# Remarks on collision manifolds and nonexistence of non self-similar collision solutions in the 3-vortex problem

By

Yasuaki HIRAOKA*

## § 1. Introduction

The $N$-vortex problem in $\mathbf{R}^2$ is described in the following differential equations:

$$\dot{z}_j = i \sum_{\substack{m=1 \\ m \neq j}}^{N} \frac{k_m}{\bar{z}_j - \bar{z}_m}, \qquad j = 1, 2, \cdots, N$$

where $i = \sqrt{-1}$, $\dot{z}_j = dz_j/dt$, $z_j = x_j + iy_j$ represents the position of the $j$-th vortex, $\bar{z}_j$ expresses the complex conjugate, and $k_j$ denotes the vorticity divided by $2\pi$ of the $j$-th vortex. These ordinary differential equations are derived as a limiting problem of the incompressible Euler equation focusing on delta function type vortices. Due to the importance of understanding motions of vortices and its simplicity rather than the Euler equation, which is a partial differential equation, the $N$-vortex problem has been studied in many fields with various viewpoints (e.g., [4]).

In the previous paper [2], the author applied the technique so called McGehee's collision manifold [3], which has been developed in the $N$-body problem, to the 3-vortex problem and discussed the regularizations of the triple collision singularity. The main results there are summarized as follows:

**Theorem 1.1.** *Under the necessary condition $k_1^{-1} + k_2^{-1} + k_3^{-1} = 0$ for the existence of triple collision solutions, if $k_1 = k_2$, then the triple collision singularity is topologically regularizable under an certain natural equivalence relation.*

**Theorem 1.2.**     *Under the necessary condition* $k_1^{-1} + k_2^{-1} + k_3^{-1} = 0$ *for the existence of triple collision solutions, if* $k_1 \neq k_2$, *then there exists a small positive constant* $\epsilon$ *such that, for* $0 < |k_2 - k_1| < \epsilon$, *the triple collision singularity can not be topologically regularizable.*

Namely, in the case of $k_1 = k_2$, a solution ending in the triple collision can be connected to a solution beginning in the triple collision so as to be continuous with respect to nearby solutions. On the other hand, in the case of $k_1 \neq k_2$, a solution ending in the triple collision can not be connected to a solution beginning in the triple collision in such a way that flow results. For the proof of these results, the collision manifold plays a central role.

In this paper, we would like to briefly discuss the nonexistence of non self-similar collision solutions from the view point of the collision manifold. The arguments are quite simple as we will see later, just counting the dimension of stable and unstable manifolds to the collision manifold. For the details of proofs in this article, we would like to refer [2].

## §2.   Settings

Let us here summarize the equations we treat in this paper. First of all, the original equations of the 3-vortex problem are given by

$$(2.1) \qquad \begin{cases} \dot{z}_1 = i \left( \dfrac{k_2}{\bar{z}_1 - \bar{z}_2} + \dfrac{k_3}{\bar{z}_1 - \bar{z}_3} \right) \\[2mm] \dot{z}_2 = i \left( \dfrac{k_1}{\bar{z}_2 - \bar{z}_1} + \dfrac{k_3}{\bar{z}_2 - \bar{z}_3} \right) \\[2mm] \dot{z}_3 = i \left( \dfrac{k_1}{\bar{z}_3 - \bar{z}_1} + \dfrac{k_2}{\bar{z}_3 - \bar{z}_2} \right). \end{cases}$$

As usual, we can derive the differential equations for the lengths $R_1, R_2$, and $R_3$ of the sides of the triangle formed by the vortices (see Figure 1) from (2.1),

$$(2.2) \qquad \begin{cases} \dot{R}_1 = \dfrac{2k_1 A}{R_1} \left( \dfrac{1}{R_2^2} - \dfrac{1}{R_3^2} \right) \\[2mm] \dot{R}_2 = \dfrac{2k_2 A}{R_2} \left( \dfrac{1}{R_3^2} - \dfrac{1}{R_1^2} \right) \\[2mm] \dot{R}_3 = \dfrac{2k_3 A}{R_3} \left( \dfrac{1}{R_1^2} - \dfrac{1}{R_2^2} \right). \end{cases}$$

