_n : \Gamma_k \hookrightarrow \pi_1(S_{v_n,\chi}) \subset \pi_1(S)$ of the restriction epimorphism $\pi_1(S) \twoheadrightarrow \Gamma_k$. Let Δ, Γ and Σ_{s_n} denote the images in $\mathrm{Aut}_{\mathbf{Z}_p}(T_p(A_\eta))$ of $\pi_1(S \times_k \overline{k})$, $\pi_1(S)$ and $s_n(\Gamma_k)$, respectively, under ρ. Since $\pi_1(S) = s_n(\Gamma_k) \cdot \pi_1(S \times_k \overline{k})$, we have $\Gamma = \Sigma_{s_n} \cdot \Delta$.

As $A_\eta = (A_\eta)_0$ is isogenous to an isotrivial abelian variety, Δ is finite. Now, $\Gamma \subset \mathrm{Aut}_{\mathbf{Z}_p}(T_p(A_\eta)) \simeq \mathrm{GL}_{2d}(\mathbf{Z}_p)$ ($d \stackrel{\text{def}}{=} \dim(A_\eta)$) is a compact p-adic Lie group, hence, in particular, it is finitely generated. Since a finitely generated profinite group admits only finitely many open subgroups of given bounded index and since $[\Gamma : \Sigma_{s_n}] \leq |\Delta| < \infty$, there are only finitely many possibilities for the $\Sigma_{s_n} \subset \Gamma$, $n \geq 0$. Thus, there exists $s \in S(k)$ such that $\Sigma_{s_n} = \Sigma_s$ for infinitely many $n \geq 0$. Write $|\Delta| = p^a m$ with $p \not| m$. Then we have:

Claim 4.4. *Let $s,\ t \in S(k)$ with $\Sigma_s = \Sigma_t$. If $\mathrm{sp}_t(v_n) \in A_t[p^n](\chi)$, then $\mathrm{sp}_s(p^a v_n) \in A_s[p^n](\chi)$.*

Indeed, the statement is trivial for $n \leq a$, so assume that $n > a$ and write $\delta(\sigma) \stackrel{\text{def}}{=} \rho(s(\sigma)t(\sigma)^{-1}) \in \Delta$. Also, since $\rho(s(\sigma)) \in \Sigma_s = \Sigma_t$, there exists $\tau = \tau_\sigma \in \Gamma_k$ such that $\rho(s(\sigma)) = \rho(t(\tau))$. As a result, one obtains $\delta(\sigma)v_n = \chi_n(\tau)\chi_n(\sigma^{-1})v_n$. In particular, the order of $\chi_n(\tau)\chi_n(\sigma^{-1}) \in (\mathbf{Z}/p^n)^*$ divides the order of $\delta(\sigma) \in \Delta$, hence divides the order $p^a m$ of Δ. On the other hand, by the assumption on χ put in Step 1, $\chi_n(\tau)\chi_n(\sigma^{-1})$ lies in the order $p^{n-i(p)}$ cyclic subgroup $1 + p^{i(p)}\mathbf{Z}/p^n\mathbf{Z}$ of $(\mathbf{Z}/p^n)^*$, when $n \geq i(p)$. Now, it follows that $\chi_n(\tau)\chi_n(\sigma^{-1}) \in 1 + p^{n-a}\mathbf{Z}/p^n$. Thus, we have

$$\rho(s(\sigma))p^a v_n = \chi_n(\tau)p^a v_n = \chi_n(\sigma)p^a v_n,$$

which completes the proof of Claim 4.4.

It follows from Claim 4.4 that, up to replacing v_n by $p^a v_{n+a}$, one may assume that $s : \mathrm{Spec}(k) \to S$ lifts to a k-rational point $s_{v_n} : \mathrm{Spec}(k) \to S_{v_n,\chi}$ for infinitely many $n \geq 0$, hence $\varprojlim S_{v_n,\chi}(k) \neq \emptyset$. This contradicts Claim 4.3(i).

