On the field intersection problem of generic polynomials: a survey

By

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Abstract

Let $k$ be a field of characteristic $\neq 2$. We survey a general method of the field intersection problem of generic polynomials via formal Tschirnhausen transformation. We announce some of our recent results of cubic, quartic and quintic cases the details of which are to appear elsewhere. In this note, we give an explicit answer to the problem in the cases of cubic and dihedral quintic by using multi-resolvent polynomials.

§1. Introduction

Let $G$ be a finite group, $k$ a field of characteristic $\neq 2$, $M$ a field containing $k$ with $\# M = \infty$, and $k(t)$ the rational function field over $k$ with $n$ indeterminates $t = (t_1, \ldots, t_n)$. Our main interest in this note is a $k$-generic polynomial for $G$ (cf. [DeM83], [Kem01], [JLY02]).

Definition. A polynomial $f_t(X) \in k(t)[X]$ is called $k$-generic for $G$ if it has the following property: the Galois group of $f_t(X)$ over $k(t)$ is isomorphic to $G$ and every $G$-Galois extension $L/M$ over an arbitrary infinite field $M \supset k$ can be obtained as $L = \text{Sp}M f_a(X)$, the splitting field of $f_a(X)$ over $M$, for some $a = (a_1, \ldots, a_n) \in M^n$.

Let $f_t^G(X) \in k(t)[X]$ be a $k$-generic polynomial for $G$. Examples of $k$-generic polynomials for $G$ are known for various pairs of $(k, G)$ (for example, see [Kem94], [DeM83], [Kem01], [JLY02]).

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Since a \( k \)-generic polynomial \( f_{G}^{G}(X) \) for \( G \) covers all \( G \)-Galois extensions over \( M \supset k \) by specializing parameters, it is natural to ask the following problem:

**Field isomorphism problem of a generic polynomial.** For a field \( M \supset k \) and \( a, b \in M^n \), determine whether \( \text{Sp}_{M}^{1}f_{a}^{G}(X) \) and \( \text{Sp}_{M}^{1}f_{b}^{G}(X) \) are isomorphic over \( M \) or not.

It would be desired to give an answer to the problem within the base field \( M \) by using the data \( a, b \in M^n \). Throughout this paper, we assume that \( f_{a}^{G}(X) \) is separable for \( a \in M^n \).

Let \( S_n \) (resp. \( D_n, C_n \)) be the symmetric (resp. the dihedral, the cyclic) group of degree \( n \). We take \( k \)-generic polynomials

\[
\begin{align*}
&f_{t}^{C_{3}}(X) := X^3 - tX^2 - (t + 3)X - 1 \in k(t)[X], \\
&f_{t}^{S_{3}}(X) := X^3 + tX + t \in k(t)[X], \\
&f_{s,t}^{D_{4}}(X) := X^4 + sX^2 + t \in k(s, t)[X]
\end{align*}
\]

for \( G = C_3, S_3, D_4 \), respectively. By using formal Tschirnhausen transformation, we showed the following theorem which is an analogue to the results of Morton [Mor94] and Chapman [Cha96].

**Theorem 1.1** ([Mor94], [Cha96], [HM]). For \( m, n \in M \), the splitting fields of \( f_{m}^{C_{3}}(X) \) and of \( f_{n}^{C_{3}}(X) \) over \( M \) coincide if and only if there exists \( z \in M \) such that

\[
n = \frac{m(z^3 - 3z - 1) - 9z(z + 1)}{mz(z + 1) + z^3 + 3z^2 - 1}
\]
or

\[
n = -\frac{m(z^3 + 3z^2 - 1) + 3(z^3 - 3z - 1)}{mz(z + 1) + z^3 + 3z^2 - 1}.
\]

We also have analogues to the above theorem for two non-abelian groups \( S_3 \) and \( D_4 \) via formal Tschirnhausen transformation.

**Theorem 1.2** ([HM07]). Assume that char \( k \neq 3 \). For \( a, b \in M \) with \( a \neq b \), the splitting fields of \( f_{a}^{S_{3}}(X) \) and of \( f_{b}^{S_{3}}(X) \) over \( M \) coincide if and only if there exists \( u \in M \) such that

\[
b = \frac{a(a^2 + 9u - 3a)^3}{(u^3 - 2au^2 - 9au - 2a^2 - 27a)^2}.
\]

**Theorem 1.3** ([HM–2]). For \( a, b \in M \), we assume that \( \text{Gal}(f_{a,b}^{D_{4}}/M) = D_4 \). Then for \( a, b, a', b' \in M \), the splitting fields of \( f_{a,b}^{D_{4}}(X) \) and of \( f_{a',b'}^{D_{4}}(X) \) over \( M \) coincide if and only if there exist \( p, q \in M \) such that either

(i) \( a' = ap^2 - 4bpq + abq^2 \), \( b' = b(p^2 - apq + bq^2)^2 \) or

(ii) \( a' = 2(ap^2 - 4bpq + abq^2) \), \( b' = (a^2 - 4b)(p^2 - bq^2)^2 \).
By applying Hilbert’s irreducibility theorem (cf. for example [JLY02, Chapter 3]) and Siegel’s theorem for curves of genus 0 (cf. for example [Lan78, Theorem 6.1]) to the theorems above respectively, we get the following corollaries:

**Corollary 1.4.** Let $f_a^G(X) = f_m^{G_3}(X)$ (resp. $f_a^{S_3}(X)$, $f_{a,b}^{D_4}(X)$) be as above in Theorem 1.1 (resp. Theorem 1.2, Theorem 1.3) with given $a \in M^n$, and suppose that $M \supset k$ is Hilbertian (e.g. a number field). Then there exist infinitely many $b \in M^n$ such that $\text{Sp}_M f_a^G(X) = \text{Sp}_M f_b^G(X)$.

**Corollary 1.5.** Let $M$ be a number field and $\mathcal{O}_M$ the ring of integers in $M$. For $f_a^G(X) = f_m^{G_3}(X)$ (resp. $f_a^{S_3}(X)$) as above in Theorem 1.1 (resp. Theorem 1.2) with a given integer $a \in \mathcal{O}_M$, there exist only finitely many integers $b \in \mathcal{O}_M$ such that $\text{Sp}_M f_a^G(X) = \text{Sp}_M f_b^G(X)$.

Indeed integers $b \in \mathcal{O}_M$ as in Corollary 1.5 are derived from some integer solutions of (finitely many) cubic Thue equations $aXY(X + Y) + X^3 + 3XY^2 - Y^3 = \lambda$ (resp. $X^3 - 2aX^2Y - 9aXY^2 - 2aY^3 - 27aY^3 = \lambda$) which are curves of genus 1 (see also the proof of [Lan78, Theorem 6.1]).

Kemper [Kem01], furthermore, showed that for a subgroup $H$ of $G$ every $H$-Galois extension over $M$ is also given by a specialization $f_a^G(X)$, $a \in M^n$, of a generic polynomial $f_t^G(X)$ for $G$. Hence a problem naturally arises.

**Field intersection problem of a generic polynomial.** For a field $M \supset k$ and $a, b \in M^n$, determine the intersection of $\text{Sp}_M f_a^G(X)$ and $\text{Sp}_M f_b^G(X)$.