Here $A$ denotes the signed area of the triangle satisfying $A^2 = s(s - R_1)(s - R_2)(s - R_3)$ with $s = (R_1 + R_2 + R_3)/2$. That is, $A$ is positive or negative when the ordering of

the vortices $1, 2, 3$ is counter-clockwise or clockwise, respectively, and $|A|$ represents the area formed by the triangle. A solution $(R_1(t), R_3(t), R_3(t))$ of (2.2) is called self-similar if $R_1(t) : R_2(t) : R_3(t) = R_1(0) : R_2(0) : R_3(0)$ for all $t \in \mathbf{R}$.
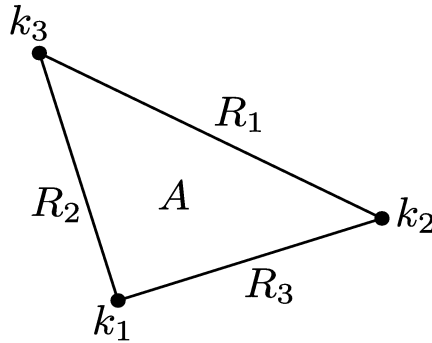


Figure 1. Three vortices

In the 3-vortex problem (2.2), the triple collision singularity is given by $S_0 = \{(R_1, R_2, R_3) = (0, 0, 0)\}$. A solution of (2.2) is said to be a collision solution if one of the $R_i, i = 1, 2, 3$, goes to zero in finite time. Let us remark that Corollary 2.6 in [2] states the nonexistence of double collision solutions in the 3-vortex problem. Moreover, it follows by Proposition 2.1 in [2] that there do not exist non collision singularities.

In this paper, we put the assumption

$$(2.3) \qquad\qquad k_1^{-1} + k_2^{-1} + k_3^{-1} = 0$$

on the vorticities, since our main interest is the collision singularities and this condition on the vorticities are necessary to have the triple collision ([1][4][5][6]). Especially, without loss of generality, we assume

$$k_3 < 0 < k_1, k_2.$$

Let us note that $A = 0$ corresponds to a solution whose configuration is collinear. Although $A = 0$ looks to be an equilibrium of (2.2), it is known that the uniqueness of the solutions fails at $A = 0$. Thus we need an additional condition $\dot{A} = 0$, where

$$\begin{aligned}
\dot{A} &= f_4(R) \\
&:= \frac{1}{2} \sum \left\{ \frac{k_1}{R_1} \left( \frac{1}{R_2^2} - \frac{1}{R_3^2} \right) \right\} \left[ (s - R_1)(s - R_2)(s - R_3) + s \sum \left\{ (s - R_2)(s - R_3) \right\} \right] \\
&\quad - s \sum \left\{ \frac{k_1}{R_1} \left( \frac{1}{R_2^2} - \frac{1}{R_3^2} \right) (s - R_2)(s - R_3) \right\},
\end{aligned}$$

in order to guarantee that a collinear configuration remains fixed for all time. Here, $R = (R_1, R_2, R_3)$ and $\Sigma \{\cdot\}$ indicates summation over a cyclic permutation of subscripts

appearing in the brace. On the previous works (e.g., [1][4][5][6]), this condition was treated separately from (2.2). However, our approach in this paper is based on the theory of dynamical systems and for this purpose, we need the uniqueness of the solutions. Therefore, instead of dealing with (2.2), we consider the following system of ordinary differential equations:

$$(2.4) \quad \begin{cases} \dot{R}_1 = f_1(R, A) = \dfrac{2k_1 A}{R_1} \left( \dfrac{1}{R_2^2} - \dfrac{1}{R_3^2} \right) \\[2mm] \dot{R}_2 = f_2(R, A) = \dfrac{2k_2 A}{R_2} \left( \dfrac{1}{R_3^2} - \dfrac{1}{R_1^2} \right) \\[2mm] \dot{R}_3 = f_3(R, A) = \dfrac{2k_3 A}{R_3} \left( \dfrac{1}{R_1^2} - \dfrac{1}{R_2^2} \right) \\[2mm] \dot{A} = f_4(R) \end{cases}$$

on

$$(R, A) \in \mathbf{R}_+^3 \times \mathbf{R}, \quad \text{where } \mathbf{R}_+ := (0, \infty).$$

On this setting, the uniqueness of the solutions of (2.4) is guaranteed even if $A = 0$.

