RIMS Kôkyûroku Bessatsu
B12 (2009), 173–182

Unramified extensions and geometric $\mathbb{Z}_p$-extensions of global function fields

By

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Abstract

We study on finite unramified extensions of global function fields (that is, function fields of one variable over a finite field). We show two results. One is an extension of Perret's result about the ideal class group problem. Another is a construction of a geometric $\mathbb{Z}_p$-extension which has a certain property.

§ 1. Main theorems

Throughout the present paper, we fix a prime number p and a finite field $\mathbb{F}$ of characteristic p. Let q be the number of elements of $\mathbb{F}$. Recall that a global function field is a function field of one variable over a finite field. Let k be a global function field with full constant field $\mathbb{F}$. We also recall that a finite algebraic extension K/k is geometric if and only if the constant field of K is also $\mathbb{F}$.

It is known that there is a finite abelian group G which is not isomorphic to the divisor class group of degree 0 of any global function field (Stichtenoth [20]). On the other hand, Perret [16] showed the following:

Theorem 1.1 ([16]). *For any given finite abelian group G, there is a finite separable geometric extension $k/\mathbb{F}(T)$ such that $\mathrm{Cl}(\mathcal{O}) \cong G$, where $\mathcal{O}$ is the integral closure of $\mathbb{F}[T]$ in k and $\mathrm{Cl}(\mathcal{O})$ is the ideal class group of $\mathcal{O}$.*

This theorem is shown by using the following:

Received March 31, 2008 Revised April 17, 2009.
2000 Mathematics Subject Classification(s): 11R58, 11R23
Key Words: global function fields, unramified extensions

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Theorem 1.2 ([16]). *For any given finite abelian group G, there is a global function field k with full constant field $\mathbb{F}$ and a non-empty finite set S of places of k such that $\mathrm{Cl}_S(k) \cong G$, where $\mathrm{Cl}_S(k)$ is the S-class group of k.*

Let S be a non-empty finite set of places of k, and $H_S(k)$ the S-Hilbert class field of k, that is, the maximal unramified abelian extension field of k in which all places of S split completely (see [17]). We note that $\mathrm{Cl}_S(k) \cong \mathrm{Gal}(H_S(k)/k)$ by class field theory. Hence Theorem 1.2 also implies the existence of k and S which satisfy $\mathrm{Gal}(H_S(k)/k) \cong G$. (More precisely, we can take k and S such that $H_S(k)/k$ is a geometric extension. See [16].)

In the present paper, we extend the above result to non-abelian finite groups. We will show the following:

Theorem 1.3. *For any given finite group G, there is a global function field k with full constant field $\mathbb{F}$ and a non-empty finite set S of places of k such that $\mathrm{Gal}(\tilde{H}_S(k)/k) \cong G$, where $\tilde{H}_S(k)$ denotes the maximal unramified Galois extension field of k in which all places of S split completely. Moreover, we can take k and S such that $\tilde{H}_S(k)/k$ is a geometric extension.*

See Ozaki [15] for the number field case.

We will prove Theorem 1.3 in section 2. Our proof is due to Perret's idea (see [16]). That is, we will construct an unramified G-extension, and take a sufficiently large set S of places such that $\mathrm{Gal}(\tilde{H}_S(k)/k) \cong G$. (We use the term "$G$-extension" as a Galois extension whose Galois group is isomorphic to G.) To construct an unramified G-extension, we shall show an analog (Theorem 2.2) of Fröhlich's classical result [4] for number fields.

In section 3, we shall apply Perret's idea to Iwasawa theory. Let k be a global function field with full constant field $\mathbb{F}$, S a non-empty finite set of places of k. We recall that a $\mathbb{Z}_p$-extension is an infinite Galois extension whose Galois group is topologically isomorphic to the additive group of the ring $\mathbb{Z}_p$ of p-adic integers. Let k_∞/k be a geometric $\mathbb{Z}_p$-extension, that is, k_∞/k is a $\mathbb{Z}_p$-extension which satisfies that every finite subextension over k is a geometric extension (see, e.g., [7]). (Recall that p is the characteristic of $\mathbb{F}$.) We assume that

(A) only finitely many places of k ramify in k_∞/k, and

(B) all places of S split completely in k_∞/k.

