Realizations of the Elliptic Polylogarithm for CM elliptic curves

By

Kenichi Bannai[*], Shinichi Kobayashi[**] and Takeshi Tsuji[***]

Abstract

In these notes, we give an overview of our paper [BKT] which gives an explicit description of the de Rham and p-adic realizations of the elliptic polylogarithm, for a general elliptic curve defined over a subfield of $\mathbb{C}$ in the de Rham case and for a CM elliptic curve defined over its field of complex multiplication and with good reduction at the primes above $p \geq 5$ in the p-adic case. As explained in the appendix of [BKT], our method also gives a simple proof of the description of the real Hodge realization of the elliptic polylogarithm for a general elliptic curve defined over $\mathbb{C}$. In these notes, we introduce the real Hodge and p-adic cases in a parallel fashion to highlight the analogy.

§1. Introduction

The classical polylogarithm for any integer $k \geq 0$ was first defined as a function given by the power series

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

on the open unit disc around the origin. These functions may be expressed as an iterated integral, of the form

$$\mathrm{Li}_{k+1}(t) = \int_0^t \mathrm{Li}_k(s) \frac{ds}{s} \quad (k \geq 0), \qquad\qquad \mathrm{Li}_0(t) = \frac{t}{1-t},$$

---

[*]Department of Mathematics, Keio University.
[**]Institute of Mathematics, Tohoku University
[***]Graduate School of Mathematics, University of Tokyo.

which shows that these functions extend to multi-valued functions on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. By definition, the polylogarithm functions satisfy the differential equations

$$(1.1) \qquad\qquad d\operatorname{Li}_{k+1}(t) = \operatorname{Li}_k(t) \frac{dt}{t}, \qquad (k \geq 0).$$

These functions were interpreted by Deligne as period functions of a certain variation of mixed Hodge structures, called the polylogarithm sheaf, on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and a motivic interpretation of this sheaf was given by Beilinson and Deligne. The cyclotomic elements in motivic cohomology, which plays an important role in the proofs of the Beilinson's conjecture as well as the Tamagawa number conjecture for Dirichlet motives, may be reinterpreted in terms of the motivic polylogarithm. The polylogarithm sheaf also plays an important role in understanding the motivic formalism underlying Zagier's conjecture [BD].

The construction of the polylogarithm sheaf was subsequently extended to the case of elliptic curves minus the identity by Beilinson and Levin [BL]. The corresponding motivic sheaf reinterprets the Eisenstein classes in motivic cohomology, and the étale realization of the elliptic polylogarithm was used by Kings [Ki] to prove the Tamagawa number conjecture for CM elliptic curves.

The main purpose of [BKT] is to explicitly describe the p-adic realization for a CM elliptic curve defined over its field of complex multiplication and with good reduction at the primes above $p \geq 5$. The key in proving our result is the explicit description of the de Rham realization of the elliptic polylogarithm, for an elliptic curve defined over a subfield of $\mathbb{C}$. Namely, we construct a family of algebraic functions $L_n(z)$ on the elliptic curve which play a role analogous to that of $\operatorname{Li}_0(t) = t/(1 - t)$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We then use $L_n(z)$ to describe the connection on the module with connection underlying the elliptic polylogarithm sheaf. Similar results were obtained by Levin and Racinet [LR] Section 5.1.3 and Besser and Solomon [BS].

As explained in the appendix of [BKT], the explicit description of the algebraic connection allows us to explicitly determine the period functions of the real Hodge realization of the elliptic polylogarithm as solutions of certain iterated differential equations analogous to (1.1). Using this result, we may prove that the specializations of the real Hodge realization of the elliptic polylogarithm sheaf to points are expressed by special values of Eisenstein-Kronecker-Lerch series, which we call the Eisenstein-Kronecker numbers. This is a result originally proved by Beilinson-Levin [BL] and Wildeshaus [Wi] using a different method. Our method of calculating the elliptic polylogarithm sheaf in terms of solutions of iterated differential equations analogous to (1.1) closely parallels the classical case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the conceptual simplicity of this method allows the consideration of a p-adic analogue.

In the p-adic case, assume that the elliptic curve has complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K. Assume in addition that

E is defined over $\boldsymbol{K}$ and has good reduction at the primes above p. In this case, we construct a p-adic analogue of Eisenstein-Kronecker numbers using p-adic interpolation. We explicitly describe the p-adic realization of the elliptic polylogarithm sheaf, and prove that the specializations of this sheaf to torsion points prime to p (more precisely, prime to $\mathfrak{p}$) are related to p-adic Eisenstein-Kronecker numbers, proving a p-adic analogue of the result of Beilinson-Levin and Wildeshaus. This result is a generalization of the result of [Ba3], where we have dealt only with the one variable case for an ordinary prime. A similar result concerning the specializations in the two-variable case was obtained in [BKi], again for ordinary primes, using a very different method. The result of the current paper is valid even when p is supersingular.

The p-adic Eisenstein-Kronecker numbers are related to special values of p-adic L-functions associated to Hecke characters of imaginary quadratic fields. The p-adic elliptic polylogarithm is expected to be the image by the syntomic regulator of the motivic elliptic polylogarithm, our result may be interpreted as a p-adic analogue of Beilinson's conjecture.

In these notes, we introduce the real Hodge and p-adic realizations of the elliptic polylogarithm in a parallel fashion to highlight the analogy.

The authors would like to thank the organizers Mamoru Asada and Hiroaki Nakamura for the opportunity to present our research at this conference. The authors would also like to thank the referee for comments. Part of this research was conducted while the first author was visiting the École Normale Supérieure in Paris, and the second author the Institut de Mathématiques de Jussieu. The authors would like to thank their hosts Yves André and Pierre Colmez for hospitality.

§ 2.   Definition of the Elliptic Polylogarithm sheaf

In this section, we review the construction by Beilinson and Levin of the elliptic polylogarithm sheaf. The construction is valid for any suitable theory of mixed sheaves – even for the conjectural theory of mixed motivic sheaves. Hence the cohomology class of the realizations of the elliptic polylogarithm sheaf in absolute Hodge and rigid syntomic cohomologies may be regarded as the image by the regulator maps of the cohomology class of the motivic elliptic polylogarithm sheaf. The main goal of [BKT] is to explicitly describe the real Hodge and the p-adic realizations of the elliptic polylogarithm sheaf.

In the Hodge case, let $S = \operatorname{Spec} \mathbb{C}$, $K = \mathbb{R}$ and $F = \mathbb{C}$. For any smooth scheme of finite type over S, we let $\mathscr{S}(X)$ be the category of *variations of mixed $\mathbb{R}$-Hodge structures* on X. In the p-adic case, we let $S = \operatorname{Spec} \mathcal{O}_K$ for the ring of integers $\mathcal{O}_K$ of a finite unramified extension K of $\mathbb{Q}_p$, and $F = K$. For any smooth scheme of finite type X over S, with smooth compactification $\overline{X}$ over S such that the complement $D = \overline{X} \setminus X$

is a normal crossing divisor relative to S, we denote by $\mathscr{S}(X)$ the category of *filtered overconvergent F-isocrystals* (previously referred to as admissible syntomic coefficients. See [Ba1] Definition 1.13 for the definition) on X.

For suitable objects $\mathscr{F} \in \mathscr{S}(X)$, we may define the absolute and relative cohomologies $H^n_{\mathscr{A}}(X, \mathscr{F})$ and $H^n(X, \mathscr{F})$ of X with coefficients in $\mathscr{F}$. The absolute cohomology is a vector space over K, where as relative cohomology is an object in $\mathscr{S}(S)$. In the Hodge case, the absolute cohomology is *absolute Hodge cohomology* and the relative cohomology is *Betti cohomology* with its mixed Hodge structure. In the p-adic case, we take for absolute cohomology *rigid syntomic cohomology* and for relative cohomology *rigid cohomology* with its structure as a filtered Frobenius module (See [Bes1] for basic facts concerning rigid syntomic cohomology, and [Ba2] for interpretation of rigid syntomic cohomology as an absolute cohomology.)

