Unramified extensions and geometric $\mathbb{Z}_p$-extensions of global function fields

By

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Abstract

We study on finite unramified extensions of global function fields (that is, function fields of one variable over a finite field). We show two results. One is an extension of Perret’s result about the ideal class group problem. Another is a construction of a geometric $\mathbb{Z}_p$-extension which has a certain property.

§ 1. Main theorems

Throughout the present paper, we fix a prime number $p$ and a finite field $\mathbb{F}$ of characteristic $p$. Let $q$ be the number of elements of $\mathbb{F}$. Recall that a global function field is a function field of one variable over a finite field. Let $k$ be a global function field with full constant field $\mathbb{F}$. We also recall that a finite algebraic extension $K/k$ is geometric if and only if the constant field of $K$ is also $\mathbb{F}$.

It is known that there is a finite abelian group $G$ which is not isomorphic to the divisor class group of degree 0 of any global function field (Stichtenoth [20]). On the other hand, Perret [16] showed the following:

**Theorem 1.1** ([16]). For any given finite abelian group $G$, there is a finite separable geometric extension $k/\mathbb{F}(T)$ such that $\text{Cl}({\mathcal{O}}) \cong G$, where $\mathcal{O}$ is the integral closure of $\mathbb{F}[T]$ in $k$ and $\text{Cl}({\mathcal{O}})$ is the ideal class group of $\mathcal{O}$.

This theorem is shown by using the following:

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Theorem 1.2 ([16]). For any given finite abelian group $G$, there is a global function field $k$ with full constant field $\mathbb{F}$ and a non-empty finite set $S$ of places of $k$ such that $\text{Cl}_S(k) \cong G$, where $\text{Cl}_S(k)$ is the $S$-class group of $k$.

Let $S$ be a non-empty finite set of places of $k$, and $H_S(k)$ the $S$-Hilbert class field of $k$, that is, the maximal unramified abelian extension field of $k$ in which all places of $S$ split completely (see [17]). We note that $\text{Cl}_S(k) \cong \text{Gal}(H_S(k)/k)$ by class field theory. Hence Theorem 1.2 also implies the existence of $k$ and $S$ which satisfy $\text{Gal}(H_S(k)/k) \cong G$. (More precisely, we can take $k$ and $S$ such that $H_S(k)/k$ is a geometric extension. See [16].)

In the present paper, we extend the above result to non-abelian finite groups. We will show the following:

Theorem 1.3. For any given finite group $G$, there is a global function field $k$ with full constant field $\mathbb{F}$ and a non-empty finite set $S$ of places of $k$ such that $\text{Gal}(\tilde{H}_S(k)/k) \cong G$, where $\tilde{H}_S(k)$ denotes the maximal unramified Galois extension field of $k$ in which all places of $S$ split completely. Moreover, we can take $k$ and $S$ such that $\tilde{H}_S(k)/k$ is a geometric extension.

See Ozaki [15] for the number field case.

We will prove Theorem 1.3 in section 2. Our proof is due to Perret’s idea (see [16]). That is, we will construct an unramified $G$-extension, and take a sufficiently large set $S$ of places such that $\text{Gal}(\tilde{H}_S(k)/k) \cong G$. (We use the term “$G$-extension” as a Galois extension whose Galois group is isomorphic to $G$.) To construct an unramified $G$-extension, we shall show an analog (Theorem 2.2) of Fröhlich’s classical result [4] for number fields.

In section 3, we shall apply Perret’s idea to Iwasawa theory. Let $k$ be a global function field with full constant field $\mathbb{F}$, $S$ a non-empty finite set of places of $k$. We recall that a $\mathbb{Z}_p$-extension is an infinite Galois extension whose Galois group is topologically isomorphic to the additive group of the ring $\mathbb{Z}_p$ of $p$-adic integers. Let $k_\infty/k$ be a geometric $\mathbb{Z}_p$-extension, that is, $k_\infty/k$ is a $\mathbb{Z}_p$-extension which satisfies that every finite subextension over $k$ is a geometric extension (see, e.g., [7]). (Recall that $p$ is the characteristic of $\mathbb{F}$.) We assume that

(A) only finitely many places of $k$ ramify in $k_\infty/k$, and

(B) all places of $S$ split completely in $k_\infty/k$.