Step 5. Case 2: $(A_\eta)_0 = 0$ — *reductions.* In this case, by Theorem 4.1 and Lemma 4.2(i), there exists an integer $N \geq 0$, such that $g_{v_n,\chi} \geq 2$ for $n > N$. Replacing $A \to S$ by $A \times_S S_{v_{N+1},\chi} \to S_{v_{N+1},\chi}$, we may assume that the genus g of the smooth compactification of $S \times_k \overline{k}$ is ≥ 2. Indeed, as we have already seen, $\tilde{K}_{A,v} = \tilde{K}_{A\times_S S_{v_{N+1},\chi},v}$, so that after this reduction A_η is still $\tilde{K}$-simple (hence, in particular, K-simple). Now, we may make one more reduction. Let C be the normal compactification of S and A_C the Néron model of A_η over C. Note that A is naturally identified with $A_C \times_C S$. Now, define $S^\sim$ to be the subset of points of C at which the fiber of $A_C \to C$ is an abelian variety. Thus, $S \subset S^\sim \subset C$, and we may regard $S^\sim$ as an open subscheme of C. Then we have $S_{v_n,\chi} \subset (S^\sim)_{v_n,\chi}$ for each $n \geq 0$. So, replacing $A \to S$ by $A_C \times_C S^\sim \to S^\sim$, we may assume that S coincides with the set of points of C at which A_η has good reduction.

If $S_{v_n,\chi}(k) \neq \emptyset$ is finite for $n \gg 0$, we have $\varprojlim S_{v_n,\chi}(k) \neq \emptyset$, which contradicts Claim 4.3(i). (In particular, this, together with Faltings' theorem (cf. Theorem 3.6), already completes the proof in characteristic 0, as in [CT].) So, we may assume that $S_{v_n,\chi}(k) \neq \emptyset$ is infinite for all $n \geq 0$.

Step 6. Case 2: $(A_\eta)_0 = 0$ — *isotriviality of* S. Our strategy is to apply Proposition 3.7, to extend all the objects in question over k to ones over a suitable model T of k over the prime field $\mathbf{F}$, to consider the fibers at a (fixed) closed point of T, and apply the above projective limit argument to the finite base field case. To do this, however, we have to check the extra assumption in Proposition 3.7 that (not only C but also) S is $\mathbf{F}$-isotrivial.

Let C_n denote the normal compactification of $S_{v_n,\chi}$ (so, $C_0 = C$), which is a proper curve over k. Then $\{C_n\}_{n\geq 0}$ naturally forms a projective system. By Theorem 3.6, C_n is $\mathbf{F}$-isotrivial, or, more explicitly, there exists a curve $C_{n,\overline{\mathbf{F}}}$ over $\overline{\mathbf{F}}$ such that $C_n \times_k \overline{k}$ is $\overline{k}$-isomorphic to $C_{n,\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} \overline{k}$. Moreover, under the identification $C_n \times_k \overline{k} = C_{n,\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} \overline{k}$, the finite $\overline{k}$-morphism $C_{n+1} \times_k \overline{k} \to C_n \times_k \overline{k}$ uniquely descends to a finite $\overline{\mathbf{F}}$-morphism $C_{n+1,\overline{\mathbf{F}}} \to C_{n,\overline{\mathbf{F}}}$. (See [T2], Lemma (1.32).) We define $S_{n,\overline{\mathbf{F}}}$ to be the image of $S_{v_n,\chi} \times_k \overline{k}$ in $C_{n,\overline{\mathbf{F}}}$.

Claim 4.5. *For each $n \geq 0$, $S_{n,\overline{\mathbf{F}}}$ is open in $C_{n,\overline{\mathbf{F}}}$, hence is regarded as an open subscheme of $C_{n,\overline{\mathbf{F}}}$. Moreover, the finite $\overline{\mathbf{F}}$-morphism $C_{n+1,\overline{\mathbf{F}}} \to C_{n,\overline{\mathbf{F}}}$ restricts to a finite, étale $\overline{\mathbf{F}}$-morphism $S_{n+1,\overline{\mathbf{F}}} \to S_{n,\overline{\mathbf{F}}}$.*

Indeed, first, as the projection $\varpi_n : C_n \times_k \overline{k} = C_{n,\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} \overline{k} \to C_{n,\overline{\mathbf{F}}}$ is an open map ([Gro], Corollaire (2.4.10)), $S_{n,\overline{\mathbf{F}}}$ is open. Next, consider the following cartesian diagram:

$$\begin{array}{ccc} C_{n+1} \times_k \overline{k} & \overset{\varpi_{n+1}}{\to} & C_{n+1,\overline{\mathbf{F}}} \\ f_{\overline{k}} \downarrow & & \downarrow f_{\overline{\mathbf{F}}} \\ C_n \times_k \overline{k} & \overset{\varpi_n}{\to} & C_{n,\overline{\mathbf{F}}} \end{array}$$