From this extension, the solutions of the original problem (2.2) correspond to those which satisfy the relationship $A^2 = s(s - R_1)(s - R_2)(s - R_3)$ for all defined time. In particular, we can easily show the following lemma:

**Lemma 2.1.** $\alpha := A^2 - s(s - R_1)(s - R_2)(s - R_3)$ *is a first integral of* (2.4).

From this lemma, the system of differential equations (2.4) on the invariant set determined by $\alpha = 0$ represents the 3-vortex problem (2.2). Moreover, this first integral $\alpha$ plays an important role for the construction of the collision manifold.

Now let us introduce a new coordinate called the trilinear coordinate $x = (x_1, x_2, x_3)$ ([5][6])

$$(2.5) \qquad \mathbf{R}_+^3 \times \mathbf{R} \ni (R, A) \mapsto (x, A, s) \in T \times \mathbf{R} \times \mathbf{R}_+$$

with $x_i = R_i / 2s$ and $s = (R_1 + R_2 + R_3)/2$, where

$$T := \left\{ x = (x_1, x_2, x_3) \in \mathbf{R}_+^3 \mid x_1 + x_2 + x_3 = 1 \right\}.$$

The system of the differential equations (2.4) is expressed by

$$(2.6) \quad \begin{cases} \dot{x}_1 = K(x, A, s) H_1(x) \\[1mm] \dot{x}_2 = K(x, A, s) H_2(x) \\[1mm] \dot{x}_3 = K(x, A, s) H_3(x) \\[1mm] \dot{A} = \bar{f}_4(x) \\[1mm] \dot{s} = f_s(x, A, s), \end{cases}$$

where

$$H_1(x) = -k_1 x_1(x_2^2 - x_3^2) + x_1 \sum \left\{ k_1 x_1(x_2^2 - x_3^2) \right\},$$

$$H_2(x) = -k_2 x_2(x_3^2 - x_1^2) + x_2 \sum \left\{ k_1 x_1(x_2^2 - x_3^2) \right\},$$

$$H_3(x) = -k_3 x_3(x_1^2 - x_2^2) + x_3 \sum \left\{ k_1 x_1(x_2^2 - x_3^2) \right\},$$

$$K(x, A, s) = \frac{A}{8 x_1^2 x_2^2 x_3^2 s^4},$$

$$\bar{f}_4(x) = f_4(2sx)$$

$$= \frac{1}{16} \sum \left\{ \frac{k_1}{x_1} \left( \frac{1}{x_2^2} - \frac{1}{x_3^2} \right) \right\} \left[ (1 - 2x_1)(1 - 2x_2)(1 - 2x_3) + \sum \left\{ (1 - 2x_2)(1 - 2x_3) \right\} \right]$$

$$- \frac{1}{8} \sum \left\{ \frac{k_1}{x_1} \left( \frac{1}{x_2^2} - \frac{1}{x_3^2} \right) (1 - 2x_2)(1 - 2x_3) \right\},$$

$$f_s(x, A, s) = \frac{A}{8s^3} \sum \left\{ \frac{k_1}{x_1} \left( \frac{1}{x_2^2} - \frac{1}{x_3^2} \right) \right\}.$$

In this coordinate, $s = 0$ corresponds to the triple collision. Here, the first integral becomes $\alpha = A^2 - s^4(1 - 2x_1)(1 - 2x_2)(1 - 2x_3)$. Let us note that if $\alpha = 0$, then $0 < x_1, x_2, x_3 \leq 1/2$ must be satisfied, which correspond that the variables $(R_1, R_2, R_3)$ satisfy the triangle inequalities. Hereafter, we fix $\alpha = 0$.