For any integer n, the category $\mathscr{S}(X)$ contains the Tate object $K(n)$, and for any $\mathscr{F} \in \mathscr{S}(X)$, we let $\mathscr{F}(n) := \mathscr{F} \otimes_K K(n)$. In both the Hodge and the p-adic cases, we have a canonical isomorphism

$$(2.1) \qquad\qquad H^n_{\mathscr{A}}(X, \mathscr{F}) = \mathrm{Ext}^n_{\mathscr{S}(X)}(K(0), \mathscr{F})$$

for $n = 0, 1$, and a short exact sequence

$$(2.2) \qquad 0 \to H^1_{\mathscr{A}}(S, H^0(X, \mathscr{F})) \to H^1_{\mathscr{A}}(X, \mathscr{F}) \to H^0_{\mathscr{A}}(S, H^1(X, \mathscr{F})) \to 0$$

relating absolute and relative cohomologies.

We will next give the definition of the elliptic polylogarithm sheaf defined by Beilinson and Levin. We first take an elliptic curve E defined over S. Then the relative cohomology $H^1(E)$ in the Hodge case is given by the pure Hodge structure arising from the isomorphism

$$H^1_B(E(\mathbb{C}), \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^1_{\mathrm{dR}}(E/\mathbb{C}),$$

and the relative cohomology in the syntomic case is given by the filtered Frobenius module

$$H^1_{\mathrm{rig}}(E_k/K) \cong H^1_{\mathrm{dR}}(E/K),$$

where k is the residue field of $\mathcal{O}_K$ and the left hand side is the rigid cohomology of $E_k := E \otimes k$.

We let $\mathscr{H} = H^1(E)^{\vee}$, where $^{\vee}$ denotes the dual, and we denote by $\mathscr{H}_E$ the pullback of $\mathscr{H}$ to E by the structure morphism. Since $H^n(E, \mathscr{H}_E) = H^n(E) \otimes \mathscr{H}$, (2.2) gives the exact sequence

$$0 \to H^1_{\mathscr{A}}(S, \mathscr{H}) \to H^1_{\mathscr{A}}(E, \mathscr{H}_E) \to H^0_{\mathscr{A}}(S, \mathscr{H}^{\vee} \otimes \mathscr{H}) \to 0$$

which splits via the pull-back with respect to the identity $[0] : S \to E$. We define the first logarithm sheaf $\mathscr{L}\mathrm{og}^{(1)}$ to be a sheaf in $\mathscr{S}(E)$ whose extension class in $H^1_{\mathscr{A}}(E, \mathscr{H}_E)$

maps to the identity in $H^0_{\mathscr{A}}(S, \mathscr{H}^\vee \otimes \mathscr{H}) = \operatorname{Hom}_{\mathscr{S}(S)}(\mathscr{H}, \mathscr{H})$ and is zero when pulled-backed to $H^1_{\mathscr{A}}(S, \mathscr{H})$. We then define the logarithm sheaf $\mathscr{L}\mathrm{og}$ to be the pro-object

$$\mathscr{L}\mathrm{og} = \varprojlim_n \operatorname{Sym}^n \mathscr{L}\mathrm{og}^{(1)}$$

in $\mathscr{S}(E)$. From the definition, the graded module of $\mathscr{L}\mathrm{og}$ with respect to the weight filtration is given by $\operatorname{Gr}^W_{-n}(\mathscr{L}\mathrm{og}) = \operatorname{Sym}^n \mathscr{H}_E$ for integers $n \geq 0$. By definition of $\mathscr{L}\mathrm{og}^{(1)}$, the pull-back of $\mathscr{L}\mathrm{og}$ by the identity gives a splitting of the weight filtration

$$(2.3) \qquad\qquad [0]^* \mathscr{L}\mathrm{og} \cong \prod_{j=0}^{\infty} \operatorname{Sym}^j \mathscr{H}$$

which is unique due to reason of weights. The following result was proved by Beilinson and Levin.

Lemma 2.1. *The relative cohomology of the logarithm sheaf is given by*

$$H^2(E, \mathscr{L}\mathrm{og}) = K(-1), \qquad\qquad H^n(E, \mathscr{L}\mathrm{og}) = 0 \quad (n \neq 2).$$

Proof. The statement was originally proved in [BL]. See also [HK] Lemma A1.4 or [Ba3] Lemma 3.4. One first calculates the cohomology of $\operatorname{Sym}^n \mathscr{L}\mathrm{og}^{(1)}$ for $n \geq 0$ by induction on n, considering the long exact sequence for cohomology associated to the short exact sequence

$$0 \to \operatorname{Sym}^{n+1} \mathscr{H} \to \operatorname{Sym}^{n+1} \mathscr{L}\mathrm{og}^{(1)} \to \operatorname{Sym}^n \mathscr{L}\mathrm{og}^{(1)} \to 0.$$

Our result is obtained by considering the inverse limit with respect to n. $\qquad\square$

Let $D = [0]$ and $U = E \setminus [0]$. Then the residue sequence associated to the inclusion $U \hookrightarrow E$

$$\cdots \to H^n(E, \mathscr{L}\mathrm{og}(1)) \to H^n(U, \mathscr{L}\mathrm{og}(1)) \xrightarrow{\mathrm{res}} H^{n-1}(D, [0]^* \mathscr{L}\mathrm{og}) \to \cdots$$

for relative cohomology gives the isomorphism $H^0(U, \mathscr{L}\mathrm{og}(1)) \overset{\cong}{\Leftarrow} H^0(E, \mathscr{L}\mathrm{og}(1)) = 0$ and the short exact sequence

$$0 \to H^1(U, \mathscr{L}\mathrm{og}(1)) \xrightarrow{\mathrm{res}} H^0(D, [0]^* \mathscr{L}\mathrm{og}) \to H^2(E, \mathscr{L}\mathrm{og}(1)) \to 0.$$

Since $H^n(U, \mathscr{H}^\vee_U \otimes \mathscr{L}\mathrm{og}(1)) = H^n(U, \mathscr{L}\mathrm{og}) \otimes \mathscr{H}^\vee(1)$, the exact sequence (2.2) gives an isomorphism

$$(2.4) \qquad H^1_{\mathscr{A}}(U, \mathscr{H}^\vee_U \otimes \mathscr{L}\mathrm{og}(1)) \xrightarrow{\cong} H^0_{\mathscr{A}}(S, H^1(U, \mathscr{H}^\vee_U \otimes \mathscr{L}\mathrm{og}(1))).$$

By Lemma 2.1 and the fact that the weight of $\mathscr{H}^\vee$ is one, we have

$$H^0_{\mathscr{A}}(S, H^2(E, \mathscr{H}^\vee_E \otimes \mathscr{L}og(1))) = H^0_{\mathscr{A}}(S, \mathscr{H}^\vee) = 0.$$

Hence $H^0_{\mathscr{A}}(S, -)$ applied to the residue sequence gives an isomorphism

$$(2.5) \qquad H^0_{\mathscr{A}}(S, H^1(U, \mathscr{H}^\vee_U \otimes \mathscr{L}og(1))) \xrightarrow{\cong} H^0_{\mathscr{A}}(S, \mathscr{H}^\vee \otimes \mathscr{H}) = \mathrm{Hom}_{\mathscr{S}(S)}(\mathscr{H}, \mathscr{H}).$$

Here, the first isomorphism follows from the splitting (2.3), which implies that

$$H^0_{\mathscr{A}}(S, H^0(D, \mathscr{H}^\vee \otimes [0]^* \mathscr{L}og)) = H^0_{\mathscr{A}}(S, \mathscr{H}^\vee \otimes \prod_{j=0}^{\infty} \mathrm{Sym}^j \mathscr{H}) = H^0_{\mathscr{A}}(S, \mathscr{H}^\vee \otimes \mathscr{H})$$

due to reason of weights. We define the residue isomorphism

$$\mathrm{res} : H^1_{\mathscr{A}}(U, \mathscr{H}^\vee_U \otimes \mathscr{L}og(1)) \cong \mathrm{Hom}_{\mathscr{S}(S)}(\mathscr{H}, \mathscr{H})$$

to be the composition of (2.4) and (2.5).