Under these assumptions, we can treat Iwasawa theory for the S-class group (see [17]). For a non-negative integer n, let k_n be the nth layer of k_∞/k. That is, k_n is the unique subfield of k_∞ which is a cyclic extension over k of degree p^n. Moreover, let A_n be the

Sylow p-subgroup of the S-class group of k_n. (Here we use the same symbol S as the set of places of k_n lying above S.) We put $X_S = \varprojlim A_n$, where the projective limit is taken with respect to the norm maps. We call X_S the Iwasawa module of k_∞/k for the S-class group. We put $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$. Note that $\Lambda \cong \mathbb{Z}_p[[T]]$. It is known that X_S is a finitely generated torsion Λ-module, and the "Iwasawa type formula" holds for A_n (see [17]). That is, there are non-negative integers λ, μ, and an integer ν such that $|A_n| = p^{\lambda n + \mu p^n + \nu}$ for all sufficiently large n. Aiba [1] studied these invariants λ, μ, and ν for certain geometric $\mathbb{Z}_p$-extensions.

There is a natural problem: characterize the Λ-modules which appear as X_S. (For the number field case, the same problem is dealt in, e.g., [14], [5].) Concerning this problem, we shall give the following result including "non-abelian" cases.

Theorem 1.4. *For any given finite p-group G, there exist a global function field k with full constant field $\mathbb{F}$, a non-empty finite set S of places of k, and a geometric $\mathbb{Z}_p$-extension k_∞/k satisfying the above assumptions* (A) *and* (B) *such that* $\mathrm{Gal}(\tilde{L}_S(k_n)/k_n) \cong G$ *(as groups) for all $n \geq 0$, where $\tilde{L}_S(k_n)$ is the maximal unramified Galois pro-p-extension field of k_n in which all places lying above S split completely.*

For the number field case, Ozaki [14] showed that every "finite Λ-module" appears as the Iwasawa module of a $\mathbb{Z}_p$-extension. Theorem 1.4 for G abelian gives a weak analog of Ozaki's result. That is, every finite Λ-module on which $\mathrm{Gal}(k_\infty/k)$ acts trivially appears as X_S. We will prove Theorem 1.4 in section 3.

The author would like to express his thanks to the referee for giving valuable comments. Especially, the referee gave many suggestions for improving the presentation.

§ 2. Proof of Theorem 1.3

§ 2.1. Function field analog of Fröhlich's result

At first, we shall show that for any finite group G, there is an unramified geometric extension K/k of global function fields such that $\mathrm{Gal}(K/k) \cong G$. Recall that any finite group can be embedded into a finite symmetric group. Hence it is sufficient to consider the case that G is a finite symmetric group. For the number field case, Fröhlich already showed the following result.

Theorem 2.1 ([4]). *For every positive integer n, there is an unramified Galois extension K/k of algebraic number fields such that* $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$, *where $\mathfrak{S}_n$ denotes the symmetric group of degree n.*

We will show the following:

Theorem 2.2. *For every positive integer n, there is a global function field k with full constant field $\mathbb{F}$ and an unramified geometric Galois extension K/k such that $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$. More precisely, there exist a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that K/k is unramified and that $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$.*

To prove this, we follow Fröhlich's original argument (see also Malinin [10]). That is, we construct a certain (ramified) $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$ and then we take a certain base change of this extension. Let ∞ be the infinite place of $\mathbb{F}(T)$.

Lemma 2.3. *There is a Galois extension k' over $\mathbb{F}(T)$ which satisfies all of the following properties.*

- *$k'/\mathbb{F}(T)$ is a geometric extension,*
- *$\mathrm{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n$, and*
- *∞ is unramified in $k'/\mathbb{F}(T)$.*

Proof. At first, we must see that there is an $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$. This follows from the fact that $\mathbb{F}(T)$ is a Hilbertian field (see, e.g, [3, Corollary 16.2.7]). We put $A = \mathbb{F}[T]$. For an element r of A, let $\deg(r)$ be the degree of r as a polynomial of T. Fix a monic separable polynomial $F(X) \in A[X]$ of degree n such that the splitting field of $F(X)$ over $\mathbb{F}(T)$ is an $\mathfrak{S}_n$-extension.