Clearly if we get an answer to the field intersection problem of a $k$-generic polynomial, we also obtain an answer to the corresponding field isomorphism problem.

The aim of this note is to survey a method to give an answer to the isomorphism problem and the intersection problem of $k$-generic polynomials via formal Tschirnhausen transformation and multi-resolvent polynomials. In Section 2, we review known results about resolvent polynomials. In Section 3, we recall a formal Tschirnhausen transformation which is given in [HM]. In Section 4, we give a general method to solve the intersection problem of $k$-generic polynomials. In Section 5, we obtain an explicit answer to the problems in the cubic case. We give a proof of Theorem 1.2 as a special case of the intersection problem of $f_t^{S_3}(X) = X^3 + tX + t$ via formal Tschirnhausen transformation. In Section 6, we take the $k$-generic polynomial

$$f_{s,t}^{D_5}(X) = X^5 + (t - 3)X^4 + (s - t + 3)X^3 + (t^2 - t - 2s - 1)X^2 + sX + t \in k(s,t)[X]$$

for $D_5$ which is called Brumer’s quintic (cf. [JLY02]). Based on the general result, we illustrate how to give an answer to the problem for $f_{s,t}^{D_5}(X)$ by multi-resolvent polynomials. We also give some numerical examples.
§2. Resolvent polynomials

In this section we review some known results in the computational aspects of Galois theory (cf. the text books [Coh93], [Ade01]). One of the fundamental tools in the determination of Galois group of a polynomial is resolvent polynomials; an absolute resolvent polynomial was introduced by Lagrange [Lag1770] and a relative one by Stauduhar [Sta73]. Several kinds of methods to compute resolvent polynomials have been developed by many mathematicians (see, for example, [Sta73], [Gir83], [SM85], [Yok97], [MM97], [AV00], [GK00] and the references therein).

Let $M(\supset k)$ be an infinite field and $\overline{M}$ a fixed algebraic closure of $M$. Let $f(X) := \prod_{i=1}^{m}(X - \alpha_i) \in M[X]$ be a separable polynomial of degree $m$ with some fixed order of the roots $\alpha_1, \ldots, \alpha_m \in \overline{M}$. The Galois group of the splitting field $\text{Spl}_M f(X)$ of $f(X)$ over $M$ may be obtained by using suitable resolvent polynomials.

Let $k[x] := k[x_1, \ldots, x_m]$ be the polynomial ring over $k$ with $m$ indeterminates $x_1, \ldots, x_m$. Put $R := k[x, 1/\Delta_x]$, where $\Delta_x := \prod_{1 \leq i < j \leq m}(x_j - x_i)$. We take a surjective evaluation homomorphism $\omega_f: R \to k(\alpha_1, \ldots, \alpha_m)$, $\Theta(x_1, \ldots, x_m) \mapsto \Theta(\alpha_1, \ldots, \alpha_m)$ for $\Theta \in R$. We note that $\omega_f(\Delta_x) \neq 0$ from the assumption that $f(X)$ is separable. The kernel of the map $\omega_f$ is the ideal $I_f = \{\Theta(x_1, \ldots, x_m) \in R \mid \Theta(\alpha_1, \ldots, \alpha_m) = 0\}$.

Let $S_m$ be the symmetric group of degree $m$. We extend the action of $S_m$ on $m$ letters $\{1, \ldots, m\}$ to that on $R$ by $\pi(\Theta(x_1, \ldots, x_m)) := \Theta(x_{\pi(1)}, \ldots, x_{\pi(m)})$. We define the Galois group of a polynomial $f(X)$ over $M$ by $\text{Gal}(f/M) := \{\pi \in S_m \mid \pi(I_f) \subseteq I_f\}$. We write $\text{Gal}(f) := \text{Gal}(f/M)$ for simplicity. The Galois group of the splitting field $\text{Spl}_M f(X)$ of the polynomial $f(X)$ over $M$ is isomorphic to $\text{Gal}(f)$. If we take another ordering of roots $\alpha_{\pi(1)}, \ldots, \alpha_{\pi(m)}$ of $f(X)$ for some $\pi \in S_m$, the corresponding realization of $\text{Gal}(f)$ is conjugate in $S_m$. Hence, for arbitrary ordering of the roots of $f(X)$, $\text{Gal}(f)$ is determined up to conjugacy in $S_m$.

Definition. For $H \leq G \leq S_m$, an element $\Theta \in R$ is called a $G$-primitive $H$-invariant if $H = \text{Stab}_G(\Theta) := \{\pi \in G \mid \pi(\Theta) = \Theta\}$. For a $G$-primitive $H$-invariant $\Theta$, the polynomial

$$\mathcal{R}_\Theta G f(X) = \prod_{\overline{\pi} \in G/H} (X - \omega_f(\pi(\Theta))) \in M[X]$$

is called the $G$-relative $H$-invariant resolvent of $f$ by $\Theta$.

The following theorem is fundamental in the theory of resolvent polynomials (cf. for example [Ade01, p.95]).
Theorem 2.1. Let $H \leq G \leq S_m$ be a tower of finite groups and $\Theta$ a $G$-primitive $H$-invariant. Assume $\text{Gal}(f) \leq G$. Suppose that $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X) = \prod_{i=1}^{l} h_{i}^{e_{i}}(X)$ gives the decomposition of $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ into a product of powers of distinct irreducible polynomials $h_{i}(X), i = 1, \ldots, l$, in $M[X]$. Then we have a bijection

$$\text{Gal}(f) \backslash G/H \quad \mapsto \quad \{h_{1}^{e_{1}}(X), \ldots, h_{l}^{e_{l}}(X)\}$$

$$\text{Gal}(f) \pi H \quad \mapsto \quad h_{\pi}(X) = \prod_{\tau H \subseteq \text{Gal}(f) \pi H} (X - \omega_{f}(\tau(\Theta)))$$

where the product runs through the left cosets $\tau H$ of $H$ in $G$ contained in $\text{Gal}(f) \pi H$, that is, through $\tau = \pi_{\sigma}\pi$ where $\pi_{\sigma}$ runs a system of representative of the left cosets of $\text{Gal}(f) \cap \pi H \pi^{-1}$; each $h_{\pi}(X)$ is irreducible or a power of an irreducible polynomial with $\text{deg}(h_{\pi}(X)) = |\text{Gal}(f) \pi H|/|H| = |\text{Gal}(f)|/|\text{Gal}(f) \cap \pi H \pi^{-1}|$.

Corollary 2.2. If $\text{Gal}(f) \leq \pi H \pi^{-1}$ for some $\pi \in G$ then $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ has a linear factor over $M$. Conversely, if $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ has a non-repeated linear factor over $M$ then there exists $\pi \in G$ such that $\text{Gal}(f) \leq \pi H \pi^{-1}$.

Remark 2.3. When the resolvent polynomial $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ has a repeated factor, there always exists a suitable Tschirnhausen transformation $\hat{f}$ of $f$ (cf. §3) over $M$ (resp. $X - \hat{\Theta}$ of $X - \Theta$ over $k$) such that $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ (resp. $\mathcal{R}\mathcal{P}_{\hat{\Theta},G,f}(X)$) has no repeated factors (cf. [Gir83], [Coh93, Alg. 6.3.4], [Col95]).