Definition 2.2 (Beilinson-Levin). The elliptic polylogarithm sheaf $\mathscr{P}$ is defined to be an extension of $\mathscr{L}og(1)$ by $\mathscr{H}$ in $\mathscr{S}(U)$, whose extension class $[\mathscr{P}]$ in

$$\mathrm{Ext}^1_{\mathscr{S}(U)}(\mathscr{H}_U, \mathscr{L}og(1)) = H^1_{\mathscr{A}}(U, \mathscr{H}^\vee_U \otimes \mathscr{L}og(1))$$

maps to the identity through the residue isomorphism.

The main object of [BKT] is to explicitly describe $\mathscr{P}$ in the p-adic case, using functions obtained as solutions of certain iterated differential equations.

§ 3.   The Kronecker theta function and the connection function

In this section, we will construct the connection functions $L_n(z)$, which are rational functions on the elliptic curve which will be used to describe the connection on the coherent module with connection underlying the elliptic polylogarithm sheaf. We let Γ be a lattice in $\mathbb{C}$. We define $\theta(z; \Gamma)$ to be the reduced theta function associated to the divisor $[0]$ on $\mathbb{C}/\Gamma$, normalized so that $\theta'(0; \Gamma) = 1$. This function may be written in terms of the Weierstrass sigma function $\sigma(z; \Gamma)$ for the lattice Γ as

$$\theta(z; \Gamma) = \exp(-e^*_{0,2} z^2/2) \sigma(z; \Gamma),$$

where $e^*_{0,2} := \lim_{s \to 0} \sum_{\gamma \in \Gamma \setminus \{0\}} \gamma^{-2} |\gamma|^{-2s}$. In what follows, we fix a lattice $\Gamma \subset \mathbb{C}$ and for simplicity, we omit Γ from the notation and write $\theta(z)$ for $\theta(z; \Gamma)$.

Definition 3.1 (Kronecker theta function).   We let

$$\Theta(z, w) := \theta(z + w)/\theta(z)\theta(w).$$

Let A be the fundamental area of Γ divided by $\pi = 3.14 \cdots$. The Kronecker theta function satisfies the transformation formula

$$(3.1) \qquad \Theta(z + \gamma_1, w + \gamma_2) = \exp\left[\frac{\gamma_1\overline{\gamma}_2}{A}\right] \exp\left[\frac{z\overline{\gamma}_2 + w\overline{\gamma}_1}{A}\right] \Theta(z, w)$$

for any γ_1, $\gamma_2 \in \Gamma$, which in particular shows that $\Theta(z, w)$ is a reduced theta function associated to the Poincaré bundle of $\mathbb{C}/\Gamma$. We let

$$F_1(z) = \lim_{w \to 0} \left(\Theta(z, w) - w^{-1}\right) = \theta'(z)/\theta(z)$$

and $\Xi(z, w) = \exp(-F_1(z)w)\Theta(z, w)$. Then (3.1) shows that $\Xi(z + \gamma, w) = \Xi(z, w)$ for any $\gamma \in \Gamma$.

Definition 3.2.   We define the connection functions $L_n(z)$ to be the coefficients

$$\Xi(z, w) = \sum_{n=0}^{\infty} L_n(z)w^{n-1}$$

of the Laurent expansion of $\Xi(z, w)$ with respect to w.

$L_n(z)$ are elliptic functions on $\mathbb{C}/\Gamma$ with poles only at $[0] \in \mathbb{C}/\Gamma$. The connection functions satisfy the following algebraicity result ([BKT] Proposition 1.6).

Lemma 3.3.   *Suppose there exists an elliptic curve $E : y^2 = 4x^3 - g_2x - g_3$ defined over a field $L \subset \mathbb{C}$, such that the pull-back of $\omega := dx/y$ by the uniformization*

$$\mathbb{C}/\Gamma \cong E(\mathbb{C}), \quad z \mapsto (\wp(z) : \wp'(z) : 1)$$

gives the invariant differential dz. Then the functions $L_n(z)$ correspond through the above uniformization to rational functions on E defined over L.

Proof.   One may prove that $\Xi(z, w) = \exp(-\zeta(z)w)\sigma(z + w)/\sigma(z)\sigma(w)$, where $\zeta(z) = \sigma'(z)/\sigma(z)$ is the Weierstrass ζ-function. The condition of the lemma implies that the Laurent expansions of $\sigma(z)$ and $\zeta(z)$ at $z = 0$ have coefficients in L, hence the same also holds true for $L_n(z)$. By definition, $L_n(z)$ is a function whose only pole on $\mathbb{C}/\Gamma$ is at $z = 0$, whose order is necessarily ≥ 2. The Laurent expansions of $\wp(z) = -\zeta'(z)$ and $\wp'(z)$ also have coefficients in L. Since the only poles of $\wp(z)$ and $\wp'(z)$ on $\mathbb{C}/\Gamma$ are at $z = 0$ of order 2 and 3, one may take a suitable polynomial $f(X, Y) \in L[X, Y]$ such that the function $L_n(z) - f(\wp(z), \wp'(z))$ has no poles on $\mathbb{C}/\Gamma$. This difference must be

a constant, necessarily in L since all the Laurent coefficients of the functions appearing in this difference is also in L. This shows that $L_n(z) \in L[\wp(z), \wp'(z)]$ as required. □

The connection functions play the role for elliptic curves of $\mathrm{Li}_0(t) = t/(1-t)$ in the classical case. We will describe the module with connection underlying the elliptic polylogarithm using these connection functions.

§4. Module with connection underlying the elliptic polylogarithm

In this section, we will use the connection function $L_n(z)$ defined in the previous section to explicitly describe the coherent module with connection underlying the elliptic polylogarithm sheaf. Let L be a field of characteristics zero, and let X_L be a scheme smooth and of finite type defined over L. We let $\mathscr{M}(X_L)$ be the category of coherent modules M on X_L with integrable connection $\nabla : M \to M \otimes \Omega^1_{X_L}$. For any $\mathcal{F} \in \mathscr{M}(X)$, we denote by $H^n_{\mathrm{dR}}(X_L, \mathcal{F})$ the de Rham cohomology of X_L with coefficients in $\mathcal{F}$. Then we have an isomorphism

(4.1) $$H^n_{\mathrm{dR}}(X_L, \mathcal{F}) = \mathrm{Ext}^n_{\mathscr{M}(X_L)}(\mathcal{O}_{X_L}, \mathcal{F})$$

for $n = 0, 1$.