We claim that there is an element $N_F \in A$ which satisfies the following property: if a monic polynomial $G(X) \in A[X]$ of degree n satisfies $G(X) \equiv F(X) \pmod{N_F}$, then the splitting field of $G(X)$ over $\mathbb{F}(T)$ is also an $\mathfrak{S}_n$-extension. We shall show this claim. By using the Chebotarev density theorem, we can take an irreducible monic polynomial p_1 such that if $G(X) \equiv F(X) \pmod{p_1}$ then $G(X)$ is irreducible and separable. Similarly, we can take distinct irreducible monic polynomials p_2, p_3 of $A = \mathbb{F}(T)$ which are distinct from p_1 and satisfy the following properties: (i) if $G(X) \equiv F(X) \pmod{p_2}$ then the Galois group of $G(X)$ contains a cycle of length $n-1$ (as a subgroup of $\mathfrak{S}_n$), and (ii) if $G(X) \equiv F(X) \pmod{p_3}$ then the Galois group of $G(X)$ contains a transposition. We put $N_F = p_1 p_2 p_3$. This N_F satisfies the above claim. Moreover, we can take N_F which is prime to T by the Chebotarev density theorem. We also fix such N_F.

To construct a geometric $\mathfrak{S}_n$-extension which is unramified at the infinite place, we take $G(X)$ as follows:

$$\begin{aligned} G(X) &\equiv F(X) && \pmod{N_F}, \\ G(X) &\equiv (\text{a product of distinct monic polynomials of degree 1}) && \pmod{r}, \textit{and} \\ G(X) &\equiv (\text{a separable polynomial}) && \pmod{T}, \end{aligned}$$

where r is a monic irreducible polynomial of $A = \mathbb{F}[T]$ such that $n < q^{\deg(r)}$, $\deg(r)$ is odd, and r is prime to TN_F. By the first congruence, we see that the splitting field k'

of $G(X)$ is an $\mathfrak{S}_n$-extension. We shall show that the constant field of k' is $\mathbb{F}$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. We note that $M := k' \cap \overline{\mathbb{F}}(T)$ is a finite cyclic extension over $\mathbb{F}(T)$. Since $\mathrm{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n$, M must be $\mathbb{F}(T)$ or the unique quadratic subfield in $k'/\mathbb{F}(T)$. If $M \neq \mathbb{F}(T)$, then no odd degree place of $\mathbb{F}(T)$ splits in M. However, we see that the place of $\mathbb{F}(T)$ corresponding to r splits completely in k' by the second congruence. It is a contradiction.

By the third congruence, we see that the place of $\mathbb{F}(T)$ corresponding to T is unramified in k'. We replace the indeterminate T by $U = 1/T$, then the infinite place of $\mathbb{F}(U)$ is unramified in k' (and the former two conditions are also satisfied). □

We shall prove Theorem 2.2. We may assume that $n \geq 2$. Fix a geometric $\mathfrak{S}_n$-extension $k'/\mathbb{F}(T)$ satisfying the properties of Lemma 2.3. We put $m = n!$. We can take a separable monic polynomial $F(X) \in A[X]$ of degree m (as a polynomial of X) whose splitting field over $\mathbb{F}(T)$ is k'. Let M' be the unique quadratic subextension field of $\mathbb{F}(T)$ contained in k'.

We define the following notation.

- $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$: the set of distinct places of $\mathbb{F}(T)$ which ramify in k' (hence are distinct from ∞).
- $\mathfrak{p}_{t+1}$: a place $\neq \infty, \mathfrak{p}_1, \ldots, \mathfrak{p}_t$ of $\mathbb{F}(T)$ which is inert in M' and has degree $> \frac{\log(m)}{\log(q)}$.
- $\mathfrak{p}_{t+2}$: a place $\neq \infty$ of $\mathbb{F}(T)$ which splits completely in k' and has **odd** degree $> \frac{\log(m)}{\log(q)}$ (hence is distinct from $\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{p}_{t+1}$).
- $p_1, \ldots, p_{t+2}$: irreducible monic polynomials of $A = \mathbb{F}[T]$ corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_{t+2}$, respectively.