Under these assumptions, we can treat Iwasawa theory for the $S$-class group (see [17]). For a non-negative integer $n$, let $k_n$ be the $n$th layer of $k_\infty/k$. That is, $k_n$ is the unique subfield of $k_\infty$ which is a cyclic extension over $k$ of degree $p^n$. Moreover, let $A_n$ be the
Sylow p-subgroup of the S-class group of \( k_n \). (Here we use the same symbol S as the set of places of \( k_n \) lying above S.) We put \( X_S = \varprojlim A_n \), where the projective limit is taken with respect to the norm maps. We call \( X_S \) the Iwasawa module of \( k_\infty/k \) for the S-class group. We put \( \Lambda = \mathbb{Z}_p[[\text{Gal}(k_\infty/k)]] \). Note that \( \Lambda \cong \mathbb{Z}_p[[T]] \). It is known that \( X_S \) is a finitely generated torsion \( \Lambda \)-module, and the “Iwasawa type formula” holds for \( A_n \) (see [17]). That is, there are non-negative integers \( \lambda, \mu, \) and an integer \( \nu \) such that \( |A_n| = p^{\lambda n + \mu p^n + \nu} \) for all sufficiently large \( n \). Aiba [1] studied these invariants \( \lambda, \mu, \) and \( \nu \) for certain geometric \( \mathbb{Z}_p \)-extensions.

There is a natural problem: characterize the \( \Lambda \)-modules which appear as \( X_S \). (For the number field case, the same problem is dealt in, e.g., [14], [5].) Concerning this problem, we shall give the following result including “non-abelian” cases.

**Theorem 1.4.** For any given finite p-group \( G \), there exist a global function field \( k \) with full constant field \( \mathbb{F} \), a non-empty finite set \( S \) of places of \( k \), and a geometric \( \mathbb{Z}_p \)-extension \( k_\infty/k \) satisfying the above assumptions (A) and (B) such that \( \text{Gal}(\bar{L}_S(k_n)/k_n) \cong G \) (as groups) for all \( n \geq 0 \), where \( \bar{L}_S(k_n) \) is the maximal unramified Galois pro-p-extension field of \( k_n \) in which all places lying above \( S \) split completely.

For the number field case, Ozaki [14] showed that every “finite \( \Lambda \)-module” appears as the Iwasawa module of a \( \mathbb{Z}_p \)-extension. Theorem 1.4 for \( G \) abelian gives a weak analog of Ozaki’s result. That is, every finite \( \Lambda \)-module on which \( \text{Gal}(k_\infty/k) \) acts trivially appears as \( X_S \). We will prove Theorem 1.4 in section 3.

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### § 2. Proof of Theorem 1.3

#### § 2.1. Function field analog of Fröhlich’s result

At first, we shall show that for any finite group \( G \), there is an unramified geometric extension \( K/k \) of global function fields such that \( \text{Gal}(K/k) \cong G \). Recall that any finite group can be embedded into a finite symmetric group. Hence it is sufficient to consider the case that \( G \) is a finite symmetric group. For the number field case, Fröhlich already showed the following result.

**Theorem 2.1** ([4]). For every positive integer \( n \), there is an unramified Galois extension \( K/k \) of algebraic number fields such that \( \text{Gal}(K/k) \cong \mathfrak{S}_n \), where \( \mathfrak{S}_n \) denotes the symmetric group of degree \( n \).

We will show the following:
Theorem 2.2. For every positive integer $n$, there is a global function field $k$ with full constant field $\mathbb{F}$ and an unramified geometric Galois extension $K/k$ such that $\text{Gal}(K/k) \cong \mathfrak{S}_n$. More precisely, there exist a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that $K/k$ is unramified and that $\text{Gal}(K/k) \cong \mathfrak{S}_n$.

To prove this, we follow Fröhlich’s original argument (see also Malinin [10]). That is, we construct a certain (ramified) $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$ and then we take a certain base change of this extension. Let $\infty$ be the infinite place of $\mathbb{F}(T)$.

Lemma 2.3. There is a Galois extension $k'$ over $\mathbb{F}(T)$ which satisfies all of the following properties.

- $k'/\mathbb{F}(T)$ is a geometric extension,
- $\text{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n$, and
- $\infty$ is unramified in $k'/\mathbb{F}(T)$.