Let the notations be as in §2. In both the Hodge and the p-adic cases, any sheaf in $\mathscr{S}(X)$ has an underlying coherent module with connection on $X_F := X \otimes_S F$. In other words, there exists a canonical functor $\mathrm{For} : \mathscr{S}(X) \to \mathscr{M}(X_F)$. For any object $\mathscr{F} \in \mathscr{S}(X)$ and underlying module $\mathcal{F} = \mathrm{For}(\mathscr{F}) \in \mathscr{M}(X_F)$, there exists a canonical isomorphism

$$H^n(X, \mathscr{F}) \otimes_K F \cong H^n_{\mathrm{dR}}(X_F, \mathcal{F}),$$

hence a natural inclusion $H^0_{\mathscr{A}}(S, H^n(X, \mathscr{F})) \hookrightarrow H^n_{\mathrm{dR}}(X_F, \mathcal{F})$. The functor For on extension classes is given by the composition $H^1_{\mathscr{A}}(X, \mathscr{F}) \to H^0_{\mathscr{A}}(S, H^1(X, \mathscr{F})) \hookrightarrow H^1_{\mathrm{dR}}(X_F, \mathcal{F})$. In other words, we have a commutative diagram

(4.2)
$$
\begin{array}{ccccc}
H^1_{\mathscr{A}}(X, \mathscr{F}) & \longrightarrow & H^0_{\mathscr{A}}(S, H^1(X, \mathscr{F})) & \longrightarrow & H^1_{\mathrm{dR}}(X_F, \mathcal{F}) \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\
\mathrm{Ext}^1_{\mathscr{S}(X)}(K(0), \mathscr{F}) & & \xrightarrow{\quad\mathrm{For}\quad} & & \mathrm{Ext}^1_{\mathscr{M}(X_F)}(\mathcal{O}_{X_F}, \mathcal{F}).
\end{array}
$$

We will calculate the module $\mathcal{P} = \mathrm{For}(\mathscr{P})$ underlying the elliptic polylogarithm sheaf.

We return to the case of the elliptic curve. We denote by $\mathcal{H}$, $\mathcal{H}_U$ and $\mathcal{L}\mathrm{og}$ the modules with connection underlying $\mathscr{H}$, $\mathscr{H}_U$ and $\mathscr{L}\mathrm{og}$. In particular, we have $\mathcal{H} = H^1_{\mathrm{dR}}(E_F)^\vee$. Since the de Rham cohomology of $H^1_{\mathrm{dR}}(E_F)$ may be calculated using differentials of the second kind on $U_F := E_F \setminus [0]$, we have

$$H^1_{\mathrm{dR}}(E_F) \cong \Gamma(E_F, \Omega^1(2[0])).$$

In the Hodge case or when the prime $p \geq 5$ in the p-adic case, we may assume that E is given by the Weierstrass equation $E : y^2 = 4x^3 - g_2 x - g_3$ defined over S. Then the differentials $\omega = dx/y$ and $\eta = x\,dx/y$ are differentials of the first and second kinds. If we denote by $\underline{\omega}$ and $\underline{\eta}$ their classes in $H^1_{\mathrm{dR}}(E_F)$, we have $H^1_{\mathrm{dR}}(E_F) \cong F\underline{\omega} \bigoplus F\underline{\eta}$.

We now let L be a subfield of $\mathbb{C}$ with a fixed embedding $L \hookrightarrow F$. We may take $L = F$ in the Hodge case. Assume that the Weierstrass model of our elliptic curve is also defined over L, and let Γ be the period lattice of E with respect to the invariant differential $\omega = dx/y$. There exists a uniformization $\mathbb{C}/\Gamma \cong E(\mathbb{C})$, $z \mapsto (\wp(z) : \wp'(z) : 1)$ such that ω corresponds to dz. If we let $\omega^* = dF_1(z)$, then ω^* is also a differential of the second kind defined over $L(e^*_{0,2})$. If we assume in addition that $e^*_{0,2} \in L$, then we have $H^1_{\mathrm{dR}}(E_L) \cong L\underline{\omega} \bigoplus L\underline{\omega}^*$ and $\mathcal{H} = L\underline{\omega}^\vee \bigoplus L\underline{\omega}^{*\vee}$, where $\underline{\omega}^*$ denotes the class of ω^*.

Remark.   In what follows, we will assume that $e^*_{0,2} \in L$. In this case, the differential form ω^* is defined over L. This condition is not necessary in describing the module with connection underlying the polylogarithm sheaf if we use $\eta = x\,dx/y$ instead of $\omega^* = dF_1$. However, since the class of ω^* is equal to that of $d\bar{z}/A$ in $H^1_{\mathrm{dR}}(E_{\mathbb{C}})$, the differential ω^* is much more amiable with calculations related to Eisenstein-Kronecker numbers.

The sheaf $\mathcal{L}\mathrm{og}$ on U_F in both the Hodge and the p-adic cases is given by

$$\mathcal{L}\mathrm{og} = \prod_{m,n \geq 0} \mathcal{O}_{U_F} \underline{\omega}^{m,n},$$

with connection given by $\nabla(\underline{\omega}^{m,n}) = \underline{\omega}^{m+1,n} \otimes \omega + \underline{\omega}^{m,n+1} \otimes \omega^*$. The weight filtration on $\mathcal{L}\mathrm{og}$ is given by $W_{-k}(\mathcal{L}\mathrm{og}) = \prod_{m,n \geq k} \mathcal{O}_{U_F} \underline{\omega}^{m,n}$, and the canonical splitting

$$\mathrm{Gr}^W_{-k} \mathcal{L}\mathrm{og}\big|_{U_F} \cong \mathrm{Sym}^k \mathcal{H}_{U_F}$$

maps $\underline{\omega}^{m,n}$ to $\underline{\omega}^{\vee m} \underline{\omega}^{*\vee n}$. Note that in the above description of $\mathcal{L}\mathrm{og}$, the connection has poles at $D_F = [0]$. This follows from the fact that the sections $\underline{\omega}^{m,n}$ do not extend to sections of $\mathcal{L}\mathrm{og}$ on E_F. However, the logarithm sheaf itself extends to a module with holomorphic connection on the whole of E_F. Since U_F is an affine scheme, the de Rham cohomology of U_F with coefficients in $\mathcal{H}^\vee_U \otimes \mathcal{L}\mathrm{og}$ may be calculated as the cohomlogy of the complex

$$\Gamma(U_F, \mathcal{H}^\vee_U \otimes \mathcal{L}\mathrm{og}) \xrightarrow{\nabla} \Gamma(U_F, \mathcal{H}^\vee_U \otimes \mathcal{L}\mathrm{og} \otimes \Omega^1_{U_F}).$$

Since we have assumed that our elliptic curve is also defined over L, the connection functions L_n of Definition 3.2 are also defined over L. We may consider these functions as rational functions on E_F through the fixed embedding $L \hookrightarrow F$. If we let pol be the section

$$\mathrm{pol} := -\underline{\omega} \otimes \underline{\omega}^{0,0} \otimes \omega^* + \sum_{n=0}^{\infty} L_{n+1}\underline{\omega} \otimes \underline{\omega}^{1,n} \otimes \omega + \sum_{n=0}^{\infty} L_n \underline{\omega}^* \otimes \underline{\omega}^{0,n} \otimes \omega$$

in $\Gamma(U_F, \mathcal{H}_U^\vee \otimes \mathcal{L}\text{og} \otimes \Omega_{U_F}^1)$, then this element determines a class $[\text{pol}]$ in $H_{\text{dR}}^1(U_F, \mathcal{H}_U^\vee \otimes \mathcal{L}\text{og})$. Then we have the following (see [BKT] Theorem 1.20).