Note that we can take $\mathfrak{p}_{t+1}$ (resp. $\mathfrak{p}_{t+2}$) by using Theorem 9.13B of [18], which is an effective version of the Chebotarev density theorem for global function fields. (See also [12], etc.) Indeed, by this theorem, there is a place of $\mathbb{F}(T)$ of arbitrary sufficiently large degree which is inert in M' (resp. splits completely in k'), as $M'/\mathbb{F}(T)$ is a geometric cyclic extension (resp. $k'/\mathbb{F}(T)$ is a geometric Galois extension).

By using Lemma 2.3, we can also construct an $\mathfrak{S}_m$-extension over $\mathbb{F}(T)$. Let $H(X)$ be a monic polynomial in $A[X]$ of degree m which gives an $\mathfrak{S}_m$-extension. Then there is an element N_H of A having the following property: if a monic polynomial $G(X) \in A[X]$ of degree m satisfies $G(X) \equiv H(X) \pmod{N_H}$, then the splitting field of $G(X)$ over $\mathbb{F}(T)$ is also an $\mathfrak{S}_m$-extension (see the proof of Lemma 2.3). We can also take N_H such that it is prime to $p_1, \ldots, p_{t+2}$.

We take a monic polynomial $G(X)$ of $A[X]$ (having degree m) which satisfies the following conditions (2.1)–(2.4).

$$G(X) \equiv H(X) \pmod{N_H}. \tag{2.1}$$

If $G(X)$ satisfies (2.1), then $G(X)$ gives an $\mathfrak{S}_m$-extension. Let L be the splitting field of $G(X)$ over $\mathbb{F}(T)$.

$$(2.2)\quad G(X) \equiv (\text{a product of distinct monic polynomials of degree } 1) \pmod{p_{t+1}}.$$

If $G(X)$ satisfies (2.1) and (2.2), then we see that $\mathfrak{p}_{t+1}$ splits in the unique quadratic subextension, say M_L, over $\mathbb{F}(T)$ contained in L. On the other hand, $\mathfrak{p}_{t+1}$ is inert in the unique quadratic subextension M' over $\mathbb{F}(T)$ contained in k'. We claim that $k' \cap L = \mathbb{F}(T)$. Indeed, suppose that $k' \cap L \neq \mathbb{F}(T)$. Then $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. If $n = 2$, this is clear. For $n \geq 3$, we have $\mathrm{Gal}(L/\mathbb{F}(T)) \cong \mathfrak{S}_m$, where $m = n! \geq 5$. Observe also that $k' \cap L \neq L$, as $m > n$. Now, since the alternating group $\mathfrak{A}_m$ is the unique nontrivial proper normal subgroup of $\mathfrak{S}_m$ when $m \geq 5$ (see, e.g., [19]), $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. Since this quadratic extension is contained in both k' and L, it must coincide with both M' and M_L at a time. This contradicts the above observation on the behavior of $\mathfrak{p}_{t+1}$ in M' and M_L. Thus, we have proved the claim. Then we see $\mathrm{Gal}(Lk'/L) \cong \mathfrak{S}_n$.

$$(2.3)\quad G(X) \equiv (\text{a product of distinct monic polynomials of degree } 1) \pmod{p_{t+2}}.$$

If $G(X)$ satisfies (2.1)–(2.3), then the odd degree place $\mathfrak{p}_{t+2}$ splits completely in $Lk'/\mathbb{F}(T)$. We claim that $Lk'/\mathbb{F}(T)$ is a geometric extension. Note that the degree of a place of k' lying above $\mathfrak{p}_{t+2}$ is also odd because $\mathfrak{p}_{t+2}$ splits completely in k'. Since $\mathrm{Gal}(Lk'/k') \cong \mathfrak{S}_m$ and an odd degree place splits completely in Lk'/k', we see that Lk'/k' is also a geometric extension. Hence the claim follows. By using Krasner's lemma, we can see that there is a positive integer s_i for each $i = 1, \ldots, t$ depending only on $F(X)$ such that if $G(X) \equiv F(X) \pmod{p_i^{s_i}}$ then $L\mathbb{F}(T)_{\mathfrak{p}_i} = k'\mathbb{F}(T)_{\mathfrak{p}_i}$, where $\mathbb{F}(T)_{\mathfrak{p}_i}$ is the completion of $\mathbb{F}(T)$ at $\mathfrak{p}_i$ (see, e.g., [13]). Hence if we take $G(X)$ satisfying (2.1)–(2.3) and