Proof. At first, we must see that there is an $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$. This follows from the fact that $\mathbb{F}(T)$ is a Hilbertian field (see, e.g., [3, Corollary 16.2.7]). We put $A = \mathbb{F}[T]$. For an element $r$ of $A$, let $\deg(r)$ be the degree of $r$ as a polynomial of $T$. Fix a monic separable polynomial $F(X) \in A[X]$ of degree $n$ such that the splitting field of $F(X)$ over $\mathbb{F}(T)$ is an $\mathfrak{S}_n$-extension.

We claim that there is an element $N_F \in A$ which satisfies the following property: if a monic polynomial $G(X) \in A[X]$ of degree $n$ satisfies $G(X) \equiv F(X) \pmod{N_F}$, then the splitting field of $G(X)$ over $\mathbb{F}(T)$ is also an $\mathfrak{S}_n$-extension. We shall show this claim. By using the Chebotarev density theorem, we can take an irreducible monic polynomial $p_1$ such that if $G(X) \equiv F(X) \pmod{p_1}$ then $G(X)$ is irreducible and separable. Similarly, we can take distinct irreducible monic polynomials $p_2, p_3$ of $A = \mathbb{F}(T)$ which are distinct from $p_1$ and satisfy the following properties: (i) if $G(X) \equiv F(X) \pmod{p_2}$ then the Galois group of $G(X)$ contains a cycle of length $n - 1$ (as a subgroup of $\mathfrak{S}_n$), and (ii) if $G(X) \equiv F(X) \pmod{p_3}$ then the Galois group of $G(X)$ contains a transposition. We put $N_F = p_1p_2p_3$. This $N_F$ satisfies the above claim. Moreover, we can take $N_F$ which is prime to $T$ by the Chebotarev density theorem. We also fix such $N_F$.

To construct a geometric $\mathfrak{S}_n$-extension which is unramified at the infinite place, we take $G(X)$ as follows:

$$
G(X) \equiv F(X) \pmod{N_F},
$$

$$
G(X) \equiv (\text{a product of distinct monic polynomials of degree 1}) \pmod{r}, \text{ and}
$$

$$
G(X) \equiv (\text{a separable polynomial}) \pmod{T},
$$

where $r$ is a monic irreducible polynomial of $A = \mathbb{F}[T]$ such that $n < q^{\deg(r)}$, $\deg(r)$ is odd, and $r$ is prime to $TN_F$. By the first congruence, we see that the splitting field $k'$
of $G(X)$ is an $\mathfrak{S}_n$-extension. We shall show that the constant field of $k'$ is $F$. Let $\overline{F}$ be the algebraic closure of $F$. We note that $M := k' \cap \overline{F}(T)$ is a finite cyclic extension over $\overline{F}(T)$. Since $\text{Gal}(k'/\overline{F}(T)) \cong \mathfrak{S}_n$, $M$ must be $\overline{F}(T)$ or the unique quadratic subfield in $k'/\overline{F}(T)$. If $M \neq \overline{F}(T)$, then no odd degree place of $\overline{F}(T)$ splits in $M$. However, we see that the place of $\overline{F}(T)$ corresponding to $r$ splits completely in $k'$ by the second congruence. It is a contradiction.

By the third congruence, we see that the place of $\overline{F}(T)$ corresponding to $T$ is unramified in $k'$. We replace the indeterminate $T$ by $U = 1/T$, then the infinite place of $\overline{F}(U)$ is unramified in $k'$ (and the former two conditions are also satisfied).

We shall prove Theorem 2.2. We may assume that $n \geq 2$. Fix a geometric $\mathfrak{S}_n$-extension $k'/\overline{F}(T)$ satisfying the properties of Lemma 2.3. We put $m = n!$. We can take a separable monic polynomial $F(X) \in A[X]$ of degree $m$ (as a polynomial of $X$) whose splitting field over $\overline{F}(T)$ is $k'$. Let $M'$ be the unique quadratic subextension field of $\overline{F}(T)$ contained in $k'$.

We define the following notation.