From this, we first see that

$$\begin{aligned} f_{\overline{\mathbf{F}}}(S_{n+1,\overline{\mathbf{F}}}) &= f_{\overline{\mathbf{F}}}(\varpi_{n+1}(S_{v_{n+1},\chi} \times_k \overline{k})) \\ &= \varpi_n(f_{\overline{k}}(S_{v_{n+1},\chi} \times_k \overline{k})) \\ &= \varpi_n(S_{v_n,\chi} \times_k \overline{k}) = S_{n,\overline{\mathbf{F}}}. \end{aligned}$$

Namely, $f_{\overline{\mathbf{F}}} : C_{n+1,\overline{\mathbf{F}}} \to C_{n,\overline{\mathbf{F}}}$ restricts to a surjective $\overline{\mathbf{F}}$-morphism $S_{n+1,\overline{\mathbf{F}}} \to S_{n,\overline{\mathbf{F}}}$. Moreover, as

$$\begin{aligned} f_{\overline{k}}^{-1}(S_{v_n,\chi} \times_k \overline{k}) &= S_{v_{n+1},\chi} \times_k \overline{k} \\ &\subset (S_{v_n,\chi} \times_k \overline{k}) \times_{S_{n,\overline{\mathbf{F}}}} S_{n+1,\overline{\mathbf{F}}} \\ &\subset f_{\overline{k}}^{-1}(S_{v_n,\chi} \times_k \overline{k}), \end{aligned}$$

we must have $S_{v_{n+1},\chi} \times_k \overline{k} = (S_{v_n,\chi} \times_k \overline{k}) \times_{S_{n,\overline{\mathbf{F}}}} S_{n+1,\overline{\mathbf{F}}}$. Namely, the diagram

$$\begin{array}{ccc} S_{v_{n+1},\chi} \times_k \overline{k} & \overset{\varpi_{n+1}}{\to} & S_{n+1,\overline{\mathbf{F}}} \\ f_{\overline{k}} \downarrow & & \downarrow f_{\overline{\mathbf{F}}} \\ S_{v_n,\chi} \times_k \overline{k} & \overset{\varpi_n}{\to} & S_{n,\overline{\mathbf{F}}} \end{array}$$

is cartesian. As $\varpi_n : C_n \times_k \overline{k} \to C_{n,\overline{\mathbf{F}}}$ is regarded as a base change of the morphism $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(\overline{\mathbf{F}})$, it is affine (hence quasi-compact) and flat. Since the open immersion $S_{v_n,\chi} \times_k \overline{k} \hookrightarrow C_n \times_k \overline{k}$ is also quasi-compact and flat, we conclude that $\varpi_n : S_{v_n,\chi} \times_k \overline{k} \to S_{n,\overline{\mathbf{F}}}$ is quasi-compact and flat. Moreover, it is surjective by definition. In summary, it is "fpqc", hence, by descent theory, the finite-étaleness of $f_{\overline{k}} : S_{v_{n+1,\chi}} \times_k \overline{k} \to S_{v_{n,\chi}} \times_k \overline{k}$ implies that of $f_{\overline{\mathbf{F}}} : S_{n+1,\overline{\mathbf{F}}} \to S_{n,\overline{\mathbf{F}}}$. Thus, the proof of Claim 4.5 is completed.

A reformulation of Claim 4.5 in terms of fundamental groups is as follows: for each $n \geq 0$, there exists a subgroup $H_n \subset \pi_1(S_{\overline{\mathbf{F}}})$ (where $S_{\overline{\mathbf{F}}} \overset{\mathrm{def}}{=} S_{0,\overline{\mathbf{F}}}$) such that the stabilizer subgroup $\pi_1(S \times_k \overline{k})_{v_n}$ at v_n is the inverse image of H_n under $\pi_1(\varpi_0) : \pi_1(S \times_k \overline{k}) \twoheadrightarrow \pi_1(S_{\overline{\mathbf{F}}})$. Moreover, for each $g \in \pi_1(S \times_k \overline{k})$, we have

$$\pi_1(S \times_k \overline{k})_{gv_n} = g\pi_1(S \times_k \overline{k})_{v_n} g^{-1} = g\pi_1(\varpi_0)^{-1}(H_n)g^{-1} = \pi_1(\varpi_0)^{-1}(\overline{g}H_n\overline{g}^{-1}),$$