Let us recall from Synge [5] that

$$(2.7) \qquad l := \{ x \in T \mid \psi(x) := k_1^{-1} x_1^2 + k_2^{-1} x_2^2 + k_3^{-1} x_3^2 = 0 \}$$

gives a hyperbolic curve (see Figure 2). Under the condition (2.3), the point

$$E := (1/3, 1/3, 1/3),$$

which determines the equilateral triangles, stays on the hyperbolic curve $l$. Let us denote the points expressing the double collision configurations by

$$P_1 = (0, 1/2, 1/2), \quad P_2(1/2, 0, 1/2), \quad P_3 = (1/2, 1/2, 0),$$

respectively. In addition, the hyperbolic curve $l$ intersects the edges $P_2 P_3$ and $P_1 P_3$ at

$$Q_1 = \left( \frac{1}{2}, \frac{k_1 + k_2 - \sqrt{K}}{2k_2}, \frac{\sqrt{K} - k_1}{2k_2} \right),$$

$$(2.8) \qquad Q_2 = \left( \frac{k_1 + k_2 - \sqrt{K}}{2k_1}, \frac{1}{2}, \frac{\sqrt{K} - k_2}{2k_1} \right),$$

respectively, where $K = k_1^2 + k_2^2 + k_1 k_2$. These correspond to the collinear configurations (see Figure 2). For later use, let us also define the point $Q_3 := (1/4, 1/4, 1/2)$, which is the middle point of the edge $P_1 P_2$.
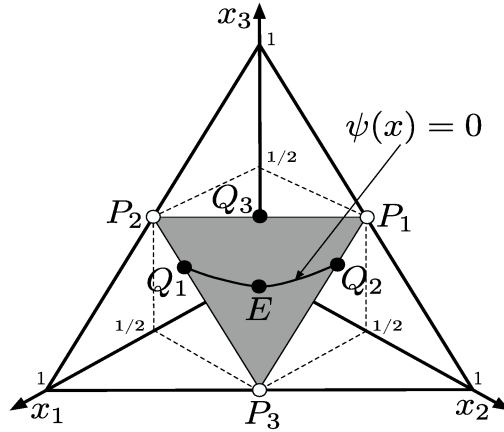
Figure 2. Trilinear coordinate

We here introduce several notations in order to describe the next lemma. Let us denote by $L_{Q_1 E}$ the interval without the end points $Q_1$ and $E$ on $l$ and by $L_{Q_2 E}$ the interval without the end points $Q_2$ and $E$ on $l$, respectively. Moreover, we introduce the following subsets in $T \times \mathbf{R} \times \mathbf{R}_+$ with $\alpha = 0$:

$$\hat{L}_{Q_1 E} := \{(x, A, s) \mid x \in L_{Q_1 E}, s \in \mathbf{R}_+, A > 0\},$$
$$\hat{L}_{Q_1 E^*} := \{(x, A, s) \mid x \in L_{Q_1 E}, s \in \mathbf{R}_+, A < 0\},$$
$$\hat{L}_{Q_2 E} := \{(x, A, s) \mid x \in L_{Q_2 E}, s \in \mathbf{R}_+, A > 0\},$$
$$\hat{L}_{Q_2 E^*} := \{(x, A, s) \mid x \in L_{Q_2 E}, s \in \mathbf{R}_+, A < 0\},$$

Then, we summarize the arguments for the solutions of (2.6) in [5][6] as follows:

**Lemma 2.2.** *Suppose $\alpha = 0$. On the hyperbolic curve (2.7), $H_1(x) = H_2(x) = H_3(x) = 0$, which mean that the solutions whose initial conditions for $x$ lie on (2.7) are self-similar. $(E, A, s), (E, -A, s), (Q_1, 0, s),$ and $(Q_2, 0, s)$ are one-parameter families of equilibria parametrized by $s \in \mathbf{R}_+$. For the initial conditions located in $\hat{L}_{Q_1 E^*}$ and $\hat{L}_{Q_2 E}$, these solutions self-similarly converge to the triple collision in positive finite time. In the negative time direction, these solutions grow up to be unbounded in infinite time. On the other hand, for the initial conditions located in $\hat{L}_{Q_1 E}$ and $\hat{L}_{Q_2 E^*}$, these solutions self-similarly converge to the triple collision in negative finite time. In the positive time direction, these solutions grow up to be unbounded in infinite time.*

## §3. Nonexistence of non self-similar collision solutions

It was explained in the previous section that the triple collision solutions are the only singular solutions. Next, we consider to blow up the triple collision singularity in

(2.6) and derive an appropriate dynamical system by pasting an invariant manifold onto the singularity.