- $\{p_1, \ldots, p_t\}$ : the set of distinct places of $\overline{F}(T)$ which ramify in $k'$ (hence are distinct from $\infty$).
- $p_{t+1}$ : a place $\neq \infty, p_1, \ldots, p_t$ of $\overline{F}(T)$ which is inert in $M'$ and has degree $> \frac{\log(m)}{\log(q)}$.
- $p_{t+2}$ : a place $\neq \infty$ of $\overline{F}(T)$ which splits completely in $k'$ and has odd degree $> \frac{\log(m)}{\log(q)}$ (hence is distinct from $p_1, \ldots, p_t, p_{t+1}$).
- $p_1, \ldots, p_{t+2}$ : irreducible monic polynomials of $A = \mathbb{F}[T]$ corresponding to $p_1, \ldots, p_{t+2}$, respectively.

Note that we can take $p_{t+1}$ (resp. $p_{t+2}$) by using Theorem 9.13B of [18], which is an effective version of the Chebotarev density theorem for global function fields. (See also [12], etc.) Indeed, by this theorem, there is a place of $\overline{F}(T)$ of arbitrary sufficiently large degree which is inert in $M'$ (resp. splits completely in $k'$), as $M'/\overline{F}(T)$ is a geometric cyclic extension (resp. $k'/\overline{F}(T)$ is a geometric Galois extension).

By using Lemma 2.3, we can also construct an $\mathfrak{S}_m$-extension over $\overline{F}(T)$. Let $H(X)$ be a monic polynomial in $A[X]$ of degree $m$ which gives an $\mathfrak{S}_m$-extension. Then there is an element $N_H$ of $A$ having the following property: if a monic polynomial $G(X) \in A[X]$ of degree $m$ satisfies $G(X) \equiv H(X) \pmod{N_H}$, then the splitting field of $G(X)$ over $\overline{F}(T)$ is also an $\mathfrak{S}_m$-extension (see the proof of Lemma 2.3). We can also take $N_H$ such that it is prime to $p_1, \ldots, p_{t+2}$.

We take a monic polynomial $G(X)$ of $A[X]$ (having degree $m$) which satisfies the following conditions (2.1)–(2.4).

\begin{equation}
G(X) \equiv H(X) \pmod{N_H}.
\end{equation}
If $G(X)$ satisfies (2.1), then $G(X)$ gives an $\mathfrak{S}_m$-extension. Let $L$ be the splitting field of $G(X)$ over $\mathbb{F}(T)$.

(2.2) $G(X) \equiv (\text{a product of distinct monic polynomials of degree } 1) \pmod{p_{t+1}}$.

If $G(X)$ satisfies (2.1) and (2.2), then we see that $p_{t+1}$ splits in the unique quadratic subextension, say $M_L$, over $\mathbb{F}(T)$ contained in $L$. On the other hand, $p_{t+1}$ is inert in the unique quadratic subextension $M'$ over $\mathbb{F}(T)$ contained in $k'$. We claim that $k' \cap L = \mathbb{F}(T)$. Indeed, suppose that $k' \cap L \neq \mathbb{F}(T)$. Then $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. If $n = 2$, this is clear. For $n \geq 3$, we have $\text{Gal}(L/\mathbb{F}(T)) \cong \mathfrak{S}_m$, where $m = n! \geq 5$. Observe also that $k' \cap L \neq L$, as $m > n$. Now, since the alternating group $\mathfrak{A}_m$ is the unique nontrivial proper normal subgroup of $\mathfrak{S}_m$ when $m \geq 5$ (see, e.g., [19]), $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. Since this quadratic extension is contained in both $k'$ and $L$, it must coincide with both $M'$ and $M_L$ at a time. This contradicts the above observation on the behavior of $p_{t+1}$ in $M'$ and $M_L$. Thus, we have proved the claim. Then we see $\text{Gal}(Lk'/L) \cong \mathfrak{S}_n$.

(2.3) $G(X) \equiv (\text{a product of distinct monic polynomials of degree } 1) \pmod{p_{t+2}}$.