where $\overline{g} \stackrel{\text{def}}{=} \pi_1(\varpi_0)(g)$. From this, we conclude that the action of $\pi_1(S \times_k \overline{k})$ on the subset $\pi_1(S \times_k \overline{k})v_n \subset A_\eta[p^n](\overline{K})$ factors through $\pi_1(S \times_k \overline{k}) \twoheadrightarrow \pi_1(S_{\overline{\mathbf{F}}})$. This further implies that the actions of $\pi_1(S \times_k \overline{k})$ on the submodules $\langle \pi_1(S \times_k \overline{k})v_n \rangle \subset A_\eta[p^n](\overline{K})$ and $T \stackrel{\text{def}}{=} \langle \pi_1(S \times_k \overline{k})v \rangle \subset T_p(A_\eta)$ also factor through $\pi_1(S \times_k \overline{k}) \twoheadrightarrow \pi_1(S_{\overline{\mathbf{F}}})$. In particular, these actions factor through $\pi_1(S \times_k \overline{k}) \twoheadrightarrow \pi_1(S_{\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} \overline{k})$. Namely, for each point x of $S_{\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} \overline{k}$, the inertia subgroup $I = I_x$ acts trivially on T. Now, by Proposition 3.4, A_η has good reduction at any such x. Recall that, in Step 5, we put the assumption that S coincides with the set of points of C at which A_η has good reduction. Thus, we conclude $S \times_k \overline{k} = S_{\overline{\mathbf{F}}} \times_{\overline{\mathbf{F}}} \overline{k}$. In particular, S is $\mathbf{F}$-isotrivial, as desired.

Step 7. Case 2: $(A_\eta)_0 = 0$ — application of Proposition 3.7 and end of proof. Now, we may apply Proposition 3.7 to obtain an $\mathbf{F}$-morphism $f : \mathcal{S} \to \mathcal{T}$ between separated, normal, integral schemes of finite type over $\mathbf{F}$, such that the following hold: (a) the function field $\mathbf{F}(\mathcal{T})$ of $\mathcal{T}$ is $\mathbf{F}$-isomorphic to k; (b) under the identification $\mathbf{F}(\mathcal{T}) = k$, S is k-isomorphic to the generic fiber $\mathcal{S}_k$ of f; and (c) under the identification $S = \mathcal{S}_k$, we have $S(k) = \mathcal{S}(\mathcal{T})$, i.e., each element of $S(k) = \mathcal{S}_k(k)$ uniquely extends to an element of $\mathcal{S}(\mathcal{T})$. Moreover, the abelian scheme A over $S = \mathcal{S}_k$ extends to one over an open subscheme of $\mathcal{S}$. More precisely, there exists an open subscheme $\mathcal{U}$ of $\mathcal{S}$ containing $S = \mathcal{S}_k$ and an abelian scheme $A_\mathcal{U}$ over $\mathcal{U}$, such that $A_\mathcal{U} \times_\mathcal{U} S$ is S-isomorphic to A (as abelian schemes). By definition, $f(\mathcal{S} \smallsetminus \mathcal{U})$ does not contain the generic point η of $\mathcal{T}$. As $f(\mathcal{S} \smallsetminus \mathcal{U})$ is constructible by Chevalley's theorem, the topological closure $Z \stackrel{\text{def}}{=} \overline{f(\mathcal{S} \smallsetminus \mathcal{U})}$ does not contain η. Now, replacing $\mathcal{T}$ by $\mathcal{T} \smallsetminus Z$ and $\mathcal{S}$ by $\mathcal{S} \times_\mathcal{T} (\mathcal{T} \smallsetminus Z)$, and considering $A_\mathcal{U} \times_\mathcal{U} (\mathcal{S} \times_\mathcal{T} (\mathcal{T} \smallsetminus Z))$, we may assume (keeping the validity of (a)-(c)) that there exists an abelian scheme $A_\mathcal{S}$ over $\mathcal{S}$, such that $A_\mathcal{S} \times_\mathcal{S} S$ is S-isomorphic to A (as abelian schemes). In particular, the action of $\pi_1(S)$ on $A_\eta[p^\infty](\overline{K})$ factors through the natural surjection $\pi_1(S) \twoheadrightarrow \pi_1(\mathcal{S})$.

As $\mathcal{T}$ is normal, the natural map $\Gamma_k \to \pi_1(\mathcal{T})$ is surjective ([GR], Exposé V, Proposition 8.2). By Lemma 3.2, (i) $\Longrightarrow$ (ii′), together with the assumption (put at the beginning of the proof of Theorem A) that the image of χ is contained in $1 + p^{i(p)}\mathbf{Z}_p \simeq \mathbf{Z}_p$, $\chi : \Gamma_k \to \mathbf{Z}_p^*$ factors through $\Gamma_k \twoheadrightarrow \Gamma_{\mathbf{F}'}$, where $\mathbf{F}'$ denotes the algebraic closure of $\mathbf{F}$ in k, hence, in particular, factors as $\Gamma_k \twoheadrightarrow \pi_1(\mathcal{T}) \stackrel{\chi_\mathcal{T}}{\to} \mathbf{Z}_p^*$.