In the case where $\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ has no repeated factors, we have the followings:

(i) For $\pi \in G$, the fixed group of the field $M(\omega_{f}(\pi(\Theta)))$ corresponds to $\text{Gal}(f) \cap \pi H \pi^{-1}$. In particular, we have $\text{Gal}(\mathcal{R}\mathcal{P}_{\Theta,G,f}) \cong \text{Gal}(f)/N$ with $N = \text{Gal}(f) \cap \bigcap_{\pi \in G} \pi H \pi^{-1}$;

(ii) let $\varphi : G \to S_{[G:H]}$ denote the permutation representation of $G$ on the set of left cosets $G/H$ given by the left multiplication. Then we have a realization of the Galois group of $\text{Sp}_{M}\mathcal{R}\mathcal{P}_{\Theta,G,f}(X)$ as a subgroup of the symmetric group $S_{[G:H]}$ by $\varphi(\text{Gal}(f))$.

§ 3. Formal Tschirnhausen transformation

We recall the geometric interpretation of a Tschirnhausen transformation which is given in [HM] (see also [HM-3]). Let $f(X)$ be a monic separable polynomial of degree $n$ in $M[X]$ with a fixed order of the roots $\alpha_{1}, \ldots, \alpha_{n}$ of $f(X)$ in $\overline{M}$. A Tschirnhausen transformation of $f(X)$ over $M$ is a polynomial of the form

$$g(X) = \prod_{i=1}^{n}(X - (c_{0} + c_{1}\alpha_{i} + \cdots + c_{n-1}\alpha_{i}^{n-1})), \quad c_{i} \in M.$$ 

Two polynomials $f(X)$ and $g(X)$ in $M[X]$ are Tschirnhausen equivalent over $M$ if they are Tschirnhausen transformations over $M$ of each other. For two irreducible separable
polynomials \( f(X) \) and \( g(X) \) in \( M[X] \), \( f(X) \) and \( g(X) \) are Tschebyshev equivalent over \( M \) if and only if the quotient fields \( M[X]/(f(X)) \) and \( M[X]/(g(X)) \) are \( M \)-isomorphic.

In order to obtain an answer to the field intersection problem of \( k \)-generic polynomials via multi-resolvent polynomials, we first treat a general polynomial whose roots are \( n \) indeterminates \( x_1, \ldots, x_n \):

\[
f_s(X) = \prod_{i=1}^{n} (X - x_i) = X^n - s_1X^{n-1} + s_2X^{n-2} + \cdots + (-1)^n s_n \in k[s][X]
\]

where \( k[x_1, \ldots, x_n]^{S_n} = k[s] := k[s_1, \ldots, s_n], s = (s_1, \ldots, s_n) \), and \( s_i \) is the \( i \)-th elementary symmetric function in \( n \) variables \( x = (x_1, \ldots, x_n) \).

Put \( R_{x,y} := k[x, y, 1/\Delta_x, 1/\Delta_y] \), where \( \Delta_x := \prod_{1 \leq i < j \leq n} (x_j - x_i) \) and \( \Delta_y := \prod_{1 \leq i < j \leq n} (y_j - y_i) \). We define the interchanging involution \( \iota_{x,y} \) which exchanges the \( x_i \)'s and the \( y_i \)'s:

\[
(3.1) \quad \iota_{x,y} : R_{x,y} \to R_{x,y}, \quad x_i \mapsto y_i, \quad y_i \mapsto x_i, \quad (i = 1, \ldots, n).
\]

We take another general polynomial \( f_t(X) := \iota_{x,y}(f_s(X)) \in k[t][X], t = (t_1, \ldots, t_n) \), whose roots are \( n \) indeterminates \( y_1, \ldots, y_n \) where \( t_i = \iota_{x,y}(s_i) \) is the \( i \)-th elementary symmetric function in \( y = (y_1, \ldots, y_n) \). We put \( K := k(s, t) \) and \( f_{s,t}(X) := f_s(X)f_t(X) \). The polynomial \( f_{s,t}(X) \) of degree \( 2n \) is defined over \( K \). We denote

\[
G_s := \text{Gal}(f_s/K), \quad G_t := \text{Gal}(f_t/K), \quad G_{s,t} := \text{Gal}(f_{s,t}/K).
\]

Then we have \( G_{s,t} = G_s \times G_t, G_s \cong G_t \cong S_n \) and \( k(x, y)^{G_{s,t}} = K \).

We intend to apply the results of the previous section for \( m = 2n, G = G_{s,t} \leq S_{2n} \) and \( f = f_{s,t} \).

Note that in the splitting field \( \text{Sp}_K f_{s,t}(X) = k(x, y) \), there exist \( n! \) Tschebyshev transformations from \( f_s(X) \) to \( f_t(X) \) with respect to \( y_{\pi(1)}, \ldots, y_{\pi(n)} \) for \( \pi \in S_n \). We shall study the field of definition of each Tschebyshev transformation from \( f_s(X) \) to \( f_t(X) \). Let \( D := [x_i^{j-1}]_{1 \leq i, j \leq n} \) be the Vandermonde matrix of size \( n \). The matrix \( D \in M_n(k(x)) \) is invertible because \( \det D = \Delta_x \). The field \( k(s)(\Delta_x) \) is a quadratic extension of \( k(s) \) which corresponds to the fixed field of the alternating group of degree \( n \). We define the \( n \)-tuple \( (u_0(x, y), \ldots, u_{n-1}(x, y)) \in (R_{x,y})^n \) by

\[
\begin{bmatrix}
u_0(x, y) \\
u_1(x, y) \\
\vdots \\
u_{n-1}(x, y)
\end{bmatrix} := D^{-1} \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]
Cramer’s rule shows us

\[ u_i(x, y) = \Delta_x^{-1} \cdot \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{i-1} & y_1 & x_1^{i+1} & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{i-1} & y_2 & x_2^{i+1} & \cdots & x_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^{i-1} & y_n & x_n^{i+1} & \cdots & x_n^{n-1} \end{bmatrix}. \]

We write \( u_i := u_i(x, y) \), \((i = 0, \ldots, n - 1)\). The Galois group \( G_{s,t} \) acts on the orbit \( \{ \pi(u_i) \mid \pi \in G_{s,t} \} \) via regular representation from the left. However this action is not faithful. We put

\[ H_{s,t} := \{ (\pi_x, \pi_y) \in G_{s,t} \mid \pi_x(i) = \pi_y(i) \text{ for } i = 1, \ldots, n \} \cong S_n. \]

If \( \pi \in H_{s,t} \) then we have \( \pi(u_i) = u_i \) for \( i = 0, \ldots, n - 1 \). Indeed we see the following:

**Lemma 3.1.** For \( i, 0 \leq i \leq n - 1 \), \( u_i \) is a \( G_{s,t} \)-primitive \( H_{s,t} \)-invariant.

Let \( \Theta := \Theta(x, y) \) be a \( G_{s,t} \)-primitive \( H_{s,t} \)-invariant. Let \( \overline{\pi} = \pi H_{s,t} \) be a left coset of \( H_{s,t} \) in \( G_{s,t} \). The group \( G_{s,t} \) acts on the set \( \{ \pi(\Theta) \mid \overline{\pi} \in G_{s,t}/H_{s,t} \} \) transitively from the left through the action on the set \( G_{s,t}/H_{s,t} \) of left cosets. Each of the sets \( \{(1, \pi_y) \mid (1, \pi_y) \in G_{s,t}\} \) and \( \{(\pi_x, 1) \mid (\pi_x, 1) \in G_{s,t}\} \) forms a complete residue system of \( G_{s,t}/H_{s,t} \), and hence the subgroups \( G_s \) and \( G_t \) of \( G_{s,t} \) act on the set \( \{ \pi(\Theta) \mid \overline{\pi} \in G_{s,t}/H_{s,t} \} \) transitively. For \( \overline{\pi} = (1, \pi_y) \in G_{s,t}/H_{s,t} \), we obtain the following equality:

\[ y_{\pi_y(i)} = \pi_y(u_0) + \pi_y(u_1)x_i + \cdots + \pi_y(u_{n-1})x_i^{n-1} \text{ for } i = 1, \ldots, n. \]

Hence the set \( \{ (\pi(u_0), \ldots, \pi(u_{n-1})) \mid \overline{\pi} \in G_{s,t}/H_{s,t} \} \) gives coefficients of \( n! \) different Tschirnhausen transformations from \( f_s(X) \) to \( f_t(X) \) each of which is defined over \( K(\pi(u_0), \ldots, \pi(u_{n-1})) \) respectively.

**Definition.** We call \( K(\pi(u_0), \ldots, \pi(u_{n-1})), (\overline{\pi} \in G_{s,t}/H_{s,t}) \), a field of formal Tschirnhausen coefficients from \( f_s(X) \) to \( f_t(X) \).

Put \( v_i := \iota_{x,y}(u_i), \) for \( i = 0, \ldots, n - 1 \). Then \( v_i \) is also a \( G_{s,t} \)-primitive \( H_{s,t} \)-invariant, and \( K(\pi(v_0), \ldots, \pi(v_{n-1})) \) gives a field of formal Tschirnhausen coefficients from \( f_t(X) \) to \( f_s(X) \).

**Proposition 3.2.** For every \( G_{s,t} \)-primitive \( H_{s,t} \)-invariant \( \Theta \), we have \( K(\pi(\Theta)) = k(x, y)^{\pi H_{s,t} \pi^{-1}} = K(\pi(u_0), \ldots, \pi(u_{n-1})) \), and \( [K(\pi(\Theta)) : K] = n! \) for each \( \overline{\pi} \in G_{s,t}/H_{s,t} \).

Hence, for each of \( n! \) fields \( K(\pi(\Theta)) \), we have \( \text{Sp} \_K(\pi(\Theta)) f_s(X) = \text{Sp} \_K(\pi(\Theta)) f_t(X) \).
Proposition 3.3. Let $\Theta$ be a $G_{s,t}$-primitive $H_{s,t}$-invariant. Then we have

(i) $K(x) \cap K(\pi(\Theta)) = K(y) \cap K(\pi(\Theta)) = K$ for $\overline{\pi} \in G_{s,t}/H_{s,t}$;
(ii) $K(x,y) = K(x,\pi(\Theta)) = K(y,\pi(\Theta))$ for $\overline{\pi} \in G_{s,t}/H_{s,t}$;
(iii) $K(x,y) = K(\pi(\Theta) \mid \overline{\pi} \in G_{s,t}/H_{s,t})$.

§ 4. Field intersection problem

In this section, we explain how to get an answer to the field intersection problem of generic polynomials via multi-resolvent polynomial (cf. [HM], [HM-3]). For $a = (a_1,\ldots,a_n)$, $b = (b_1,\ldots,b_n) \in M^n$, we take some fixed order of the roots $\alpha_1,\ldots,\alpha_n$ of $f_a(X)$ and $\beta_1,\ldots,\beta_n$ of $f_b(X)$ in $\overline{M}$, respectively, and denote $L_a := M(\alpha_1,\ldots,\alpha_n)$ and $L_b := M(\beta_1,\ldots,\beta_n)$. We put $f_{a,b}(X) := f_a(X)f_b(X) \in M[X]$ and define a specialization homomorphism $\omega_{f_a,b}$ by

$$\omega_{f_a,b} : R_{x,y} \rightarrow k(\alpha_1,\ldots,\alpha_n, \beta_1,\ldots,\beta_n), \quad \Theta(x,y) \mapsto \Theta(\alpha_1,\ldots,\alpha_n, \beta_1,\ldots,\beta_n).$$

Put $\Delta_a := \omega_{f_a,b} (\Delta_x)$ and $\Delta_b := \omega_{f_a,b} (\Delta_y)$. We assume that both of the polynomials $f_a(X)$ and $f_b(X)$ are separable over $M$, i.e. $\Delta_a \cdot \Delta_b \neq 0$. Put $G_a := \text{Gal}(f_a/M)$, $G_b := \text{Gal}(f_b/M)$ and $G_{a,b} := \text{Gal}(f_{a,b}/M)$. Then we may naturally regard $G_{a,b}$ as a subgroup of $G_{s,t}$. For $\overline{\pi} \in G_{s,t}/H_{s,t}$, we put $c_{j,\pi} := \omega_{f_{a,b}}(\pi(u_j))$, $d_{j,\pi} := \omega_{f_{a,b}}(\pi(x,y)(u_j))$, $(j = 0,\ldots,n-1)$. Then for each $i = 1,\ldots,n$, we have

$$\beta_{\pi(y)(i)} = c_{0,\pi} + c_{1,\pi} \alpha_{\pi(x)(i)} + \cdots + c_{n-1,\pi} \alpha_{\pi(x)(i)}^{n-1},$$

$$\alpha_{\pi(x)(i)} = d_{0,\pi} + d_{1,\pi} \beta_{\pi(y)(i)} + \cdots + d_{n-1,\pi} \beta_{\pi(y)(i)}^{n-1}.$$ 

For each $\overline{\pi} \in G_{s,t}/H_{s,t}$, there exists a Tschirnhausen transformation from $f_a(X)$ to $f_b(X)$ over the field of Tschirnhausen coefficients $M(c_{0,\pi},\ldots,c_{n-1,\pi})$. From the assumption $\Delta_a \cdot \Delta_b \neq 0$, we first see the following lemma (cf. [JLY02, p. 141], [HM]).

Lemma 4.1. Let $M'/M$ be a field extension. If $f_b(X)$ is a Tschirnhausen transformation of $f_a(X)$ over $M'$, then $f_a(X)$ is a Tschirnhausen transformation of $f_b(X)$ over $M'$. In particular, we have $M(c_{0,\pi},\ldots,c_{n-1,\pi}) = M(d_{0,\pi},\ldots,d_{n-1,\pi})$ for every $\overline{\pi} \in G_{s,t}/H_{s,t}$.

To obtain an answer to the field intersection problem of $f_a(X)$ we study the $n!$ fields $M(c_{0,\pi},\ldots,c_{n-1,\pi})$ of Tschirnhausen coefficients from $f_a(X)$ to $f_b(X)$ over $M$.

