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On oscillatory solutions in ultradiscrete system

By

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Abstract

We discuss oscillatory solutions in ultradiscrete systems of linear and nonlinear equations. Firstly, existence of oscillatory solutions is shown in the ultradiscrete system corresponding to the second-order linear difference equation. For the ultradiscrete Sine–Gordon equation, we construct oscillatory solutions which are considered to be a counterpart of the breather solutions.

§ 1. Introduction

Ultradiscretization is a limiting procedure constructing a cellular automaton from a given difference equation. To apply this procedure, we first transform a dependent variable in a given equation x_n to a new variable X_n by

$$x_n = e^{X_n/\varepsilon}, \tag{1.1}$$

where $\varepsilon > 0$ is a parameter. Then we take the limit $\varepsilon \to +0$. As a result, multiplication, division and addition for the original variables are replaced by addition, subtraction and max-function for the new ones, respectively. However, it is not known how to cover variables with nondefinite sign. Hence, a serious difficulty arises in ultradiscretization of the trigonometric functions, which are often employed for describing oscillatory phenomena. In this paper, we report one method to capture oscillatory phenomena in ultradiscrete systems [1, 2]. We first discuss an ultradiscretization of a second-order linear difference equation and its solution in section 2. It is shown that behaviour of solutions in the ultradiscrete system is classified by system parameters, as is in continuous systems.

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Moreover, we present an oscillatory solution whose origin is not the trigonometric functions. In section 3, we apply this result to a nonlinear system, the Sine–Gordon (SG) equation. We construct oscillatory solutions of the ultradiscrete SG equation. These solutions are considered to be a counterpart of the breather solutions.

§ 2. Linear System

As is well known, the general solution of the second-order linear differential equation

$$\frac{d^2x}{dt^2} - (\lambda+\mu)\frac{dx}{dt} + \lambda\mu x = 0 \tag{2.1}$$

is classified by constants λ and μ:

$$x(t) = c_1 e^{\lambda t} + c_2 e^{\mu t} \qquad (\lambda, \mu \in \mathbb{R},\ \lambda \neq \mu), \tag{2.2}$$

$$x(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) \qquad (\lambda = \mu^* = \alpha + i\beta \in \mathbb{C}). \tag{2.3}$$

We remark that (2.2) includes the case of $\lambda = \mu \in \mathbb{R}$ as its limit. Similarly, the general solution of the second-order linear difference equation

$$x_{n+1} - (\lambda+\mu)x_n + \lambda\mu x_{n-1} = 0 \tag{2.4}$$

is classified as

$$x_n = c_1\lambda^n + c_2\mu^n \qquad (\lambda, \mu \in \mathbb{R},\ \lambda \neq \mu), \tag{2.5}$$

$$x_n = c_1(\alpha+i\beta)^n + c_2(\alpha-i\beta)^n \qquad (\lambda = \mu^* = \alpha + i\beta \in \mathbb{C}). \tag{2.6}$$

In both systems, essential behaviour of solutions is determined only by the values of system parameters: exponentially *growth* (or decay) or *oscillation*.

Let us construct an ultradiscrete analogue of the difference equation

$$x_{n+1} + bx_n + cx_{n-1} = 0 \quad (n \geq 0), \tag{2.7}$$

where b and c are constants [1]. We assume $x_n \geq 0$ for $\forall n$. We first consider the case of b, $c < 0$. Setting

$$b = -e^{B/\varepsilon}, \quad c = -e^{C/\varepsilon}, \quad x_n = e^{X_n/\varepsilon}, \tag{2.8}$$

transposing the negative terms and taking the limit $\varepsilon \to +0$, we have the ultradiscrete system

$$X_{n+1} = \max(X_n + B,\ X_{n-1} + C). \tag{2.9}$$

Solving (2.9) for given initial values X_0 and X_1, we obtain solutions

$$\text{(2.10)} \quad \text{if } 2B \geq C, \quad X_n = \max(X_1 + B,\ X_0 + C) + (n-2)B,$$

$$\text{(2.11)} \quad \text{if } 2B < C, \quad X_n = \begin{cases} \max(X_1 + B,\ X_0 + C) + (k-1)C & (n = 2k), \\ \max(X_1,\ X_0 + B) + kC & (n = 2k+1). \end{cases}$$

We have two types of solutions, a linear growth type (2.10) and an oscillating type (2.11). Essential behaviour of solutions is again determined only by the values of system parameters B and C. Typical behaviour of these solutions is shown in Figure 1–3. We find in Figure 2 and 3 that the solution (2.11) actually describes an oscillatory phenomenon.