As in Step 2, we obtain a connected finite étale cover $\mathcal{S}_{v_n,\chi_\mathcal{T}}$ of $\mathcal{S}$ for each $n \geq 0$, such that $\mathcal{S}_{v_n,\chi_\mathcal{T}} \times_\mathcal{S} S = S_{v_n,\chi}$. More precisely, write $\pi_1(\mathcal{S})_{\langle v_n \rangle}$ for the stabilizer of $\langle v_n \rangle$ under $\pi_1(\mathcal{S})$ and $\mathcal{S}_{\langle v_n \rangle} \to \mathcal{S}$ for the resulting connected finite étale cover. Consider the projection morphism $\mathrm{pr}_{\langle v_n \rangle} : \pi_1(\mathcal{S}_{\langle v_n \rangle}) \to \pi_1(\mathcal{T})$ and the natural representation $\rho_{\langle v_n \rangle} : \pi_1(\mathcal{S}_{\langle v_n \rangle}) \to \mathrm{Aut}_{\mathbf{Z}/p^n}(\langle v_n \rangle)$. These, together with $\chi_{\mathcal{T},n} = \chi_\mathcal{T} \bmod p^n$, define a representation

$$\rho_{v_n,\chi_\mathcal{T}} : \pi_1(\mathcal{S}_{\langle v_n \rangle}) \to \mathrm{Aut}_{\mathbf{Z}/p^n}(\langle v_n \rangle),\ \gamma \mapsto \chi_{\mathcal{T},n}(\mathrm{pr}_{\langle v_n \rangle}(\gamma))^{-1}\rho_{\langle v_n \rangle}(\gamma).$$

Now, define $\mathcal{S}_{v_n,\chi} \to \mathcal{S}_{\langle v_n \rangle}$ to be the connected finite étale Galois cover corresponding to the open normal subgroup $\mathrm{Ker}(\rho_{v_n,\chi_T}) \subset \pi_1(\mathcal{S}_{\langle v_n \rangle})$. Observe that $S_{v_n,\chi}(k) = \mathcal{S}_{v_n,\chi_T}(T)$ for each $n \geq 0$. Indeed, this follows from condition (c) (i.e., $S(k) = \mathcal{S}(T)$), together with the fact that T is normal.

Finally, fix a closed point t of T. Note that the residue field $k(t)$ at t is a finite field. Consider the fiber $A_{\mathcal{S}_t} \to \mathcal{S}_t \to \mathrm{Spec}(k(t))$ of $A_{\mathcal{S}} \to \mathcal{S} \to T$ at $t \in T$ and the specialization isomorphism $\mathrm{sp}_t : A_\eta[p^\infty](\overline{K}) \tilde{\to} A_{\mathcal{S}_t}[p^\infty](\overline{K_t})$, where K_t denote the function field of $\mathcal{S}_t$. Then it is easy to see that $\mathcal{S}_{v_n,\chi_T} \times_T \mathrm{Spec}(k(t)) \simeq (\mathcal{S}_t)_{\mathrm{sp}_t(v_n),\chi_t}$ over $k(t)$. Here, $\mathrm{Spec}(k(t)) \to T$ is the natural morphism with image t, and $\chi_t : \Gamma_{k(t)} \to \mathbf{Z}_p^*$ denotes the character obtained by taking the composite of $\chi_T : \pi_1(T) \to \mathbf{Z}_p^*$ and the natural map $\Gamma_{k(t)} \to \pi_1(T)$ associated with $\mathrm{Spec}(k(t)) \to T$. By Lemma 3.3, χ_t is non-Tate. As $\mathcal{S}_{v_n,\chi_T}(T) = S_{v_n,\chi}(k)$ is nonempty for any $n \geq 0$, so is $\mathcal{S}_{v_n,\chi_T}(k(t)) = (\mathcal{S}_t)_{\mathrm{sp}_t(v_n),\chi_t}(k(t))$. Now, as $k(t)$ is a finite field, $(\mathcal{S}_t)_{\mathrm{sp}_t(v_n),\chi_t}(k(t))$ is finite for any $n \geq 0$, hence we conclude $\varprojlim (\mathcal{S}_t)_{\mathrm{sp}_t(v_n),\chi_t}(k(t)) \neq \emptyset$, which contradicts Claim 4.3(i).

Thus, the proof of Theorem A is completed. $\square$

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