Theorem 4.1. *The class* $[\text{pol}] \in H_{\text{dR}}^1(U_F, \mathcal{H}_U^\vee \otimes \mathcal{L}\text{og})$ *maps to the identity in* $\text{Hom}_F(\mathcal{H}, \mathcal{H}) = \mathcal{H}^\vee \otimes \mathcal{H} \subset \mathcal{H}^\vee \otimes \prod_{j=0}^\infty \text{Sym}^j \mathcal{H}$ *through the residue map*

$$\text{res} : H_{\text{dR}}^1(U_F, \mathcal{H}_U^\vee \otimes \mathcal{L}\text{og}) \to H_{\text{dR}}^0(D_F, \mathcal{H}^\vee \otimes [0]^* \mathcal{L}\text{og}) = \mathcal{H}^\vee \otimes \prod_{j=0}^\infty \text{Sym}^j \mathcal{H}.$$

The compatibility of the residue maps for de Rham and relative cohomologies gives the commutative diagram

$$
\begin{array}{ccccc}
\text{Ext}_{\mathscr{S}(U)}^1(K(0), \mathscr{H}_U^\vee \otimes \mathscr{L}\text{og}(1)) & \xrightarrow{\cong} & H_{\mathscr{A}}^1(U, \mathscr{H}_U^\vee \otimes \mathscr{L}\text{og}(1)) & \xrightarrow{\text{res}} & \text{Hom}_{\mathscr{S}(S)}(\mathscr{H}, \mathscr{H}) \\
\text{For} \downarrow \cap & & \cong \downarrow & & \| \downarrow \\
 & & H_{\mathscr{A}}^0(S, H^1(U, \mathscr{H}_U^\vee \otimes \mathscr{L}\text{og}(1))) & \xrightarrow{\text{res}} & H_{\mathscr{A}}^0(S, \mathscr{H}^\vee \otimes \mathscr{H}) \\
\downarrow & & \cap \downarrow & & \cap \downarrow \\
\text{Ext}_{\mathscr{M}(U)}^1(\mathcal{O}_{U_F}, \mathcal{H}_U^\vee \otimes \mathcal{L}\text{og}) & \xrightarrow{\cong} & H_{\text{dR}}^1(U_F, \mathcal{H}_U^\vee \otimes \mathcal{L}\text{og}) & \xrightarrow{\text{res}} & \mathcal{H}^\vee \otimes \prod_{j \geq 0} \text{Sym}^j \mathcal{H}.
\end{array}
$$

Hence the cohomology class $[\text{pol}]$ corresponds to the class of $\mathcal{P} = \text{For}(\mathscr{P})$, where $\mathscr{P}$ is the polylogarithm sheaf. The relation between extensions and classes in de Rham cohomology gives the following corollary (see [BKT] Corollary 1.21).

Corollary 4.2. *The* $\mathcal{O}_{U_F}$*-module with connection* $\mathcal{P}$ *in* $\mathscr{M}(U_F)$ *is the extension*

$$0 \to \mathcal{L}\text{og} \to \mathcal{P} \to \mathcal{H}_U \to 0$$

whose underlying module is $\mathcal{P} = \mathcal{H}_U \oplus \mathcal{L}\text{og}$ *and whose connection is given by*

$$\nabla(\underline{\omega}^\vee) = -\underline{\omega}^{0,0} \otimes \omega^* + \sum_{n=0}^\infty L_{n+1} \underline{\omega}^{1,n} \otimes \omega, \qquad \nabla(\underline{\omega}^{*\vee}) = \sum_{n=0}^\infty L_n \underline{\omega}^{0,n} \otimes \omega.$$

The above result gives the explicit description of the module with connection $\mathcal{P}$ underlying the polylogarithm sheaf $\mathscr{P}$.

§ 5.   Eisenstein-Kronecker numbers

In this section, we define the complex and p-adic version of Eisenstein-Kronecker numbers. The complex Eisenstein-Kronecker numbers are defined as special values of Eisenstein-Kronecker series, where as the p-adic Eisenstein-Kronecker numbers which are defined for CM elliptic curves are constructed using p-adic interpolation.

We first review the definition of Eisenstein-Kronecker-Lerch series following [We]. Let $\Gamma \subset \mathbb{C}$ be as before and let a be an integer. Then the Eisenstein-Kronecker-Lerch series is defined as the sum

$$K_a^*(z_0, w_0, s) = \sum_{\gamma \in \Gamma \backslash \{-z_0\}} \frac{(\overline{z}_0 + \overline{\gamma})^a}{|z_0 + \gamma|^{2s}} \langle \gamma, w_0 \rangle,$$

where $\langle z, w \rangle = \exp((z\overline{w} - \overline{z}w)/A)$ and A is the fundamental area of Γ divided by $\pi = 3.1415\cdots$. The sum above converges absolutely for $\mathrm{Re}(s) > a/2 + 1$, and $K_a^*(z_0, w_0, s)$ continues meromorphically to a function on the whole s-plane, with a simple pole only at $s = 1$ if $a = 0$ and $w_0 \in \Gamma$ ([BKT] Proposition 2.2). We define the complex Eisenstein-Kronecker numbers as follows.

Definition 5.1. Let a and b be integers, and let z_0 and w_0 be points in $\mathbb{C}$ satisfying $w_0 \notin \Gamma$ when $(a, b) = (-1, 1)$. We define the Eisenstein-Kronecker number $e_{a,b}^*(z_0, w_0)$ by $e_{a,b}^*(z_0, w_0) := K_{a+b}^*(z_0, w_0, b)$. In addition, for $z_0 \notin \Gamma$, we let

$$e_{a,b}^*(z_0) := e_{a,b}^*(0, z_0) = K_{a+b}^*(0, z_0, b).$$

The numbers $e_{a,b}^*(z_0)$ will be used to express the specializations to torsion points of the real Hodge realization of the elliptic polylogarithm sheaf. We will next define the p-adic analogues of $e_{a,b}^*(z_0)$ by p-adic interpolation, when $\Gamma \subset \mathbb{C}$ corresponds to a CM elliptic curve defined over the field of complex multiplication, with good reduction at p. The crucial fact for the construction is the following theorem, which asserts that the Eisenstein-Kronecker numbers $e_{a,b+1}^*(z_0, w_0)$ for $a, b \geq 0$ are generated by the Kronecker theta function ([BK1] Theorem 1.17. See also [BK2] Theorem 2.11).

Theorem 5.2. *For any z_0, $w_0 \in \mathbb{C}$, let*

$$\Theta_{z_0, w_0}(z, w) := \exp\left[-\frac{z_0 \overline{w}_0}{A}\right] \exp\left[-\frac{z\overline{w}_0 + w\overline{z}_0}{A}\right] \Theta(z + z_0, w + w_0).$$

Then we have

$$\Theta_{z_0, w_0}(z, w) = \frac{\delta_{z_0}}{z} \langle w_0, z_0 \rangle + \frac{\delta_{w_0}}{w} + \sum_{a,b \geq 0} (-1)^{a+b} \frac{e_{a,b+1}^*(z_0, w_0)}{a! A^a} z^b w^a,$$

where $\delta_x = 1$ if $x \in \Gamma$ and $\delta_x = 0$ otherwise.

Remark. A similar expansion was proved in [Zag] §3 Theorem for $z_0 = w_0 = 0$, for a slightly different normalization of the theta function.

We first define $F_{z_0, b}(z)$ to be meromorphic functions on $\mathbb{C}$ defined as the coefficients

$$\Theta_{z_0, 0}(z, w) = \sum_{b=0}^{\infty} F_{z_0, b}(z) w^{b-1}$$

of the Laurent expansion of $\Theta_{z_0,0}(z,w)$ with respect to the variable w, and we define $F_b(z) := F_{0,b}(z)$. Note that $F_0(z) \equiv 1$ and $F_1(z)$ is equal to $\theta'(z)/\theta(z)$ as before. The function $F_{z_0,b}(z)$ depends only on the choice of z_0 modulo Γ, hence the z_0 of $F_{z_0,b}(z)$ will denote its class in $\mathbb{C}$ modulo Γ.