Proposition 4.2. Under the assumption $\Delta_a \cdot \Delta_b \neq 0$, we have

(i) $\text{Sp} \ M(c_{0,\pi},\ldots,c_{n-1,\pi})f_a(X) = \text{Sp} \ M(c_{0,\pi},\ldots,c_{n-1,\pi})f_b(X)$ for each $\overline{\pi} \in G_{s,t}/H_{s,t}$;
(ii) $L_a L_b = L_a M(c_{0,\pi},\ldots,c_{n-1,\pi}) = L_b M(c_{0,\pi},\ldots,c_{n-1,\pi})$ for each $\overline{\pi} \in G_{s,t}/H_{s,t}$.
Applying the specialization \( \omega_{fa,b} \), we take the \( G_{s,t} \)-relative \( H_{s,t} \)-invariant resolvent polynomial of \( fa,b \) by a \( G_{s,t} \)-primitive \( H_{s,t} \)-invariant \( \Theta \):

\[
\mathcal{R}P_{\Theta,G_{s,t},fa,b}(X) = \prod_{\overline{\pi} \in G_{s,t}/H_{s,t}} (X - \omega_{fa,b}(\pi(\Theta))) \in M[X].
\]

The resolvent polynomial \( \mathcal{R}P_{\Theta,G_{s,t},fa,b}(X) \) is also called an (absolute) multi-resolvent (cf. [GLV88], [RV99], [Val]).

**Proposition 4.3.** For \( a,b \in M^n \) with \( \Delta_a \cdot \Delta_b \neq 0 \), suppose that the polynomial \( \mathcal{R}P_{\Theta,G_{s,t},fa,b}(X) \) has no repeated factors. Then the following two assertions hold:

(i) \( M(c_0,\pi_1,\ldots,c_{n-1},\pi) = M(\omega_{fa,b}(\pi(\Theta))) \) for each \( \overline{\pi} \in G_{s,t}/H_{s,t} \);

(ii) \( \text{Spl}_M f_{a,b}(X) = M(\omega_{fa,b}(\pi(\Theta)) \mid \overline{\pi} \in G_{s,t}/H_{s,t}) \).

**Definition.** For a separable polynomial \( f(X) \in M[X] \) of degree \( d \), the decomposition type of \( f(X) \) over \( M \), denoted by \( DT(f/M) \), is defined as the partition of \( d \) induced by the degrees of the irreducible factors of \( f(X) \) over \( M \). We define the decomposition type \( DT(\mathcal{R}P_{\Theta,G,f}/M) \) of \( \mathcal{R}P_{\Theta,G,f}(X) \) over \( M \) by \( DT(\mathcal{R}P_{\Theta,G,f}/M) \) where \( \hat{f}(X) \) is a Tschirnhausen transformation of \( f(X) \) over \( M \) which satisfies that \( \mathcal{R}P_{\Theta,G,f}(X) \) has no repeated factors (cf. Remark 2.3).

We write \( DT(f) := DT(f/M) \) for simplicity. From Theorem 2.1, the decomposition type \( DT(\mathcal{R}P_{\Theta,G_{s,t},fa,b}) \) coincides with the partition of \( n! \) induced by the lengths of the orbits of \( G_{s,t}/H_{s,t} \) under the action of \( \text{Gal}(f_{a,b}) \). Hence, by Proposition 4.3, \( DT(\mathcal{R}P_{\Theta,G_{s,t},fa,b}) \) gives the degrees of \( n! \) fields of Tschirnhausen coefficients from \( f_a(X) \) to \( f_b(X) \) over \( M \).

We conclude that \( DT(\mathcal{R}P_{\Theta,G_{s,t},fa,b}) \) gives us information about the field intersection problem for \( f_a(X) \) through the degrees of the fields of Tschirnhausen coefficients \( M(c_0,\pi_1,\ldots,c_{n-1},\pi) \) over \( M \) and is determined by the degeneration of the Galois group \( \text{Gal}(f_{a,b}) \) under the specialization \((s,t) \mapsto (a,b)\). As a special case of the field intersection problem, we get the followings:

**Theorem 4.4 ([HM-3]).** For \( a,b \in M^n \) with \( \Delta_a \cdot \Delta_b \neq 0 \), the quotient fields \( M[X]/(f_a(X)) \) and \( M[X]/(f_b(X)) \) are \( M \)-isomorphic if and only if the decomposition type \( DT(\mathcal{R}P_{\Theta,G_{s,t},fa,b}) \) over \( M \) includes 1.

**Corollary 4.5** (The field isomorphism problem). For \( a,b \in M^n \) with \( \Delta_a \cdot \Delta_b \neq 0 \), we assume that both of \( G_a \) and \( G_b \) are isomorphic to a transitive subgroup \( G \leq S_n \) and that all subgroups of \( G \) with index \( n \) are conjugate in \( G \). Then \( DT(\mathcal{R}P_{\Theta,G_{s,t},fa,b}) \) over \( M \) includes 1 if and only if \( \text{Spl}_M f_a(X) \) and \( \text{Spl}_M f_b(X) \) coincide.
§ 5. Field intersection problem: the case of $S_3$

We take $f_{a}^{S_3}(X) = X^3 + sX + s \in k(s)[X]$. For $a, b \in M$, we put $L_a := \text{Spl}_M f_{a}^{S_3}(X)$ and $G_a := \text{Gal}(f_{a}^{S_3}/M), G_{a,b} := \text{Gal}(f_{a}^{S_3}f_{b}^{S_3}/M)$. For $a, b \in M$, we assume that $ab(4a + 27)(4b + 27) \neq 0$ because the discriminant of $f_{a}^{S_3}(X)$ equals $-s^2(4s + 27)$.

In the case of char $k \neq 3$, we take a $G_{s,t}$-primitive $H_{s,t}$-invariant $\Theta := 3u_1/u_2$ where $u_1$ and $u_2$ are formal Tschirnhausen coefficients which are given in Section 3. Then we may evaluate

$$F_{s,t}(X) := (s-t) \cdot \mathcal{R}\mathcal{P}_{\Theta, G_{s,t}, f_{s}^{S_3}f_{t}^{S_3}}(X) = (s-t) \cdot \prod_{\pi \in G_{s,t}/H_{s,t}} (X - \pi(\Theta)) \in k(s, t)[X]$$

The discriminant of $F_{s,t}(X)$ is $s^{10}t^{4}(4s + 27)^{15}(4t + 27)^{3}$. Note that, for $a, b \in M$ with $ab(4a + 27)(4b + 27) \neq 0$, $F_{a,b}(X)$ has no repeated factors. In the case of char $k = 3$, we may take $\Theta = u_0$, the constant term of a formal Tschirnhausen transformation, as a suitable $G_{s,t}$-primitive $H_{s,t}$-invariant. Then we get squarefree $\mathcal{R}\mathcal{P}_{u_0, G_{s,t}, f_{s}^{S_3}f_{t}^{S_3}}(X)$ as we mentioned in Remark 2.3 (see [HM]).

**Theorem 5.1** ([HM]). Assume that char $k \neq 3$. For $a, b \in M$, we also assume that $a \neq b$ and $\#G_a \geq \#G_b > 1$. Then an answer to the field intersection problem for $f_{s}^{S_3}(X)$ is given by DT($F_{a,b}$) as Table 1 shows.