$$(2.4)\qquad G(X) \equiv F(X) \pmod{p_i^{s_i}} \ \text{ for } i = 1, \ldots, t,$$

then we can see that Lk'/L is unramified at all places.

We can take $G(X)$ satisfying (2.1)–(2.4). By the above arguments, the extension Lk'/L satisfies the assertion of Theorem 2.2. □

Remark. When G is abelian, an unramified geometric G-extension was constructed by Angles [2]. Moret-Bailly [11] also gives a result which is very close to ours. Probably, it seems that one can prove our main theorems by using the result given in [11] instead of Theorem 2.2.

§ 2.2. Proof of Theorem 1.3

Since G is embedded into $\mathfrak{S}_n$ for some $n > 0$, Theorem 2.2 implies that there exists a global function field k with full constant field $\mathbb{F}$ and an unramified geometric Galois extension K/k such that $\mathrm{Gal}(K/k) \cong G$.

Proposition 2.4. *There is a non-empty finite set S of places of k such that* (i) *all places of S split completely in K, and* (ii) *$\tilde{H}_S(k)/k$ is a finite extension.*

Proof. The crucial point of this proposition is choosing a set S to satisfy (ii). For a positive integer N, we put

$$B_N = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a place of } k \text{ having degree } N,\ \mathfrak{p} \text{ splits completely in } K/k.\}.$$

Since K/k is a geometric extension, Theorem 9.13B of [18] implies that

$$|B_N| = \frac{q^N}{|G|N} + O\left(\frac{q^{N/2}}{N}\right)$$

(recall that q is the number of elements of $\mathbb{F}$). In particular, if N is sufficiently large, then we obtain the inequality

$$|B_N| > \frac{q^{N/2}-1}{N}\mathrm{Max}(g-1,0),$$

where g is the genus of k. We fix an integer N which satisfies the above inequality. According to Ihara's theorem [8, Theorem 1(FF)], if $S \supset B_N$, then $\tilde{H}_S(k)/k$ is a finite extension. Hence we can take S to satisfy the conditions (i) and (ii). □

The rest of the proof of Theorem 1.3 is quite similar to Perret's argument given in [16]. We choose a set S of places which satisfies the conditions of Proposition 2.4. We remark that K is contained in $\tilde{H}_S(k)$. For a nontrivial element σ of $\mathrm{Gal}(\tilde{H}_S(k)/K)$, we can take a place $\mathfrak{P}$ of $\tilde{H}_S(k)$ corresponding to σ by the Chebotarev density theorem. We can take $\mathfrak{P}$ which is unramified in $\tilde{H}_S(k)/K$. Let $\mathfrak{p}$ be the place of k which is lying below $\mathfrak{P}$. Since the decomposition field of $\mathfrak{P}$ in $\tilde{H}_S(k)/k$ contains K and K/k is a Galois extension, we see that $\mathfrak{p}$ splits completely in K/k. Then we see $\tilde{H}_S(k) \supsetneq \tilde{H}_{S\cup\{\mathfrak{p}\}}(k) \supset K$. Replacing $S \cup \{\mathfrak{p}\}$ by S and repeating the above operation, we can see that $\tilde{H}_S(k) = K$ for some finite set S. This implies $\mathrm{Gal}(\tilde{H}_S(k)/K) \cong G$.