We now assume that $\mathbb{C}/\Gamma$ has a model $E : y^2 = 4x^3 - g_2 x - g_3$ mapping the invariant differential dz to $\omega = dx/y$, such that E is defined over the ring of integers $\mathcal{O}_{\boldsymbol{K}}$ of an imaginary quadratic field $\boldsymbol{K}$ with good reduction at the primes above $p \geq 5$. We assume in addition that E has complex multiplication by $\mathcal{O}_{\boldsymbol{K}}$. In this case, Damerell's theorem asserts that the numbers $e^*_{a,b+1}(z_0, w_0)/A^a$ for z_0, $w_0 \in \Gamma \otimes \mathbb{Q}$ and $a, b \geq 0$ are in $\overline{\mathbb{Q}}$. Then for any non-zero torsion point $z_0 \in (\Gamma \otimes \mathbb{Q})/\Gamma$ and $b \geq 0$, the Taylor expansion of $F_{z_0,b}(z)$ at $z = 0$ has coefficients in $\overline{\mathbb{Q}}$.

We denote by $\widehat{E}$ the formal group of E associated to the parameter $s = -2x/y$, and we denote by $\lambda(s)$ the formal logarithm of $\widehat{E}$ normalized so that $\lambda'(0) = 1$. We let

$$\widehat{F}_{z_0,b}(s) := F_{z_0,b}(z)|_{z=\lambda(s)} \in \overline{\mathbb{Q}}[[s]]$$

be the composition as formal power series of the Taylor expansion of $F_{z_0,b}(z)$ at $z = 0$ with $\lambda(s)$. We fix a prime $\mathfrak{p}$ of $\mathcal{O}_{\boldsymbol{K}}$ above p, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ such that the completion of $\boldsymbol{K}$ in $\mathbb{C}_p$ is $\boldsymbol{K}_{\mathfrak{p}}$. Then we have the following.

Lemma 5.3. *Suppose z_0 is a non-zero torsion point of $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. Then the power series $\widehat{F}_{z_0,b}(s) \in \mathbb{C}_p[[s]]$ converges on the open unit disc $B^-(0,1) := \{s \in \mathbb{C}_p \mid |s|_p < 1\}$.*

We will use the above theorem and the theory of Schneider and Teitelbaum [ST] to construct our p-adic distribution. The formal group $\widehat{E}$ is a Lubin-Tate group with action by $\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}$. We have a $\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}$-linear isomorphism

$$\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}} \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}}(\widehat{E}, \widehat{\mathbb{G}}_m),$$

which is not canonical and depends on the choice of a p-adic period as follows. There exists $\Omega_{\mathfrak{p}} \in \mathbb{C}_p^\times$ such that the formal power series $\exp(\lambda(s)/\Omega_{\mathfrak{p}})$ is an element in $\mathcal{O}_{\mathbb{C}_p}[[s]]$. For a suitable choice of $\Omega_{\mathfrak{p}}$, we may construct an isomorphism as above by associating to any $x \in \mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}$ the homomorphism of formal groups defined by the power series $\exp(x\lambda(s)/\Omega_{\mathfrak{p}})$. In what follows, we fix a choice of $\Omega_{\mathfrak{p}}$. Let $C^{\mathrm{an}}(\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}, \mathbb{C}_p)$ be the set of locally $\boldsymbol{K}_{\mathfrak{p}}$-analytic functions on $\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}$. We define the distribution $\mu_{z_0,b}$ as follows.

Definition 5.4. Let z_0 be a non-zero torsion point of $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. For any integer $b \geq 0$, we define $\mu_{z_0,b}$ to be the p-adic distribution on $C^{\mathrm{an}}(\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}, \mathbb{C}_p)$ corresponding by the theory of [ST] to the power series $\widehat{F}_{z_0,b}(s)$, satisfying

$$\int_{\mathcal{O}_{\boldsymbol{K}_{\mathfrak{p}}}} \exp(x\lambda(s)/\Omega_{\mathfrak{p}})d\mu_{z_0,b}(x) = \widehat{F}_{z_0,b}(s).$$

Since $\partial_{\log,s} := \partial_s/\lambda'(s) = \partial_z$, we have by construction

$$\Omega_{\mathfrak{p}}^{-a} \int_{\mathcal{O}_{K_{\mathfrak{p}}}} x^a d\mu_{z_0,b}(x) = \partial_{\log,s}^a \widehat{F}_{z_0,b}(s)\Big|_{s=0} = (-1)^{a+b+1} \frac{e_{a,b}^*(z_0)}{A^a}$$

for any integer $a \geq 0$. We will use this distribution to construct the p-adic Eisenstein-Kronecker numbers.

Definition 5.5. Let z_0 be a non-zero torsion point of $E(\overline{\mathbb{Q}})$ of order prime to $\mathfrak{p}$. We define the p-adic Eisenstein-Kronecker numbers for integers $a, b \in \mathbb{Z}$ with $b \geq 0$ by

$$e_{a,b}^{(p)}(z_0) = \Omega_{\mathfrak{p}}^{b-1} \int_{\mathcal{O}_{K_{\mathfrak{p}}}^{\times}} x^a d\mu_{z_0,b}(x).$$

Remark. Let the notations be as above, and let $\psi := \psi_{E/K}$ be the Grossencharacter of $\boldsymbol{K}$ associated to E, and we let $\pi := \psi(\mathfrak{p})$. The interpolation property of $\mu_{z_0,b}$ and the distribution relation for $F_{z_0,b}(z)$ show that we have

$$\frac{e_{a,b}^{(p)}(z_0)}{\Omega_{\mathfrak{p}}^{a+b-1}} = (-1)^{a+b-1} \left(\frac{e_{a,b}^*(z_0)}{A^a} - \frac{\pi^a e_{a,b}^*(\pi z_0)}{\overline{\pi}^b A^a} \right)$$

for any $a, b \geq 0$. This is why we view $e_{a,b}^{(p)}(z_0)$ as p-adic analogues of Eisenstein-Kronecker numbers.

§6. The main results

In this section, we state the main results of [BKT] in the real Hodge and the p-adic cases. We first consider the real Hodge case. Let z_0 be a point in $E(\mathbb{C})$ and let $i_{z_0} : S \hookrightarrow E$ be the natural inclusion. The calculation of absolute Hodge cohomology in the real Hodge case gives an isomorphism

$$H^1_{\mathscr{A}}(S, \mathscr{H}^{\vee} \otimes \mathrm{Sym}^j \mathscr{H}(1)) \cong M_{\mathbb{R}}^{(j)}/M_{\mathbb{R}}^{(j)} \cap \left(M_{\mathbb{R}}^{(j)}(1) + F^1 M_{\mathbb{C}}^{(j)} \right),$$

where $M^{(j)} := \mathscr{H}^{\vee} \otimes \mathrm{Sym}^j \mathscr{H}$. Hence we have an isomorphism

$$(6.1) \qquad H^1_{\mathscr{A}}(S, i_{z_0}^*(\mathscr{H}_U^{\vee} \otimes \mathscr{L}\mathrm{og}(1))) \cong \prod_{j \geq 0} M_{\mathbb{R}}^{(j)}/M_{\mathbb{R}}^{(j)} \cap \left(M_{\mathbb{R}}^{(j)}(1) + F^1 M_{\mathbb{C}}^{(j)} \right).$$

We note that when restricted to z_0, the mixed Hodge structure $i_{z_0}^*(\mathscr{H}_U^{\vee} \otimes \mathscr{L}\mathrm{og}(1))$ splits into pure components. The main result in this case is the following.

Theorem 6.1. *Denote by $i_{z_0}^* \mathscr{P}$ be the pull-back of $\mathscr{P}$ by i_{z_0} to S. Then the image of $[i_{z_0}^* \mathscr{P}]$ in $H^1_{\mathscr{A}}(S, i_{z_0}^*(\mathscr{H}_U^{\vee} \otimes \mathscr{L}\mathrm{og}(1)))$ through the isomorphism (6.1) is*

$$\sum_{m,k \geq 1} (-1)^{k-1} \frac{e_{-m,k}^*(z_0)}{A^{-m}} \left(\underline{\omega} \otimes \underline{\omega}^{m,k-1} + \underline{\omega}^* \otimes \underline{\omega}^{m-1,k} \right).$$

The above result was originally proved in [BL]. See also [Wi] III Theorem 4.8. In [BKT] Theorem A.17, we give a different proof using the explicit form of $\mathcal{P}$ obtained by solving explicit iterated integrals.