Let us introduce a new variable $B = \dfrac{A}{s^2}$:

$$(3.1) \qquad T \times \mathbf{R} \times \mathbf{R}_+ \ni (x, A, s) \mapsto (x, B, s) \in T \times \mathbf{R} \times \mathbf{R}_+$$

and a new time variable by $t = s^2 x_1^2 x_2^2 x_3^2 \tau$. For our purpose to blow up the triple collision singularity, we do not need to add the term $x_1^2 x_2^2 x_3^2$ in the time scaling. However, for the simplicity of the expression of transformed differential equations, we here adopt this time scaling. Essentially, this time transformation acts to slow down the orbits for small $s$ so that a solution ending in the triple collision takes infinite time to reach it. Then the differential equations (2.6) take the following form:

$$(3.2) \quad \begin{cases} \dfrac{dx_1}{d\tau} = \dfrac{k_1 x_1 B}{8}(k_3 x_3 - k_2 x_2)\psi(x) \\[2mm] \dfrac{dx_2}{d\tau} = \dfrac{k_2 x_2 B}{8}(k_1 x_1 - k_3 x_3)\psi(x) \\[2mm] \dfrac{dx_3}{d\tau} = \dfrac{k_3 x_3 B}{8}(k_2 x_2 - k_1 x_1)\psi(x) \\[2mm] \dfrac{dB}{d\tau} = g(x) - \dfrac{k_3 B^2}{4}\left\{ x_2^2 - x_1^2 + (k_1 x_1 - k_2 x_2)\psi(x) \right\} \\[2mm] \dfrac{ds}{d\tau} = \dfrac{k_3 s B}{8}\left\{ x_2^2 - x_1^2 + (k_1 x_1 - k_2 x_2)\psi(x) \right\}, \end{cases}$$

where

$$g(x) = \dfrac{k_3}{16}\left( x_2^2 - x_1^2 + (k_1 x_1 - k_2 x_2)\psi(x) \right)\left[ (1 - 2x_1)(1 - 2x_2)(1 - 2x_3) + \sum\{(1 - 2x_2)(1 - 2x_3)\} \right]$$
$$- \dfrac{1}{8}\sum\{k_1 x_1 (x_3^2 - x_2^2)(1 - 2x_2)(1 - 2x_3)\},$$

and we used the condition (2.3) for the derivation.

Note that the differential equations (3.2) do not have singularities at $s = 0$ and define a vector field on $\{(x, B, s) \in T \times \mathbf{R} \times [0, \infty)\}$. Thus, we have extended the differential equations (2.6) to include the triple collision of vortices. Let us also note that the set determined by $s = 0$ is an invariant set. It means that the set of orbits ending in the triple collision is now the set of orbits asymptotic to the invariant set.

From the restriction $\alpha = s^4 \left\{ B^2 - (1 - 2x_1)(1 - 2x_2)(1 - 2x_3) \right\} = 0$ and the differential equations for $x$ and $B$, which are independent of $s$, the set

$$\left\{ (x, B, s) \mid x \in T, B^2 = (1 - 2x_1)(1 - 2x_2)(1 - 2x_3) \right\}$$

becomes an invariant set, especially

$$\mathcal{C} := \left\{ (x, B, 0) \mid x \in T, B^2 = (1 - 2x_1)(1 - 2x_2)(1 - 2x_3) \right\}$$

is an invariant set. We call $\mathcal{C}$ the collision manifold in this paper, which plays a similar role for the analysis of the regularization problems in [3].