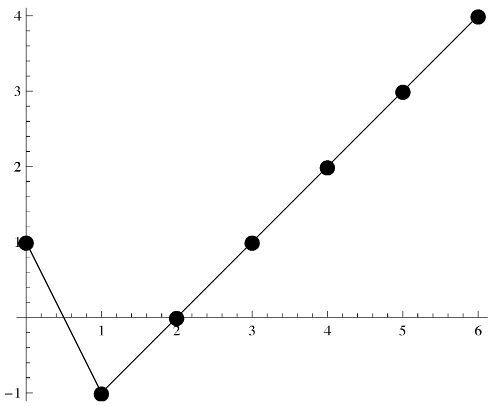

Figure 1. An example of linear growth type solution. Values of the system parameter and initial values are $B = 1$, $C = -1$, $X_0 = 1$, $X_1 = -1$.

We next study the case of $b > 0$ and $c < 0$. Setting $b = e^{B/\varepsilon}$, $c = -e^{C/\varepsilon}$, $x_n = e^{X_n/\varepsilon}$, transposing the negative term and taking the limit $\varepsilon \to +0$, we have

$$\text{(2.12)} \qquad \max(X_{n+1},\ X_n + B) = X_{n-1} + C.$$

We again consider the initial value problem of (2.12). Here, we require existence and uniqueness of the solution at each step. Then, we have the unique solution for $n \geq 0$ only in the case that $2B < C$ and X_1 satisfies $X_0 + B < X_1 < X_0 + C - B$ for arbitrary X_0,

$$\text{(2.13)} \qquad X_n = \begin{cases} X_0 + kC & (n = 2k \geq 0), \\ X_1 + kC & (n = 2k+1 \geq 1), \end{cases}$$

which is an oscillatory type.

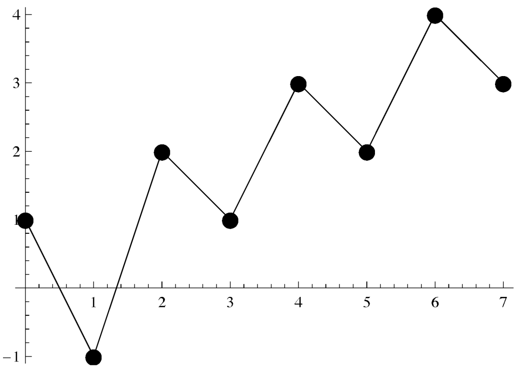

Figure 2. An example of oscillating type solution. Values of the system parameter and initial values are $B = -1$, $C = 1$, $X_0 = 1$, $X_1 = -1$.

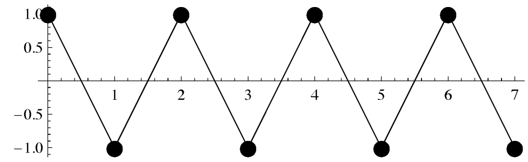

Figure 3. An example of oscillating type solution. Values of the system parameter and initial values are $B = -2$, $C = 0$, $X_0 = 1$, $X_1 = -1$.

In the case of $b < 0$ and $c > 0$, we have the ultradiscrete system

$$\max(X_{n+1},\ X_{n-1} + C) = X_n + B. \tag{2.14}$$

Again we consider the initial value problem and require existence and uniqueness of solution. The unique solution, which is a linear growth type,

$$X_n = X_1 + (n-1)B \quad (n \geq 1) \tag{2.15}$$

exists only in the case of $2B > C$ and $X_1 > X_0 + C - B$.

Finally, in the case of b, $c > 0$, we have the ultradiscrete system

$$\max(X_{n+1},\ X_n + B,\ X_{n-1} + C) = -\infty, \tag{2.16}$$

whose solution is a *trivial* solution $X_n = -\infty$ for $n \geq 0$.