We next state our main result in the p-adic case. We again assume that $\mathbb{C}/\Gamma$ has a model $E : y^2 = 4x^3 - g_2 x - g_3$ defined over the ring of integers $\mathcal{O}_{\boldsymbol{K}}$ of an imaginary quadratic field $\boldsymbol{K}$, with good reduction at the primes above $p \geq 5$. We assume in addition that E has complex multiplication by $\mathcal{O}_{\boldsymbol{K}}$. We fix a prime $\mathfrak{p}$ of $\boldsymbol{K}$ above p, and we denote by K a finite unramified extension of $\boldsymbol{K}_{\mathfrak{p}}$.

Let z_0 be a K-rational torsion point of order prime to $\mathfrak{p}$, and denote again by $i_{z_0} : S \hookrightarrow E$ the inclusion defined by z_0. We define the absolute cohomology in the p-adic case, using the lifting of the absolute Frobenius in the ordinary case and using the square Frobenius, which is the Frobenius whose action on the residue field is the square of the absolute Frobenius, in the supersingular case. This gives an isomorphism

$$H^1_{\mathscr{A}}(S, \mathscr{H}^{\vee} \otimes \mathrm{Sym}^j \mathscr{H}(1)) \cong (\mathscr{H}^{\vee} \otimes \mathrm{Sym}^j \mathscr{H})/K\underline{\omega} \otimes \underline{\omega}^{*\vee j}.$$

Hence we have an isomorphism

$$(6.2) \qquad H^1_{\mathscr{A}}(S, i_{z_0}^{*}(\mathscr{H}_U^{\vee} \otimes \mathscr{L}\mathrm{og}(1))) \cong \prod_{j \geq 0} (\mathscr{H}^{\vee} \otimes \mathrm{Sym}^j \mathscr{H})/K\underline{\omega} \otimes \underline{\omega}^{0,j}.$$

We note that when restricted to *torsion* points z_0, the filtered Frobenius module $i_{z_0}^{*}(\mathscr{H}_U^{\vee} \otimes \mathscr{L}\mathrm{og}(1)))$ splits into pure components. Our main result in the p-adic case is the following.

Theorem 6.2 ([BKT] Theorem 5.6). *Denote by $i_{z_0}^{*}\mathscr{P}$ be the pull-back of $\mathscr{P}$ by i_{z_0} to S. Then the image of $[i_{z_0}^{*}\mathscr{P}]$ in $H^1_{\mathscr{A}}(S, i_{z_0}^{*}(\mathscr{H}_U^{\vee} \otimes \mathscr{L}\mathrm{og}(1)))$ through the isomorphism* (6.2) *is*

$$-\sum_{m,k \geq 1} \frac{e_{-m,k}^{(p)}(z_0)}{\Omega_{\mathfrak{p}}^{k-m-1}} \left(\underline{\omega} \otimes \underline{\omega}^{m,k-1} + \underline{\omega}^{*} \otimes \underline{\omega}^{m-1,k} \right).$$

Remark. When $p \geq 5$ is an ordinary prime, our result for $m = k = 1$ essentially coincides with the calculation of the syntomic regulator for CM elliptic curves given in [CD] and [Bes2].

§ 7.   The elliptic polylogarithm function

The proof of the main theorems are given by solving explicit differential equations in each realization. In the real Hodge case, holomorphic functions giving the $\mathbb{R}$-Hodge structure of the elliptic polylogarithm satisfy the iterated differential equations. In the

p-adic case, the p-adic overconvergent functions giving the Frobenius structure of the elliptic polylogarithm sheaf satisfy an analogous iterated differential equations.

In the real Hodge case, the differential equation giving the $\mathbb{R}$-Hodge structure underlying the elliptic polylogarithm sheaf is given by

$$dD_{m,n}(z) = -D_{m-1,n}(z)dz - D_{m,n-1}(z)dF_1(z)$$

for $m > 0$, $n \geq 0$, with $D_{0,n}(z) = L_n(z)$, and $D_{m,n}(z) = 0$ for $n < 0$. This is the elliptic analogue of the differential equation (1.1), and we solve this equation in terms of functions related to the Eisenstein-Kronecker-Lerch series. See the original paper [BKT] for details.

We next consider the p-adic case. In the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the p-adic analogues $\ell^{(p)}(t)$ of $\mathrm{Li}_k(t)$ are given as the unique solutions as overconvergent functions of the differential equations

$$\ell_{k+1}^{(p)}(t) = \ell_k^{(p)}(t)\frac{dt}{t} \qquad (k \geq 0),$$

where

$$\ell_0^{(p)}(t) = \frac{t}{1-t} - \frac{t^p}{1-t^p}.$$

These equations are the p-adic analogues of (1.1). We consider such an analogue for elliptic curves with complex multiplication. Let the notations be as in §6, before Theorem 6.2. In addition, let $\pi := \psi_{E/\boldsymbol{K}}(\mathfrak{p})$ as in the remark after Definition 5.5. We let $\Theta^{(p)}(z,w) := \Theta(z,w) - \overline{\pi}^{-1}\Theta(\pi z, w/\overline{\pi})$ and

$$\Xi^{(p)}(z,w) := \exp(-F_1(z)w)\Theta^{(p)}(z,w).$$

Definition 7.1. We define the p-adic version $L_n^{(p)}(z)$ of the connection functions as the coefficients

$$\Xi^{(p)}(z,w) = \sum_{n=0}^{\infty} L_n^{(p)}(z)w^{n-1}$$

of the Laurent expansion of $\Xi^{(p)}(z,w)$ with respect to w.

The functions $L_n^{(p)}(z)$ correspond to a rational function of E defined over K. The differential equation for the p-adic elliptic polylogarithm function is given by

$$(7.1) \qquad dD_{m,n}^{(p)} = -D_{m-1,n}^{(p)}dz - D_{m,n-1}^{(p)}dF_1$$

for $m > 0$, $n > 0$, with $D_{0,n}^{(p)} = L_n^{(p)}$ and $D_{m,n}^{(p)} = 0$ for $n \leq 0$.

Proposition 7.2. *The differential equation (7.1) has a unique system of solutions as overconvergent functions on E minus the residue disc around $[0]$.*

The proof of the above result follows from the following stronger result.