If $G(X)$ satisfies (2.1)–(2.3), then the odd degree place $p_{t+2}$ splits completely in $Lk'/\mathbb{F}(T)$. We claim that $Lk'/\mathbb{F}(T)$ is a geometric extension. Note that the degree of a place of $k'$ lying above $p_{t+2}$ is also odd because $p_{t+2}$ splits completely in $k'$. Since $\text{Gal}(Lk'/k') \cong \mathfrak{S}_m$ and an odd degree place splits completely in $Lk'/k'$, we see that $Lk'/k'$ is also a geometric extension. Hence the claim follows. By using Krasner’s lemma, we can see that there is a positive integer $s_i$ for each $i = 1, \ldots, t$ depending only on $F(X)$ such that if $G(X) \equiv F(X) \pmod{p_i^{s_i}}$ then $L\mathbb{F}(T)_{p_i} = k'\mathbb{F}(T)_{p_i}$, where $\mathbb{F}(T)_{p_i}$ is the completion of $\mathbb{F}(T)$ at $p_i$ (see, e.g., [13]). Hence if we take $G(X)$ satisfying (2.1)–(2.3) and

(2.4) $G(X) \equiv F(X) \pmod{p_i^{s_i}}$ for $i = 1, \ldots, t$,

then we can see that $Lk'/L$ is unramified at all places.

We can take $G(X)$ satisfying (2.1)–(2.4). By the above arguments, the extension $Lk'/L$ satisfies the assertion of Theorem 2.2. \hfill \Box

Remark. When $G$ is abelian, an unramified geometric $G$-extension was constructed by Angles [2]. Moret-Bailly [11] also gives a result which is very close to ours. Probably, it seems that one can prove our main theorems by using the result given in [11] instead of Theorem 2.2.

§ 2.2. Proof of Theorem 1.3

Since $G$ is embedded into $\mathfrak{S}_n$ for some $n > 0$, Theorem 2.2 implies that there exists a global function field $k$ with full constant field $\mathbb{F}$ and an unramified geometric Galois extension $K/k$ such that $\text{Gal}(K/k) \cong G$. 
Proposition 2.4. There is a non-empty finite set $S$ of places of $k$ such that (i) all places of $S$ split completely in $K$, and (ii) $H_S(k)/k$ is a finite extension.

Proof. The crucial point of this proposition is choosing a set $S$ to satisfy (ii). For a positive integer $N$, we put

$$B_N = \{ \mathfrak{p} | \mathfrak{p} \text{ is a place of } k \text{ having degree } N, \mathfrak{p} \text{ splits completely in } K/k \}.$$  

Since $K/k$ is a geometric extension, Theorem 9.13B of [18] implies that

$$|B_N| = \frac{q^N}{|G|N} + O\left(\frac{q^{N/2}}{N}\right)$$

(recall that $q$ is the number of elements of $\mathbb{F}$). In particular, if $N$ is sufficiently large, then we obtain the inequality

$$|B_N| > \frac{q^{N/2} - 1}{N} \max(g - 1, 0),$$

where $g$ is the genus of $k$. We fix an integer $N$ which satisfies the above inequality. According to Ihara’s theorem [8, Theorem 1(FF)], if $S \supset B_N$, then $H_S(k)/k$ is a finite extension. Hence we can take $S$ to satisfy the conditions (i) and (ii). 

The rest of the proof of Theorem 1.3 is quite similar to Perret’s argument given in [16]. We choose a set $S$ of places which satisfies the conditions of Proposition 2.4. We remark that $K$ is contained in $H_S(k)$. For a nontrivial element $\sigma$ of $\text{Gal}(H_S(k)/K)$, we can take a place $\mathfrak{P}$ of $H_S(k)$ corresponding to $\sigma$ by the Chebotarev density theorem. We can take $\mathfrak{P}$ which is unramified in $H_S(k)/K$. Let $\mathfrak{p}$ be the place of $k$ which is lying below $\mathfrak{P}$. Since the decomposition field of $\mathfrak{P}$ in $H_S(k)/k$ contains $K$ and $K/k$ is a Galois extension, we see that $\mathfrak{p}$ splits completely in $K/k$. Then we see $H_S(k) \supset H_{S \cup \{ \mathfrak{p} \}}(k) \supset K$. Replacing $S \cup \{ \mathfrak{p} \}$ by $S$ and repeating the above operation, we can see that $H_S(k) = K$ for some finite set $S$. This implies $\text{Gal}(H_S(k)/K) \cong G$.

We recall that $K/k$ is a geometric extension. Hence the final part of the theorem follows. 