The solution in the case $b < 0$, $c < 0$ has the richest structure among all the cases. In order to clarify the origin of this solution, we derive it by taking the limit of a solution of (2.7). The characteristic roots of (2.7) are

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{e^{B/\varepsilon} \pm \sqrt{e^{2B/\varepsilon} + 4e^{C/\varepsilon}}}{2}, \tag{2.17}$$

where we put $b = -e^{B/\varepsilon}$ and $c = -e^{C/\varepsilon}$. The general solution is given by

$$x_n = a_1(\varepsilon){\lambda_+}^n + a_2(\varepsilon){\lambda_-}^n, \tag{2.18}$$

where $a_1(\varepsilon)$ and $a_2(\varepsilon)$ do not depend on n. If $2B \geq C$, we have

$$\lambda_+ \sim e^{B/\varepsilon}, \quad \lambda_- \sim -e^{(C-B)/\varepsilon} \quad (\varepsilon \to +0). \tag{2.19}$$

As a sufficient condition of $x_n > 0$, we assume $a_1 = e^{A_1/\varepsilon}$, $a_2 = e^{A_2/\varepsilon}$ and $A_1 \gg A_2$. Then

$$x_n \sim e^{(A_1+nB)/\varepsilon} \quad (\varepsilon \to +0) \tag{2.20}$$

and its ultradiscrete limit gives a growth type solution

$$X_n = A_1 + nB. \tag{2.21}$$

If $2B < C$, we have

$$\lambda_+ \sim e^{C/2\varepsilon}, \quad \lambda_- \sim -e^{C/2\varepsilon} \quad (\varepsilon \to +0). \tag{2.22}$$

Here we set $a_1 = e^{A_1/\varepsilon} + e^{A_2/\varepsilon}$ and $a_2 = e^{A_2/\varepsilon}$, which is valid for our assumption $x_n > 0$, and further assume $A_1 < A_2$. Then, in $\varepsilon \to +0$, we have

$$\begin{aligned} x_n &\sim (e^{A_1/\varepsilon} + e^{A_2/\varepsilon})e^{nC/2\varepsilon} + (-1)^n e^{A_2/\varepsilon} e^{nC/2\varepsilon} \\ &\sim \begin{cases} (e^{A_1/\varepsilon} + 2e^{A_2/\varepsilon})e^{nC/2\varepsilon} & (n\text{: even}), \\ e^{A_1/\varepsilon} e^{nC/2\varepsilon} & (n\text{: odd}), \end{cases} \\ &\sim \begin{cases} e^{(A_2+nC/2)/\varepsilon} & (n\text{: even}), \\ e^{(A_1+nC/2)/\varepsilon} & (n\text{: odd}). \end{cases} \end{aligned} \tag{2.23}$$

Its ultradiscrete limit gives an oscillatory type solution

$$X_n = \begin{cases} A_2 + nC/2 & (n\text{: even}), \\ A_1 + nC/2 & (n\text{: odd}). \end{cases} \tag{2.24}$$

We comment that the origin of an oscillatory solution is not the trigonometric function. A point constructing an oscillatory type solution is to consider a pair of the exponential functions with positive and negative roots, respectively.

§ 3. Nonlinear System

We apply the result in the previous section to a nonlinear system [2]. We consider the SG equation

$$\frac{\partial^2 \varphi}{\partial x \partial t} = \sin \varphi. \tag{3.1}$$

As is well known, the SG equation possesses the N-soliton solution, which is usually called the multi-kink solution. The two-soliton solution is written as

$$\varphi = 4 \tan^{-1} \left(\frac{e^{\eta_1} + e^{\eta_2}}{1 - \frac{(p_1-p_2)^2}{(p_1+p_2)^2} e^{\eta_1+\eta_2}} \right), \tag{3.2}$$

$$\eta_j := p_j x + \frac{t}{p_j} + d_j. \tag{3.3}$$

The breather solution, which describes an oscillatory phenomenon, is also a well-known special solution of (3.1). It is obtained as a specific case of (3.2). If we put $p_1 = a + ib$, $p_2 = a - ib$ $(a, b \in \mathbb{R})$ and $d_1 = d_2 = d \in \mathbb{R}$ for simplicity, (3.2) is reduced to the breather solution

$$\varphi = 4\tan^{-1}\left(\frac{2e^{ax+\frac{at}{a^2+b^2}+d}\cos(bx - \frac{bt}{a^2+b^2} + d)}{1 + \frac{b^2}{a^2}e^{2(ax+\frac{at}{a^2+b^2}+d)}}\right). \tag{3.4}$$