Proposition 7.3. *We let $D_{m,n}^{(p)}$ for $m = 0$ or $n \leq 0$ as above. Then for integers $m,\ n > 0$, there exists a unique system of overconvergent functions $D_{m,n}^{(p)}$ on E minus the residue disc around $[0]$ iteratedly satisfying*
(1) The differential equation $dD_{m,n}^{(p)} = -D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^*$.
(2) The distribution relation

$$\sum_{z_1 \in E[\mathfrak{p}](K)} G_{m,n}^{(p)}(z + z_1) = 0, \tag{7.2}$$

where $F_1^{(p)}(z) := F_1(z) - \overline{\pi}^{-1}F_1(\pi z)$ and

$$G_{m,n}^{(p)} = \sum_{k=0}^{n} \frac{(F_1^{(p)})^{n-k}}{(n-k)!} D_{m,k}^{(p)}.$$

Proof. See [BKT] Proposition 3.4 for details. The proof is done by induction on $N = m + n$. The distribution relation for $n = 0$ follows from the fact that $G_{m,0}^{(p)} = D_{m,0}^{(p)} = 0$, and the result for $m = 0$ follows from the definition of $\Theta_{z_0,0}^{(p)}(z,w)$ and the distribution property of $\Theta_{z_0,0}(z,w)$ ([BKT] Proposition 2.14). Let N be an integer ≥ 0 and suppose $D_{a,b}^{(p)}$ exists for any a, b such that $a + b < N$. For any integer m, $n > 0$ such that $m + n = N$, the differential

$$-D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^*$$

defines a class in $H_{\mathrm{rig}}^1(U_k/K)$, where $U_k := U \otimes k$ for $k := \mathcal{O}_{K_\mathfrak{p}}/\mathfrak{p}$ and $H_{\mathrm{rig}}^1(U_k/K)$ is the rigid cohomology of U_k. It is known that $H_{\mathrm{rig}}^1(U_k/K) = K\omega \oplus K\omega^*$ and that rigid cohomology give the obstruction of integration of differential forms by overconvergent functions. Hence for suitable constants $c_{m,n}$, $c_{m,n}^* \in K$, the differential form

$$-D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^* + c_{m,n}\omega + c_{m,n}^*\omega^*$$

is integrable by some overconvergent function $\widetilde{D}_{m,n}$. From the distribution relation (7.2) for $D_{m-1,n}^{(p)}$ and $D_{m,n-1}^{(p)}$ and the fact that the classes of ω and ω^* in H_{rig}^1 are translation invariant, we may prove that $c_{m,n} = c_{m,n}^* = 0$. If we let $D_{m,n}^{(p)} := \widetilde{D}_{m,n} + c$ for some constant $c \in K$, then $D_{m,n}^{(p)}$ also satisfies the distribution relation (7.2). This proves our assertion. □

The Frobenius structure (or the square-Frobenius structure in the supersingular case) of the elliptic polylogarithm sheaf may be described explicitly using the functions $D_{m,n}^{(p)}(z)$. The main ingredient in the proof of Theorem 6.2 is the following proposition, which gives the relation between the p-adic elliptic polylogarithm functions $D_{m,n}^{(p)}$ and the p-adic distribution used in defining the p-adic Eisenstein-Kronecker numbers.

Proposition 7.4. *Let $D_{m,n}^{(p)}$ be as above and let z_0 be a non-zero torsion point of $E(K)$ of order prime to $\mathfrak{p}$. Denote by*

$$\widehat{D}_{z_0,m,n}^{(p)}(s) := D_{m,n}^{(p)}(z+z_0)\mid_{z=\lambda(s)}$$

the expansion of $D_{m,n}^{(p)}(z+z_0)$ with respect to the formal parameter $s = -2x/y$ at the origin. Then we have the equality

$$\sum_{n=0}^{b} \frac{\widehat{F}_{z_0,1}(s)^{b-n}}{(b-n)!} \widehat{D}_{z_0,m,n}^{(p)}(s) = (-\Omega_\mathfrak{p})^m \int_{\mathcal{O}_{K_\mathfrak{p}}^{\times}} x^{-m} \exp(x\lambda(s)/\Omega_\mathfrak{p}) d\mu_{z_0,b}(x)$$

of formal power series in $\mathbb{C}_p[[s]]$.

Proof. See [BKT] Proposition 3.9 for details of the proof. We prove the proposition for any integer $b \geq 0$ by induction on m. The case for $m = 0$ follows from the formula

$$\sum_{n=0}^{b} \frac{(-F_1(z))^{b-n}}{(b-n)!} L_n(z) = F_b(z)$$

and the calculation of the restriction of the measure $\mu_{z_0,b}$ from $\mathcal{O}_{K_\mathfrak{p}}$ to $\mathcal{O}_{K_\mathfrak{p}}^{\times}$, noting that $D_{0,n}(z) = L_n(z)$ and that the distribution $\mu_{z_0,b}$ is defined using $F_b(z)$. Suppose that the statement is true for $m > 0$. We first prove that the two sides for $m + 1$ differ only by a constant, using the induction hypothesis and the fact that both sides satisfy the differential equation $df_{m+1} = -f_m\omega$, where $\omega = \lambda'(s)ds$ on each residue disc. We then prove that the constant is zero, using that both sides of the equality satisfy a distribution relation on each residue disc with respect to π-torsion points (derived from (7.2) for the left hand side, and derived from the fact that $\mu_{z_0,b}$ is a distribution for the right hand side). $\qquad\square$

References

[Ba1] Bannai, K., Rigid syntomic cohomology and p-adic polylogarithms, J. Reine Angew. Math. **529** (2000), 205-237.

[Ba2] Bannai, K., Syntomic cohomology as a p-adic absolute Hodge cohomology, Math. Z. **242** (2002), 443-480.

[Ba3] Bannai, K., On the p-adic realization of elliptic polylogarithms for CM-elliptic curves, Duke Math. J., **113** (2002), no. 2, 193-236.

[BKi] Bannai, K. and Kings, G., p-adic elliptic polylogarithm, p-adic Eisenstein series and Katz measure, preprint 2007, arXiv:0707.3747v1 [math.NT].

[BK1] K. Bannai and S. Kobayashi, Algebraic theta functions and p-adic interpolation of Eisenstein-Kronecker numbers, preprint, arXiv:math.NT/0610163v4, December 11, 2007.

[BK2] Bannai, K., Kobayashi, S, Algebraic theta functions and Eisenstein-Kronecker numbers, RIMS Kôkyûroku Bessatsu B4: *Proceedings of the Symposium on Algebraic Number theory and Related Topics*, eds. K. Hashimoto, Y. Nakajima and H. Tsunogai, December (2007), 63–78; Available at arXiv:0709.0640v1 [math.NT].

[BKT] Bannai, K., Kobayashi, S. and Tsuji, T., On the de Rham and p-adic realizations of the elliptic polylogarithm for CM elliptic curves, preprint 2007, arXiv:0711.1701v2 [math.NT].

[BD] Beilinson, A. and Deligne, P., Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs, In: *Motives* (Seattle, WA, 1991), Proc. Symp. in Pure Math. **55** (1994), 97-121.

[BL] Beilinson, A. and Levin, A., The Elliptic Polylogarithm, In: *Motives* (Seattle, WA, 1991), Proc. Symp. in Pure Math. **55** (1994), 123-192.

[Bes1] Besser, A., Syntomic regulators and p-adic integration I: Rigid syntomic regulators, Israel Journal of Math. **120** (2000), 291-334.

[Bes2] Besser, A., Syntomic regulators and p-adic integration II: K_2 of curves, Israel Journal of Math. **120** (2000), 335-360.

[BS] A. Besser and N. Solomon, in preparation.

[CD] Coleman, R. and de Shalit, E., p-adic regulators on curves and special values of p-adic L-functions, Invent. Math. **93** (1988), no. 2, 239–266.

[HK] Huber, A. and Kings, G., Degeneration of l-adic Eisenstein classes and of the elliptic polylog, Invent. Math. **135** (1999), no. 3, 545–594.

[Ki] Kings, G., The Tamagawa number conjecture for CM elliptic curves, Invent. Math. **143** No. 3 (2001), 571–627.

[LR] Levin, A. and Racinet, G., Towards multiple elliptic polylogarithm, preprint March 8, 2007, arXiv:math/0703237.

[ST] Schneider, P. and Teitelbaum, J., p-adic Fourier Theory, Documenta Math. **6** (2001), 447-481.

[We] Weil, A., *Elliptic Functions according to Eisenstein and Kronecker*, Springer-Verlag, 1976.

[Wi] Wildeshaus, J., *Realizations of Polylogarithms*, SLN **1650**, Springer-Verlag, 1997.

[Zag] Zagier, D., Periods of modular forms and Jacobi theta functions, Inv. math. **104** (1991), 449-465.