<table>
<thead>
<tr>
<th>$G_a$</th>
<th>$G_b$</th>
<th>$G_{a,b}$</th>
<th>$L_a \cap L_b = M$</th>
<th>DT($F_{a,b}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_3$</td>
<td>$S_3$</td>
<td>$S_3 \times S_3$</td>
<td>$L_a \cap L_b = M$</td>
<td>6</td>
</tr>
<tr>
<td>$(C_3 \times C_3) \times C_2$</td>
<td>$C_3 \times C_3$</td>
<td>$L_a = L_b$</td>
<td>3, 2, 1</td>
<td></td>
</tr>
<tr>
<td>$C_3$</td>
<td>$C_3$</td>
<td>$C_3 \times C_3$</td>
<td>$L_a \neq L_b$</td>
<td>6</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$C_2$</td>
<td>$C_2 \times C_2$</td>
<td>$L_a = L_b$</td>
<td>3, 1, 1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$C_2$</td>
<td>$C_2 \times C_2$</td>
<td>$L_a \neq L_b$</td>
<td>2, 2, 1, 1</td>
</tr>
</tbody>
</table>

As a special case of Theorem 5.1 (cf. Theorem 4.4 and Corollary 4.5), we get Theorem 1.2 which we introduced in Section 1.

**Proof of Theorem 1.2.** Because the polynomial $F_{s,t}(X) = s(X^2 + 9X - 3s)^3 - t(X^3 - 2sX^2 - 9sX - 2s^2 - 27s)^2$ is linear in $t$, we see that $F_{a,b}(X)$ has a root in $M$ if and only if there exists $u \in M$ such that

$$b = \frac{a(u^2 + 9u - 3a)^3}{(u^3 - 2au^2 - 9au - 2a^2 - 27a)^2}.$$
§ 6. Field intersection problem: the case of $D_5$

Let $\sigma := (12345)$, $\rho := (1243)$, $\tau := \rho^2$, $\omega := (12) \in S_5$ act on $k(x_1, \ldots, x_5)$ by $\pi(x_i) = x_{\pi(i)}$, $\pi \in S_5$. For simplicity, we write

$$C_5 = \langle \sigma \rangle, \quad D_5 = \langle \sigma, \tau \rangle, \quad F_{20} = \langle \sigma, \rho \rangle, \quad S_5 = \langle \sigma, \omega \rangle.$$

We take the cross-ratios

$$x := \xi(x_1, \ldots, x_5) = \frac{x_1 - x_4}{x_1 - x_3} / \frac{x_2 - x_4}{x_2 - x_3}, \quad y := \eta(x_1, \ldots, x_5) = \frac{x_2 - x_5}{x_2 - x_4} / \frac{x_3 - x_5}{x_3 - x_4}.$$

Then $S_5$ acts faithfully on $k(x, y)$ as

$$\sigma : x \mapsto y, \quad \rho : x \mapsto x/(x-1), \quad \tau : x \mapsto (x-1)/y, \quad \omega : x \mapsto 1/x.$$

We take a $k$-generic polynomial $f_{s,t}^{D_5}(X) \in k(s, t)[X]$ as the formal $D_5$-relative $\langle \tau \rangle$-invariant resolvent polynomial by:

$$f_{s,t}^{D_5}(X) := \mathcal{R}\mathcal{P}_{x,D_5}(X) = (X - x)(X - y)(X - \frac{1-y}{x})(X - \frac{x+y-1}{xy})(X - \frac{1-x}{y})$$

$$= X^5 + (t-3)X^4 + (s-t+3)X^3 + (t^2-t-2s-1)X^2 + sX + t.$$

Note that $k(s, t) = k(x, y)^{D_5}$. We take two fields $k(x) = k(x, y)$ and $k(x') = k(x', y')$ where $x' := \xi(y_1, \ldots, y_5)$, $y' := \eta(y_1, \ldots, y_5)$ and the interchanging involution

$$\iota : k(x, x') \rightarrow k(x, x'), \quad (x, y, x', y') \mapsto (x', y', x, y)$$

which is induced by the $\iota_{x,y}$ of (3.1). We put $(s', t', d') := \iota(s, t, d)$, $s' := (s', t')$ and $(\sigma', \tau', \rho') := (\iota^{-1} \sigma \iota, \iota^{-1} \tau \iota, \iota^{-1} \rho \iota) \in \text{Aut}_k(k(x', y'))$ and write

$$D_5' = \langle \sigma', \tau' \rangle, \quad F_{20}' = \langle \sigma', \rho' \rangle \quad \text{and} \quad D_5'' = \langle \sigma \sigma', \tau \tau' \rangle.$$

We now take the $D_5 \times D_5'$-primitive $D_5''$-invariant

$$P := xx' + yy' + \frac{(y-1)(y'-1)}{xx'} + \frac{(x+y-1)(x' + y' - 1)}{xx'yy'} + \frac{(x-1)(x'-1)}{yy'}$$

and the formal $D_5 \times D_5'$-relative $D_5''$-invariant resolvent polynomial by $P$:

$$F_{s,s'}^{1}(X) := \mathcal{R}\mathcal{P}_{P,D_5 \times D_5'}(X) = \prod_{\pi \in (D_5 \times D_5')/D_5''} (X - \pi(P))$$

$$= \left( X^5 - (t-3)(t'-3)X^4 + c_3X^3 + \frac{c_2}{2}X^2 + \frac{c_1}{2}X + \frac{c_0}{2} \right)^2$$

$$- \frac{d^2d'^2}{4} \left( X^2 + (t+t'-1)X + (s-t+s'-t'+tt'+2) \right)^2.$$
here \(c_3, c_2, c_1, c_0 \in k(s, t, s', t')\) are given by

\[
c_3 = [2s - 2t + 3t^2 - 2ts' + t^2s' - t^2t'] + 3s - 3ss' + 5tt',
\]
\[
c_2 = [-20s + 112t + 8st - 32t^2 + 2t^3 + 5ts' - 13sts' - 12s^2s' - 4t^3s' - 15stt']
+ 14t^2t' + 2t^3t' + 8t^2s't' - 2t^3t'^2] - 102 + 27ss' - 119tt' - sts't' + 6t^2t'^2,
\]
\[
c_1 = [32s + 2s^2 - 128t - 60t^2 + 4st^2 - 8t^3 - 6s^2s' - 7ts' + 38sts' + 9t^2s' - 5st^2s'
- 12t^3s' + 2t^4s' - 20ts^2s' - 2sts'^2 + 6ts'^3 + 2t^3s'^2 + 2st't' - 77t^2t' + 3st^2t' + 8t^3t' - 29t^2s't'
+ st^2s't' + 18t^3s't' - 2st^2t'^2 + 10t^3t'^2] + 80 - 37ss' + 145tt' - 45sts't' + 24t^2t'^2 - 8t^3t'^3,
\]
\[
c_0 = [-16s - 2s^2 + 56t + 24st + 2s^2t - 38st^2 + 8t^3 + 5s^2s' - 2ts' - 38sts' - 7s^2ts'
+ 5t^2s' + 13s^2s' + 8t^3s' + 2st^3s' - 4ts' - 21ts^2 - 11sts^2 - 2t^2s'^2 + 2st^2s'^2 + 4t^3s'^2
- 104sts' - 33s^2tt' + 105t^2t' + 35st^2t' + 4t^3t' + 16st^3t' - 6t^4t' - 2t^5t' - 2t^6s't'
- 14st^2s't' - 6t^3s't' + 6ts^3t' + 8ts^2t'^2 - 37st^2t'^2 + 22t^3t'^2 - 2st^3t'^2 + 8t^4t'^2 + 8ts^3t'^2
- 2t^4t'^3] - 24 + 14ss' - 8s^2s'^2 - 224tt' + st^2st' - 101t^2t'^2 - st^2s'^2t'^2 - 8s^3t'^3,
\]

where \([a] := a + \iota(a)\) for \(a \in k(s, t, s', t')\), and \(d^2 \in k(s, t)\) is given by the formula

\[
d^2 = s^2 - 4s^3 + 4t - 14st - 30s^2t - 91t^2 - 34st^2 + s^2t^2 + 40t^3 + 24st^3 + 4t^4 - 4t^5.
\]