Let us investigate the shape of $\mathcal{C}$. Recall that, due to $B^2 = (1 - 2x_1)(1 - 2x_2)(1 - 2x_3)$, we have $0 < x_1, x_2, x_3 \leq 1/2$. On each line given by $x_1 = 1/2, x_2 = 1/2$, or $x_3 = 1/2$, we have $B = 0$. In addition, $E = (1/3, 1/3, 1/3)$ gives the unique maximum $(B = \frac{1}{3\sqrt{3}})$ and minimum $(B = -\frac{1}{3\sqrt{3}})$ points. Therefore $\mathcal{C}$ is homeomorphic to a two dimensional sphere minus three points.
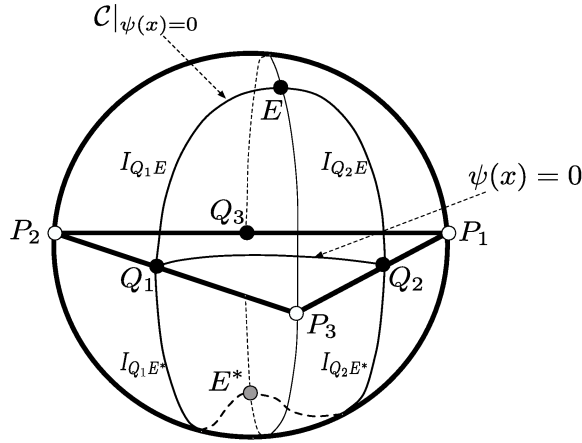


Figure 3. Collision manifold

Taking the variable changes (2.5) and (3.1) into account, we use the same symbols $P_1, P_2, P_3, Q_1, Q_2, Q_3$ for the points on $\bar{\mathcal{C}}$ as those for the corresponding points on $\bar{T}$. Here, $\bar{\mathcal{C}}$ and $\bar{T}$ represent these closures. In addition, we denote by $E$ and $E^*$ the maximum and minimum points of $B$ on $\mathcal{C}$, respectively (see Figure 3).

Let us investigate the flow on $\mathcal{C}$. At first, we can easily show the following lemma:

**Lemma 3.1.** *Under the condition (2.3), all points on the closed curve determined by $\{(x, B, 0) \in \mathcal{C} \mid \psi(x) = 0\}$ are equilibria.*

Obviously, this property on $\mathcal{C}$ is induced by Lemma 2.2.

Next, we study the stabilities of these equilibria. Let us denote by $I_{Q_1 E}$ the interval without the end points $Q_1$ and $E$ on $\{(x, B, 0) \in \mathcal{C} \mid \psi(x) = 0, B > 0\}$. The interval without the endpoints $Q_2$ and $E$ on $\{(x, B, 0) \in \mathcal{C} \mid \psi(x) = 0, B > 0\}$ is denoted by $I_{Q_2 E}$. Similarly, $I_{Q_1 E^*}$ and $I_{Q_2 E^*}$ are defined for the case of $B < 0$ (see Figure 3).

**Lemma 3.2.** *Each point $(q, 0) \in I_{Q_2 E}$ and $(r^*, 0) \in I_{Q_1 E^*}$ has the one dimensional unstable manifold on $\mathcal{C}$. Similarly, each point $(q^*, 0) \in I_{Q_2 E^*}$ and $(r, 0) \in I_{Q_1 E}$ has the one dimensional stable manifold on $\mathcal{C}$. In addition, these stable and unstable manifolds are transverse to the curve $\{(x, B, 0) \in \mathcal{C} \mid \psi(x) = 0\}$.*

Now, we are in the position to explain the nonexistence of non self-similar collision solutions. From the fact that there do not exist double collision solutions, we can show that all triple collision solutions converge to the hyperbolic curve $\psi(x) = 0$ on the collision manifold $\mathcal{C}$. Namely, the triple collision solutions asymptotically converge to equilibria, whose energy levels are determined by the initial conditions. However all equilibria which correspond to triple collision ending have one dimensional unstable manifolds on $\mathcal{C}$. Therefore, they do not have another directions for converging to $\mathcal{C}$ except for self-similar collision solutions, whose initial states themselves start from $\psi(x) = 0$. This is the geometrical reasoning of the nonexistence of non self-similar collision solutions.

From this argument, it follows that if some equilibria of the collision ending type have stable manifolds on $\mathcal{C}$, non self-similar collision solutions can exist. This suggests a promising method to a challenging question in the $N$-vortex problem with $N \geq 4$ about existence of non self-similar collision solutions. Namely, we first construct a collision manifold by a similar manner in [2], and if there exists a stable manifold, it implies the existence of such solutions.

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