If we further introduce new axes (ξ, s) by

$$\xi = x + t, \quad s = x - t \tag{3.5}$$

and put $a^2 + b^2 = 1$ for simplicity, (3.4) is deformed into a well-known form,

$$\varphi = 4\tan^{-1}\left(\left|\frac{a}{b}\right| \operatorname{sech}(a\xi - \log\left|\frac{a}{b}\right|)\cos(bs)\right). \tag{3.6}$$

An integrable discrete analogue of (3.1) is given by Hirota [3]

$$\sin\left(\frac{\phi_{n+1}^{m+1} + \phi_{n-1}^{m-1} - \phi_{n+1}^{m-1} - \phi_{n-1}^{m+1}}{4}\right) = \delta^2 \sin\left(\frac{\phi_{n+1}^{m+1} + \phi_{n-1}^{m-1} + \phi_{n+1}^{m-1} + \phi_{n-1}^{m+1}}{4}\right) \tag{3.7}$$

through the bilinearizing technique. In order to construct an ultradiscrete analogue of the SG equation, coworkers and two of the authors (S.I and J.S) proposed another discrete SG equation [4]

$$\begin{vmatrix} \left(1-\delta^2\right)u_{n-1}^{m-1} - 1 & \left(1+\delta^2\right)/u_{n-1}^{m+1} - 1 \\ \left(1+\delta^2\right)/u_{n+1}^{m-1} - 1 & \left(1-\delta^2\right)u_{n+1}^{m+1} - 1 \end{vmatrix} = 0. \tag{3.8}$$

If we introduce a new variable f_j^t by

$$u_n^m = \frac{f_{n+1}^{m+1} f_{n-1}^{m-1}}{f_{n+1}^{m-1} f_{n-1}^{m+1}}, \tag{3.9}$$

(3.8) is reduced to the trilinear form

$$\begin{vmatrix} \left(1-\delta^2\right)f_{n-2}^{m-2} & f_{n-2}^{m} & \left(1+\delta^2\right)f_{n-2}^{m+2} \\ f_n^{m-2} & f_n^m & f_n^{m+2} \\ \left(1+\delta^2\right)f_{n+2}^{m-2} & f_{n+2}^{m} & \left(1-\delta^2\right)f_{n+2}^{m+2} \end{vmatrix} = 0. \tag{3.10}$$

For the purpose of ultradiscretization, setting

$$\delta = \tanh(L/2\varepsilon), \; f_n^m = e^{F_n^m/\varepsilon}, \; u_n^m = e^{U_n^m/\varepsilon} \tag{3.11}$$

and taking the limit $\varepsilon \to +0$, we have an ultradiscrete analogue of the SG (udSG) equation for U_n^m

$$\begin{aligned}&\max\Big[-|L| + U_{n+1}^{m+1} + U_{n-1}^{m-1},\, |L| - U_{n+1}^{m-1},\, |L| - U_{n-1}^{m+1}\Big]\\ &= \max\Big[|L| - U_{n+1}^{m-1} - U_{n-1}^{m+1},\, U_{n+1}^{m+1},\, U_{n-1}^{m-1}\Big]\end{aligned} \tag{3.12}$$

from (3.8) and for F_n^m

$$\begin{aligned}&\max\Big[-|L| + F_{n+2}^{m+2} + F_n^m + F_{n-2}^{m-2},\, |L| + F_{n+2}^{m-2} + F_n^{m+2} + F_{n-2}^m,\\ &\qquad |L| + F_{n+2}^m + F_n^{m-2} + F_{n-2}^{m+2}\Big]\\ &= \max\Big[|L| + F_{n+2}^{m-2} + F_n^m + F_{n-2}^{m+2},\, F_{n+2}^{m+2} + F_n^{m-2} + F_{n-2}^m,\\ &\qquad F_{n+2}^m + F_n^{m+2} + F_{n-2}^{m-2}\Big]\end{aligned} \tag{3.13}$$

from (3.10) and the relation between F_n^m and U_n^m

$$U_n^m = F_{n+1}^{m+1} + F_{n-1}^{m-1} - F_{n+1}^{m-1} - F_{n-1}^{m+1} \tag{3.14}$$

from (3.9). Refer to [4] for more details about the udSG equation and its soliton solutions.