We also take \(\rho(P) \in k(s, t)\) which is conjugate of \(P\) under the action of \(F_{20} \times F'_{20}\) but not so under the action of \(D_5 \times D'_5\). Put

\[
F^2_{s,s'}(X) := R^P_{\rho(P),D_5 \times D'_5}(X) = F^1_{\rho(s),\rho(t),s',t'}(X) = \rho(F^1_{s,s'}(X)).
\]

For \(a = (a_1, a_2) \in M^2\), we denote \(L_a := \text{Spl}_M f^D_{\text{a}}(X)\), \(G_a := \text{Gal}(f^D_{\text{a}})\) and \(G_{a,a'} := \text{Gal}(f^D_{\text{a}} f^D_{\text{a}'})\). We now state the result of the dihedral quintic case.

**Theorem 6.1 ([HM-3])**. For \(a = (a_1, a_2), a' = (a_1', a_2') \in M^2\), we assume \(#G_a \geq #G_{a'} > 1\). An answer to the field intersection problem for \(f^D_{s,t}(X)\) is given by the decomposition types \(DT(F^1_{a,a'})\) and \(DT(F^2_{a,a'})\) over \(M\) as Table 2 shows.

<table>
<thead>
<tr>
<th>(G_a)</th>
<th>(G_{a,a'})</th>
<th>(G_{a,a'})</th>
<th>(DT(F^1_{a,a'}))</th>
<th>(DT(F^2_{a,a'}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_5 \times D_5)</td>
<td>(L_a \cap L_{a'} = M)</td>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>(C_5 \times C_5)</td>
<td>([L_a \cap L_{a'} : M] = 2)</td>
<td>5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>(C_5 \times C_5)</td>
<td>(L_a = L_{a'})</td>
<td>5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>(D_5)</td>
<td>(L_a \supset L_{a'})</td>
<td>5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>(C_5 \times C_5)</td>
<td>(L_a \supset L_{a'})</td>
<td>5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>(C_5 \times C_5)</td>
<td>(L_a \neq L_{a'})</td>
<td>5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>(C_5)</td>
<td>(L_a = L_{a'})</td>
<td>5, 5</td>
<td>5, 5</td>
<td></td>
</tr>
<tr>
<td>(C_5)</td>
<td>(L_a \cap L_{a'} = M)</td>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>(C_2)</td>
<td>(C_2 \cap C_2)</td>
<td>(L_a \neq L_{a'})</td>
<td>4, 4, 2</td>
<td>4, 4, 2</td>
</tr>
<tr>
<td>(C_2)</td>
<td>(L_a = L_{a'})</td>
<td>2, 2, 2, 2, 1, 1</td>
<td>2, 2, 2, 2, 1, 1</td>
<td></td>
</tr>
</tbody>
</table>
We checked the decomposition types on Table 2 using the computer algebra system GAP [GAP].

**Remark 6.2.** In the cases of $G_{a} = D_{5}$, $C_{2}$, the quadratic subextension of $L_{s_{1},t_{1}}$ over $M$ for $(s_{1}, t_{1}) \in M^{2}$ is given by

$$M\left(\sqrt{s_{1}^{2} - 4s_{1}^{3} + 4t_{1} - 14s_{1}t_{1} - 30s_{1}^{2}t_{1} - 91t_{1}^{2} - 34s_{1}t_{1}^{2} + 24s_{1}t_{1}^{3} + 4t_{1}^{4} - 4t_{1}^{5}}\right).$$

**Remark 6.3.** As is well known, for a suitable integer $l$, we may have an answer to the field intersection problem via the formal $D_{5} \times D_{5}'$-relative $(\tau) \times (\tau')$-invariant resolvent polynomial $G_{s,s'}(X) := R_{x+lx',D_{5} \times D_{5}'}(X)$ by $x + lx'$ (cf. [Coh93], [Coh00]). Although the degree of the multi-resolvent polynomials $F_{s,s'}^{1}(X)$ and $F_{s,s'}^{1}(X)$ with respect to $X$ is 10, the degree of $G_{s,s'}(X)$ is 25 and an explicit formula of $G_{s,s'}(X)$ in terms of $s$ and $s'$ is very complicated. We remark that to construct a suitable explicit formula is significant to investigate the structure of all $G$-Galois extensions over $M$ (cf. Section 1 and Corollaries 1.4 and 1.5).

**Example 6.4.** Take $M = \mathbb{Q}$, $a = (0,3)$ and $a' = (10,3)$. Then we have

$$F_{a,a'}^{1}(X) = (X + 5)^{3}(X^{2} - 15X + 150)(X^{5} - 625X^{2} - 9375),$$

$$F_{a,a'}^{2}(X) = \left(X^{5} - \frac{125}{9}X^{3} + \frac{6250}{81}X - \frac{3125}{27}\right)\left(X^{5} - \frac{125}{9}X^{3} + \frac{6250}{81}X^{2} + \frac{15625}{81}X + \frac{15625}{27}\right).$$

The decomposition type of $F_{a,a'}^{1}(X)$ over $\mathbb{Q}$ should be 5, 2, 2, 1 (cf. also Theorem 2.1); and hence we conclude that $\text{Sp}_{\mathbb{Q}}f_{a}^{D_{5}}(X) = \text{Sp}_{\mathbb{Q}}f_{a'}^{D_{5}}(X)$ and $G_{a} = D_{5}$.

From the viewpoint of Diophantine geometry, we give some numerical examples of the field isomorphism problem of $f_{s,t}^{D_{5}}(X)$ over $M = \mathbb{Q}$ and for integral points $a, a' \in \mathbb{Z}^{2}$ using Theorem 6.1 and the explicit formula (6.1). We do not know, however, for a given $a \in \mathbb{Z}^{2}$ whether there exist only finitely many $a' \in \mathbb{Z}^{2}$ such that $\text{Sp}_{\mathbb{Q}}f_{a}^{D_{5}}(X) = \text{Sp}_{\mathbb{Q}}f_{a'}^{D_{5}}(X)$ or not (cf. Corollary 1.5).