§ 3. Proof of Theorem 1.4

Firstly, we shall show the following:

Theorem 3.1. Let $k$ be a finite Galois extension over $\mathbb{F}(T)$. Then, there exist a non-empty finite set $S$ of places of $\mathbb{F}(T)$ and a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ which satisfy the following properties.
\begin{itemize}
\item $F_\infty \cap k = \mathbb{F}(T)$,
\item all places of $S$ split completely in $k$,
\item both of $F_\infty /\mathbb{F}(T)$ and $F_\infty k/k$ satisfy the assumptions (A) and (B) in section 1, and
\item the Sylow $p$-subgroup of $\text{Cl}_S(F_n k)$ is trivial for all $n \geq 0$,
\end{itemize}

where $F_n$ is the $n$th layer of $F_\infty /\mathbb{F}(T)$. (We use the same symbol $S$ as the set of places lying above $S$.)

Proof. We take a place $p_0$ of $\mathbb{F}(T)$ which splits completely in $k$. We also take a place $r$ of $\mathbb{F}(T)$ which is distinct from $p_0$ and unramified in $k$. We claim that there is a geometric $\mathbb{Z}_p$-extension $F_\infty /\mathbb{F}(T)$ unramified outside $r$ which satisfies that

\begin{itemize}
\item $r$ is totally ramified,
\item $p_0$ splits completely.
\end{itemize}

We shall show this claim. Let $M$ be the maximal pro-$p$ abelian extension over $\mathbb{F}(T)$ which is unramified outside $r$. We know that $\text{Gal}(M/\mathbb{F}(T))$ is isomorphic to a countable infinite product of the additive group of $\mathbb{Z}_p$ (see [21], [9]). Hence there are infinitely many geometric $\mathbb{Z}_p$-extensions which satisfy the above conditions.

By the above choice of $F_\infty$, we see $F_1 \cap k = \mathbb{F}(T)$. We put $k_1 = F_1 k$. Then $k_1 /\mathbb{F}(T)$ is a Galois extension, and $p_0$ splits completely in $k_1$. We set $S_0 = \{p_0\}$, and we use the same symbol to denote the set of places lying above $p_0$. We can see that $H_{S_0}(k_1)$ is a finite Galois extension over $\mathbb{F}(T)$. We take a nontrivial element $\sigma_1$ of $\text{Gal}(H_{S_0}(k_1)/k_1)$.

By using the above argument, we can take a geometric $\mathbb{Z}_p$-extension $F'_\infty /\mathbb{F}(T)$ unramified outside $r$ which satisfies

\begin{itemize}
\item $F'_\infty \cap F_\infty = \mathbb{F}(T)$,
\item $r$ is totally ramified in $F'_\infty F_\infty$, and
\item $p_0$ splits completely in $F'_\infty$.
\end{itemize}

Let $F'_1$ be the initial layer of $F'_\infty /\mathbb{F}(T)$. Then we see that $F'_1 \cap k_1 = \mathbb{F}(T)$ and $k_1 F'_1 \cap H_{S_0}(k_1) = k_1$. We note that

$$\text{Gal}(F'_1 H_{S_0}(k_1)/k_1) \cong \text{Gal}(F'_1 k_1/k_1) \times \text{Gal}(H_{S_0}(k_1)/k_1), \quad \text{Gal}(F'_1 k_1/k_1) \cong \text{Gal}(F'_1 /\mathbb{F}(T)).$$

Hence there is an isomorphism

$$\text{Gal}(F'_1 /\mathbb{F}(T)) \times \text{Gal}(H_{S_0}(k_1)/k_1) \xrightarrow{\sim} \text{Gal}(F'_1 H_{S_0}(k_1)/k_1).$$
Let $\tau$ be a generator of the cyclic group $\text{Gal}(F'_{1}/F(T))$, and $\tau_1$ an element of $\text{Gal}(F'_{1}H_{S_0}(k_1)/k_1)$ which is the image of $(\tau, \sigma_1)$ under the above isomorphism. We can regard $\tau$ as an element of $\text{Gal}(F'_{1}H_{S_0}(k_1)/F(T))$. By the Chebotarev density theorem, there is a place $\mathfrak{p}_1$ of $F'_{1}H_{S_0}(k_1)$ which corresponds to $\tau_1$. Let $p_1$ be the place of $F(T)$ lying below $\mathfrak{p}_1$. We can take $\mathfrak{p}_1$ such that $p_1$ is not ramified in $F'_{1}H_{S_0}(k_1)$. Then we see that $p_1$ splits completely in $k_1$ and is inert in $F'_{1}$. We put $S_1 = S_0 \cup \{p_1\}$.