We recall that K/k is a geometric extension. Hence the final part of the theorem follows. □

§ 3. Proof of Theorem 1.4

Firstly, we shall show the following:

Theorem 3.1. *Let k be a finite Galois extension over $\mathbb{F}(T)$. Then, there exist a non-empty finite set S of places of $\mathbb{F}(T)$ and a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ which satisfy the following properties.*

- *$F_\infty \cap k = \mathbb{F}(T)$,*
- *all places of S split completely in k,*
- *both of $F_\infty/\mathbb{F}(T)$ and $F_\infty k/k$ satisfy the assumptions (A) and (B) in section 1, and*
- *the Sylow p-subgroup of $\mathrm{Cl}_S(F_n k)$ is trivial for all $n \geq 0$,*

where F_n is the nth layer of $F_\infty/\mathbb{F}(T)$. (We use the same symbol S as the set of places lying above S.)

Proof. We take a place $\mathfrak{p}_0$ of $\mathbb{F}(T)$ which splits completely in k. We also take a place $\mathfrak{r}$ of $\mathbb{F}(T)$ which is distinct from $\mathfrak{p}_0$ and unramified in k. We claim that there is a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ unramified outside $\mathfrak{r}$ which satisfies that

- $\mathfrak{r}$ is totally ramified, and
- $\mathfrak{p}_0$ splits completely.

We shall show this claim. Let M be the maximal pro-p abelian extension over $\mathbb{F}(T)$ which is unramified outside $\mathfrak{r}$. We know that $\mathrm{Gal}(M/\mathbb{F}(T))$ is isomorphic to a countable infinite product of the additive group of $\mathbb{Z}_p$ (see [21], [9]). Hence there are infinitely many geometric $\mathbb{Z}_p$-extensions which satisfy the above conditions.

By the above choice of F_∞, we see $F_1 \cap k = \mathbb{F}(T)$. We put $k_1 = F_1 k$. Then $k_1/\mathbb{F}(T)$ is a Galois extension, and $\mathfrak{p}_0$ splits completely in k_1. We set $S_0 = \{\mathfrak{p}_0\}$, and we use the same symbol to denote the set of places lying above $\mathfrak{p}_0$. We can see that $H_{S_0}(k_1)$ is a finite Galois extension over $\mathbb{F}(T)$. We take a nontrivial element σ_1 of $\mathrm{Gal}(H_{S_0}(k_1)/k_1)$.

By using the above argument, we can take a geometric $\mathbb{Z}_p$-extension $F'_\infty/\mathbb{F}(T)$ unramified outside $\mathfrak{r}$ which satisfies

- $F'_\infty \cap F_\infty = \mathbb{F}(T)$,
- $\mathfrak{r}$ is totally ramified in $F'_\infty F_\infty$, and
- $\mathfrak{p}_0$ splits completely in F'_∞.

Let F'_1 be the initial layer of $F'_\infty/\mathbb{F}(T)$. Then we see that $F'_1 \cap k_1 = \mathbb{F}(T)$ and $k_1 F'_1 \cap H_{S_0}(k_1) = k_1$. We note that

$$\mathrm{Gal}(F'_1 H_{S_0}(k_1)/k_1) \cong \mathrm{Gal}(F'_1 k_1/k_1) \times \mathrm{Gal}(H_{S_0}(k_1)/k_1), \quad \mathrm{Gal}(F'_1 k_1/k_1) \cong \mathrm{Gal}(F'_1/\mathbb{F}(T)).$$

Hence there is an isomorphism

$$\mathrm{Gal}(F'_1/\mathbb{F}(T)) \times \mathrm{Gal}(H_{S_0}(k_1)/k_1) \xrightarrow{\sim} \mathrm{Gal}(F'_1 H_{S_0}(k_1)/k_1).$$

Let τ be a generator of the cyclic group $\mathrm{Gal}(F_1'/\mathbb{F}(T))$, and τ_1 an element of $\mathrm{Gal}(F_1'H_{S_0}(k_1)/k_1)$ which is the image of (τ, σ_1) under the above isomorphism. We can regard τ as an element of $\mathrm{Gal}(F_1'H_{S_0}(k_1)/\mathbb{F}(T))$. By the Chebotarev density theorem, there is a place $\mathfrak{P}_1$ of $F_1'H_{S_0}(k_1)$ which corresponds to τ_1. Let $\mathfrak{p}_1$ be the place of $\mathbb{F}(T)$ lying below $\mathfrak{P}_1$. We can take $\mathfrak{P}_1$ such that $\mathfrak{p}_1$ is not ramified in $F_1'H_{S_0}(k_1)$. Then we see that $\mathfrak{p}_1$ splits completely in k_1 and is inert in F_1'. We put $S_1 = S_0 \cup \{\mathfrak{p}_1\}$.