**Example 6.5.** Take $M = \mathbb{Q}$ and $t := 1$. Then we have $f_{s_{1},t}^{D_{5}}(X) = X^{5} - 2X^{4} + (s + 2)X^{3} - (2s + 1)X^{2} + sX + 1$. For $s_{1}, s'_{1} \in \mathbb{Z}$ in the range $-10000 \leq s_{1} < s'_{1} \leq 10000$, $\text{Sp}_{\mathbb{Q}}f_{s_{1},1}^{D_{5}}(X) = \text{Sp}_{\mathbb{Q}}f_{s'_{1},1}^{D_{5}}(X)$ if and only if $(s_{1}, s'_{1}) \in X_{1} \cup X_{2}$ where

$$X_{1} = \{(-6,0), (-1,41), (-94, -10)\},$$

$$X_{2} = \{(-1,0), (-6,-1), (-18, -7), (1,34), (0,41), (-6,41), (-167, -8)\}.$$ 

It was directly checked by Theorem 6.1 that, in the range $-10000 \leq s_{1} < s'_{1} \leq 10000$, $(s_{1}, s'_{1}) \in X_{i}$ if and only if $\text{DT}(F_{s_{1},1,s'_{1},1}^{i}/\mathbb{Q})$ includes 1, for each of $i = 1, 2$.

**Example 6.6.** Kida-Renault-Yokoyama [KRY] showed that there exist infinitely many $b \in \mathbb{Q}$ such that $\text{Sp}_{\mathbb{Q}}f_{0,1}^{D_{5}}(X) = \text{Sp}_{\mathbb{Q}}f_{b,1}^{D_{5}}(X)$. Their method enables us to construct such $b$’s explicitly via rational points of an associated elliptic curve (cf. [KRY]).
They also pointed out that in the range \(-400 \leq s_1, t_1 \leq 400\) there are 25 pairs \((s_1, t_1) \in \mathbb{Z}^2\) such that \(\text{Spl}_{Q}f_{s_1, t_1}^{D_5}(X) = \text{Spl}_{Q}f_{0,1}^{D_5}(X)\). We may classify the 25 pairs by the polynomials \(F_{0,1,s_1,t_1}^{1}(X)\) and \(F_{0,1,s_1,t_1}^{2}(X)\). In the range above, for \(i = 1, 2\), \(\text{DT}(F_{0,1,s_1,t_1}^{i}/\mathbb{Q})\) includes 1 if and only if \((s_1, t_1) \in X_i\) where

\[
X_1 = \{(0, 1), (4, -1), (4, 5), (-6, 1), (-24, 19), (34, 11), (36, -5), (46, -1), (-188, 23), (264, 31), (372, 5), (378, 43)\},
\]

\[
X_2 = \{(-1, -1), (-1, 1), (5, -1), (41, 1), (-43, 5), (47, 13), (59, -5), (59, 19), (101, 19), (125, 23), (149, 11), (155, 25), (-169, 55)\}.
\]

By Theorem 6.1, we checked such pairs in the range \(-20000 \leq s_1, t_1 \leq 20000\); and added six pairs \((526, 41), (952, 113), (2302, 95), (6466, 311), (7180, 143), (7480, 169)\) to \(X_1\) and just four pairs \((785, 25), (3881, 29), (11215, 299), (19739, 281)\) to \(X_2\).

**Example 6.7.** Take the \(k\)-generic polynomial \(g_{A,B}^{C_5}(X) \in k(A, B)[X]\) for \(C_5\) which is constructed by Hashimoto-Tsunogai [HT03]:

\[
g_{A,B}^{C_5}(X) = X^5 - \frac{P}{Q^2}(A^2 - 2A + 15B^2 + 2)X^3 + \frac{P^2}{Q^3}(2BX^2 - (A-1)X - 2B)
\]

where \(P = (A^2 - A - 1)^2 + 25(A^2 + 1)B^2 + 125B^4\), \(Q = (A+7)B^2 - A + 1\). We may apply Theorem 6.1 to \(g_{A,B}^{C_5}(X)\) since there exist \(s_1, t_1 \in k(A, B)\) such that \(\text{Spl}_{k(A,B)}f_{s_1, t_1}^{D_5}(X) = \text{Spl}_{k(A,B)}g_{A,B}^{C_5}(X)\) (cf. [HT03], [HM-3]). For \(a = (a, b), a' = (a', b') \in \mathbb{Z}^2\), if \(a' = (a, \pm b)\) or \(\{a, a'\} = \{(-1, \pm b), (1, \pm b)\}\) then \(\text{Spl}_{\mathbb{Q}}g_{a,b}^{C_5}(X) = \text{Spl}_{\mathbb{Q}}g_{a,b}^{C_5}(X)\) (cf. [HM-3]). For \(a, a' \in \mathbb{Z}^2\) in the range \(-50 \leq a, a' \leq 50, 0 \leq b \leq b' \leq 50\) with \(a \neq a', \{a, a'\} \neq \{(-1, b), (1, b)\}\) we see that \(\text{Spl}_{\mathbb{Q}}g_{a,b}^{C_5}(X) = \text{Spl}_{\mathbb{Q}}g_{a,b'}^{C_5}(X)\) if and only if \((a, b, a', b') \in X_1 \cup X_2\) where

\[
X_1 = \{(-3, 1, -3, 11), (3, 3, 23, 3), (23, 3, 3, 3), (7, 3, 27, 9), (2, 2, -28, 14), (8, 11, 33, 14), (23, 5, 35, 7), (41, 11, -15, 17)\},
\]

\[
X_2 = \{(-2, 1, 3, 2), (4, 1, -6, 2), (3, 1, 13, 7), (16, 2, -12, 5), (2, 3, 18, 4), (31, 1, 1, -19, 7), (-3, 3, -33, 3), (-33, 3, -3, 3), (-16, 13, 34, 19), (-2, 3, 43, 6), (12, 4, 46, 10)\}.
\]

By Theorem 6.1, it can be checked, in the range above and for each of \(i = 1, 2\), that \((a, b, a', b') \in X_i\) if and only if the decomposition type of \(F_{a,a'}^{i}(X)\) over \(\mathbb{Q}\) includes 1.

**Example 6.8.** Let \(h_n(X)\) be Lehmer’s simplest quintic polynomial

\[
h_n(X) = X^5 + n^2X^4 - (2n^3 + 6n^2 + 10n + 10)X^3
\]

\[
+ (n^4 + 5n^3 + 11n^2 + 15n + 5)X^2 + (n^3 + 4n^2 + 10n + 10)X + 1
\]

(cf. [Leh88]), and take \(M = \mathbb{Q}\). We regard \(n\) as an independent parameter over \(\mathbb{Q}\). By the result in [HR], for Brumer’s quintic \(f_{s,t}^{D_5}(X)\), we see that \(\text{Spl}_{\mathbb{Q}(n)}h_n(X) = \text{Spl}_{\mathbb{Q}(n)}f_{s,t}^{D_5}(X)\) for each of \(i = 1, 2\), that \((a, b, a', b') \in X_i\) if and only if the decomposition type of \(F_{a,a'}^{i}(X)\) over \(\mathbb{Q}\) includes 1.
$\text{Sp}_{Q(n)} f_{s,t}^{D_{5}}(X)$ where $s = -20 - 5n + 10n^2 + 12n^3 + 5n^4 + n^5$, $t = -7 - 10n - 5n^2 - n^3$. By Theorem 6.1, we checked pairs $(n, n') \in \mathbb{Z}^2$ in the range $-10000 \leq n < n' \leq 10000$ to confirm that $\text{Sp}_{Q} h_n(X) = \text{Sp}_{Q} h_{n'}(X)$ if and only if $(n, n') = (-2, -1)$.

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References