In general, $p_1$ may not split completely in $F_\infty$. This is a problem because we need the assumption (B). We remark that $F_\infty F'_{1}/F(T)$ is a $\mathbb{Z}_p$-extension unramified outside $\tau$. We recall that $p_1$ does not split in $F'_{1}$. Hence the decomposition field of $F_\infty F'_{1}/F(T)$ for $p_1$ is a $\mathbb{Z}_p$-extension over $F(T)$. We denote it by $F''_{\infty}$. We also note that $F''_{\infty}/F(T)$ is the unique $\mathbb{Z}_p$-extension contained in $F_\infty F_\infty'$ such that $p_1$ splits completely. Then the initial layer of $F''_{\infty}/F(T)$ must coincide with $F_1$. We replace $F_\infty$ by $F''_{\infty}$.

We note that $H_{S_0}(k_1) \supseteq H_{S_1}(k_1)$ by the definition of $p_1$. Similarly, we can choose a place $p_2$, put $S_2 = S_1 \cup \{p_2\}$, and modify the $\mathbb{Z}_p$-extension such that all places of $S_2$ splits completely. Repeating this operation, we see that $H_{S_1}(k_1) = k_1$ for some finite set $S_t$. From the above construction, we see that $F_\infty \cap k = F(T)$ and that $F_\infty k/k$ satisfies the assumptions (A) and (B).

Finally, we shall give an Iwasawa-theoretic argument. In $F_\infty k/k$, all ramified places (which are lying above $\tau$) are totally ramified. From this, we also see $H_{S_1}(k) = k$. Let $A_n$ be the Sylow $p$-subgroup of $\text{Cl}_{S_1}(kF_n)$. By the above results, we see that both of $A_0$ and $A_1$ are trivial. In this situation, we can use the method given in Fukuda [6]. Namely, if all places which ramify in $F_\infty k/k$ are totally ramified and both of $A_0$ and $A_1$ are trivial, then $A_n$ is trivial for all $n \geq 0$. (See [6, Theorem 1]. We note that the same method is also applicable for our situation.) Hence we see that $A_n$ is trivial for all $n \geq 0$.

We shall show Theorem 1.4. We fix a finite $p$-group $G$. By using Theorem 2.2, we can take a geometric Galois extension $K/F(T)$ and a subextension $k/F(T)$ of $K/F(T)$ such that $K/k$ is unramified and $\text{Gal}(K/k) \cong G$. By Theorem 3.1, we can take a geometric $\mathbb{Z}_p$-extension $F_\infty/F(T)$ and a set $S$ of places of $F(T)$ such that $F_\infty \cap K = F(T)$, all places of $S$ split completely in $K$, both of $F_\infty/F(T)$ and $F_\infty K/K$ satisfy the assumptions (A) and (B), and $A_n$ is trivial for all $n \geq 0$ (where $A_n$ is the Sylow $p$-subgroup of $\text{Cl}_S(F_nK)$, and $F_n$ is the $n$th layer of $F_\infty/F(T)$). We note that $F_\infty k/k$ also satisfies the assumptions (A) and (B). We claim that $\tilde{L}_S(F_nK) = F_nK$ for all $n \geq 0$. Indeed, if $\tilde{L}_S(F_nK)/F_nK$ is nontrivial, then there is a nontrivial finite Galois $p$-subextension over $F_nK$. Moreover, there is a nontrivial finite abelian $p$-subextension over $F_nK$ because every $p$-group is solvable. Since $A_n$ is trivial, it is a contradiction. We have shown the above claim. This implies that $\tilde{L}_S(F_nK) = F_nK$ because $F_nK/F_nK$ is unramified and all places of $F_nK$ lying above $S$ split completely in $F_nK$. Hence
Gal(\tilde{L}_S(F_nk)/F_nk) \cong G for all \( n \geq 0 \). Then the theorem follows. \qed

References