In general, $\mathfrak{p}_1$ may not split completely in F_∞. This is a problem because we need the assumption (B). We remark that $F_\infty F_\infty'/\mathbb{F}(T)$ is a $\mathbb{Z}_p^2$-extension unramified outside $\mathfrak{r}$. We recall that $\mathfrak{p}_1$ does not split in F_1'. Hence the decomposition field of $F_\infty F_\infty'/\mathbb{F}(T)$ for $\mathfrak{p}_1$ is a $\mathbb{Z}_p$-extension over $\mathbb{F}(T)$. We denote it by F_∞''. We also note that $F_\infty''/\mathbb{F}(T)$ is the unique $\mathbb{Z}_p$-extension contained in $F_\infty F_\infty'$ such that $\mathfrak{p}_1$ splits completely. Then the initial layer of $F_\infty''/\mathbb{F}(T)$ must coincide with F_1. We replace F_∞ by F_∞''.

We note that $H_{S_0}(k_1) \supsetneq H_{S_1}(k_1)$ by the definition of $\mathfrak{p}_1$. Similarly, we can choose a place $\mathfrak{p}_2$, put $S_2 = S_1 \cup \{\mathfrak{p}_2\}$, and modify the $\mathbb{Z}_p$-extension such that all places of S_2 splits completely. Repeating this operation, we see that $H_{S_t}(k_1) = k_1$ for some finite set S_t. From the above construction, we see that $F_\infty \cap k = \mathbb{F}(T)$ and that $F_\infty k/k$ satisfies the assumptions (A) and (B).

Finally, we shall give an Iwasawa-theoretic argument. In $F_\infty k/k$, all ramified places (which are lying above $\mathfrak{r}$) are totally ramified. From this, we also see $H_{S_t}(k) = k$. Let A_n be the Sylow p-subgroup of $\mathrm{Cl}_{S_t}(kF_n)$. By the above results, we see that both of A_0 and A_1 are trivial. In this situation, we can use the method given in Fukuda [6]. Namely, if all places which ramify in $F_\infty k/k$ are totally ramified and both of A_0 and A_1 are trivial, then A_n is trivial for all $n \geq 0$. (See [6, Theorem 1]. We note that the same method is also applicable for our situation.) Hence we see that A_n is trivial for all $n \geq 0$. $\square$

We shall show Theorem 1.4. We fix a finite p-group G. By using Theorem 2.2, we can take a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that K/k is unramified and $\mathrm{Gal}(K/k) \cong G$. By Theorem 3.1, we can take a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ and a set S of places of $\mathbb{F}(T)$ such that $F_\infty \cap K = \mathbb{F}(T)$, all places of S split completely in K, both of $F_\infty/\mathbb{F}(T)$ and $F_\infty K/K$ satisfy the assumptions (A) and (B), and A_n is trivial for all $n \geq 0$ (where A_n is the Sylow p-subgroup of $\mathrm{Cl}_S(F_nK)$, and F_n is the nth layer of $F_\infty/\mathbb{F}(T)$). We note that $F_\infty k/k$ also satisfies the assumptions (A) and (B). We claim that $\tilde{L}_S(F_nK) = F_nK$ for all $n \geq 0$. Indeed, if $\tilde{L}_S(F_nK)/F_nK$ is nontrivial, then there is a nontrivial finite Galois p-subextension over F_nK. Moreover, there is a nontrivial finite abelian p-subextension over F_nK because every p-group is solvable. Since A_n is trivial, it is a contradiction. We have shown the above claim. This implies that $\tilde{L}_S(F_nk) = F_nK$ because F_nK/F_nk is unramified and all places of F_nk lying above S split completely in F_nK. Hence

$\mathrm{Gal}(\tilde{L}_S(F_nk)/F_nk) \cong G$ for all $n \geq 0$. Then the theorem follows. $\square$

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