For the purpose of our discussion, we consider the two-soliton solution of (3.10),

$$f_n^m = 1 + a_1 x_1 + a_2 x_2 + a_1 a_2 b_{12} x_1 x_2, \tag{3.15}$$

$$x_j := {p_j}^n {q_j}^m, \tag{3.16}$$

$$b_{jk} := \frac{({p_j}^2 - {p_k}^2)^2}{((p_j p_k)^2 - 1)^2}, \tag{3.17}$$

where $p_j,\ q_j$ are parameters satisfying the dispersion relation

$$\delta^2({p_j}^2 + 1)({q_j}^2 + 1) = ({p_j}^2 - 1)({q_j}^2 - 1) \tag{3.18}$$

and a_j's are arbitrary phase constants. Let us construct a 2-periodic solution. If we set

$$p_2 = -p_1,\ q_2 = q_1,\ a_1 = \alpha_1 + \alpha_2,\ a_2 = \alpha_2, \tag{3.19}$$

then (3.15) is reduced to

$$f_n^m = \begin{cases} 1 + (\alpha_1 + 2\alpha_2)x_1 & (n : \text{even}),\\ 1 + \alpha_1 x_1 & (n : \text{odd}).\end{cases} \tag{3.20}$$

In order to ultradiscretize (3.20), we put

$$p_1 = e^{P_1/\varepsilon},\quad q_1 = e^{Q_1/\varepsilon},\quad \alpha_1 = e^{A_1/\varepsilon},\quad \alpha_2 = e^{A_2/\varepsilon}\quad (A_1 < A_2) \tag{3.21}$$

and take the limit $\varepsilon \to +0$. Then we have the ultradiscrete analogue of (3.20),

$$F_n^m = \begin{cases} \max(0,\ P_1 n + Q_1 m + A_2) & (n : \text{even}), \\ \max(0,\ P_1 n + Q_1 m + A_1) & (n : \text{odd}), \end{cases} \tag{3.22}$$

where P_1 and Q_1 satisfy the dispersion relation

$$|P_1 + Q_1| = |L| + |P_1 - Q_1|. \tag{3.23}$$

We have a solution of (3.12) by substituting (3.22) into (3.14). In order to exaggerate its periodic behaviour, we introduce new independent variables (k, l) by

$$n = k - l,\ m = k + l \tag{3.24}$$

and consider a specific case $P = Q = |L|/2$.

Tables 1, 2 show behaviour of U_n^m for various values of parameters A_1, A_2. In both cases, the solutions describe localized pulses oscillating in period 2 for l. This behaviour is similar to that of the breather solution.

	$\cdots$	-2	-1	0	1	2	3	$\cdots$	$\to k$
l: even	$\cdots$	0	0	1	0	0	0	$\cdots$	
l: odd	$\cdots$	0	0	2	1	0	0	$\cdots$	

Table 1. Behaviour of oscillatory solution. $L = 2$, $P_1 = Q_1 = 1$, $A_1 = -1$, $A_2 = 0$.

	$\cdots$	-4	-3	-2	-1	0	1	2	3	$\cdots$	$\to k$
l: even	$\cdots$	0	0	0	1	1	0	0	0	$\cdots$	
l: odd	$\cdots$	0	0	1	0	0	1	0	0	$\cdots$	

Table 2. Behaviour of oscillatory solution. $L = 2$, $P_1 = Q_1 = 1$, $A_1 = -1$, $A_2 = 3$.

We can construct the oscillatory solutions with richer structure by starting from the four-soliton solution. Refer to [2] for details of these solutions.

§ 4. Concluding Remarks

We have given the ultradiscrete analogue of the second-order linear equation. The solutions of the ultradiscrete system are classified to two types, linear growth and oscillating types. This classification depends only on the values of system parameters. We have also given oscillating solutions of the udSG equation. They are considered to be a counterpart of the breather solutions. We comment that these solutions are essentially 2-periodic ones due to its construction.

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